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Capacity of attractors

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Abstract. Let f be a diffeomorphism of a manifold and Λ be an f-invariant set supporting an ergodic Borel probability measure μ with certain properties. A lower bound on the capacity of Λ is given in terms of the μ-Lyapunov exponents. This applies in particular to Axiom A attractors and their Bowen–Ruelle measure.

Let $f: M \circlearrowleft$ be a diffeomorphism of a manifold. Suppose $U \subset M$ is an open set with $f\bar{U} \subset U$. This tells us that there is an attracting set Λ in U. A natural question is: how big is Λ? Intuitively the size of Λ is related to the relative strengths of expansion and contraction of f near this attracting set. We proceed to formulate a theorem in this direction.

First, we must decide on a notion of 'size'. Riemannian volume seems most natural, but attractors in 'dissipative' systems often have Riemannian measure zero. Hausdorff dimension is a classical tool for distinguishing between sets of measure zero. Our results are in terms of *capacity*, a notion very similar to Hausdorff dimension. Precise definitions are given later.

THEOREM. *Let $f: M^p \circlearrowleft$ be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism of a compact Riemannian manifold and let μ be an ergodic Borel probability measure on M with Lyapunov exponents $\lambda_1 \geq \cdots \geq \lambda_p$. Suppose also that $\lambda_u > 0 > \lambda_{u+1}$ and that μ has absolutely continuous conditional measures on unstable leaves. Then the lower capacity of the support of μ,*

$$\underline{C}(\operatorname{supp} \mu) \geq u + \frac{\lambda_1 + \cdots + \lambda_u}{|\lambda_p|}.$$

COROLLARY. *If Ω is an attractor of a C^2 Axiom A diffeomorphism and μ is the Bowen–Ruelle measure* [**1**] *on Ω, then*

$$\underline{C}(\Omega) \geq u + \frac{\lambda_1 + \cdots + \lambda_u}{|\lambda_p|},$$

where $\lambda_1 \geq \cdots \geq \lambda_p$ are the μ-exponents and $u = \dim E^u$.

This work is motivated by a conjecture of Yorke's [**3**], [**4**] and complements the work of Douady, Oesterlé [**2**] and Ledrappier [**6**] who have given upper bounds

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for the Hausdorff dimension of attractors. Mallet-Paret [**8**], Manning [**10**], Mañé [**9**] and Takens [**15**] have also done work on the dimension of invariant sets.

The author thanks C. Robinson for the references in the proof of lemma 3 and the referee for his comments.

Remark. After this note was written the author found out that F. Ledrappier had independently arrived at a very similar result [**7**].

1. *Definitions*

Let X be a compact metric space. For $\varepsilon > 0$, let $N(\varepsilon)$ be the smallest number of ε-balls that cover X. Define the *capacity* or *upper capacity of* X, $\bar{C}(X)$ to be

$$\limsup_{\varepsilon\to 0} \frac{\log N(\varepsilon)}{\log (1/\varepsilon)}$$

and the *lower capacity* of X, $\underline{C}(X)$ to be

$$\liminf_{\varepsilon\to 0} \frac{\log N(\varepsilon)}{\log (1/\varepsilon)}.$$

(See for instance [**3**], [**9**] and [**15**].)

It is easy to verify that

$$d < \underline{C}(X) \Rightarrow \lim_{\varepsilon\to 0} N(\varepsilon)\varepsilon^d = \infty,$$

$$\underline{C}(X) < d < \bar{C}(X) \Rightarrow \begin{cases} \limsup_{\varepsilon\to 0} N(\varepsilon)\varepsilon^d = \infty, \\ \liminf_{\varepsilon\to 0} N(\varepsilon)\varepsilon^d = 0, \end{cases}$$

and

$$d > \bar{C}(X) \Rightarrow \lim_{\varepsilon\to 0} N(\varepsilon)\varepsilon^d = 0.$$

Recall that if $N(\varepsilon)\varepsilon^d$ is replaced by

$$\alpha(\varepsilon, d) = \operatorname*{Inf}_{\substack{\text{covers } \mathfrak{A} \\ \text{diam } \mathfrak{A} < \varepsilon}} \sum_{A\in\mathfrak{A}} (\text{diam } A)^d,$$

then $\lim_{\varepsilon\to 0} \alpha(\varepsilon, d)$ always exists and the unique number d_0 with

$$d < d_0 \Rightarrow \lim_{\varepsilon\to 0} \alpha(\varepsilon, d) = \infty,$$

and

$$d > d_0 \Rightarrow \lim_{\varepsilon\to 0} \alpha(\varepsilon, d) = 0$$

is the *Hausdorff dimension* of X.

In general,

$$HD(X) \le \underline{C}(X) \le \bar{C}(X).$$

Examples of compact sets with strict inequalities can easily be constructed but I do not know if these numbers coincide for nice invariant sets such as Axiom A basic sets.

Capacity can also be defined in terms of separated sets. Let $S(\varepsilon)$ denote the maximum number of ε-separated points in X. Then $S(2\varepsilon)\le N(\varepsilon)\le S(\varepsilon)$. Hence

$$\bar{C}(X)=\limsup_{\varepsilon\to 0}\frac{\log S(\varepsilon)}{\log(1/\varepsilon)},$$

and

$$\underline{C}(X)=\liminf_{\varepsilon\to 0}\frac{\log S(\varepsilon)}{\log(1/\varepsilon)}.$$

Next we turn to the definition of Lyapunov exponents and stable manifolds. Let $f: M \circlearrowleft$ be a $C^{1+\alpha}$ diffeomorphism. Let $x\in M$ and $v\in T_xM$, the tangent space at x. Define the *Lyapunov exponent* of f at x in the direction v to be the number

$$\chi(x, v)=\lim_{n\to\infty}(1/n)\log\|Df_x^n v\|$$

if this limit exists.

Let μ be any f-invariant Borel probability measure. It is a fact that μ-a.e. x is 'regular', which implies in particular that $\chi(x, v)$ is well-defined for every $v\in T_xM$. For a regular point x, let

$$\chi(x)=\max\{\chi(x,v):\chi(x,v)<0\}.$$

Then

$$W^s(x)=\{y\in M:\limsup_{n\to\infty}(1/n)\log d(f^nx, f^ny)\le\chi(x)\}$$

is an immersed submanifold of M called the *stable manifold* of x. The *unstable manifold* $W^u(x)$ is defined similarly with f^{-1} playing the role of f. For a detailed account of this theory, see [**12**] or [**14**].

2. *Facts from Pesin theory*

In this section we fix some notations and recall some relevant facts. We follow mainly the approach in [**11**], a summary of which appears in [**5**].

Let M be a p-dimensional compact Riemannian manifold and $f: M\circlearrowleft$ be a $C^{1+\alpha}$ ($\alpha>0$) diffeomorphism. Consider an ergodic Borel probability measure μ with Lyapunov exponents $\lambda_1\ge\cdots\ge\lambda_p$. Assume that $\lambda_u>0>\lambda_{u+1}$. Let

$$\lambda=\min\{\lambda_u, |\lambda_{u+1}|\}$$

and fix small $\varepsilon>0$. For $l=1,2,\cdots$ let Λ_l be the Pesin sets, that is,

$$\Lambda_l=\Big\{x\in M:\forall m\in\mathbb{Z}\text{ there is a splitting of the tangent space } T_{f^mx}M=E^u_{f^mx}\oplus E^s_{f^mx}$$

satisfying the following conditions for all $n\ge 0$, $m\in\mathbb{Z}$:

$$(1)\quad v\in E^s_{f^mx}\Rightarrow\|Df^n_{f^mx}v\|\le l\exp\{\varepsilon|m|\}\exp\{-(\lambda-\varepsilon)n\}\|v\|,$$
$$\|Df^{-n}_{f^mx}v\|\ge(l\exp\{\varepsilon|m|\})^{-1}\exp\{(\lambda-\varepsilon)n\}\|v\|,$$

$$(2)\quad v\in E^u_{f^mx}\Rightarrow\|Df^n_{f^mx}v\|\ge(l\exp\{\varepsilon|m|\})^{-1}\exp\{(\lambda-\varepsilon)n\}\|v\|,$$
$$\|Df^{-n}_{f^mx}v\|\le l\exp\{\varepsilon|m|\}\exp\{-(\lambda-\varepsilon)n\}\|v\|,$$

$$\text{and (3)}\quad \measuredangle(E^u_{f^mx}, E^s_{f^mx})\ge(l\exp\{\varepsilon|m|\})^{-1}.\Big\}$$

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It is well-known that

$$\mu\Big(\bigcup_l \Lambda_l\Big)=1.$$

Some of our estimates are more transparent in 'Lyapunov charts'. 'Lyapunov charts' are non-autonomous changes in coordinates via which f becomes uniformly hyperbolic. We collect a few facts that will be useful later.

Let $B_r^u=\{z\in\mathbb{R}^u : \|z\|<r\}$. For each fixed l there are positive numbers K_l and δ_l such that for every $x\in\Lambda_l$ there is a neighbourhood $B(x)$ of x in M and a diffeomorphism

$$\psi_x : B_1^u\times B_1^{p-u}\to B(x)$$

having the following properties:

(1) If $\langle\cdot,\cdot\rangle'$ denotes the Riemannian metric on M and $\langle\cdot,\cdot\rangle'_x$ denotes the metric in $B(x)$ induced by ψ_x from the Euclidean metric on $B_1^u\times B_1^{p-u}$, then

$$K\le\frac{\|\cdot\|'_x}{\|\cdot\|'}\le K_l$$

for some universal constant K.

(2) Let $W^u_{\text{loc}}(x)$ be the component of $W^u(x)\cap B(x)$ containing x. In the x-chart, $\psi_x^{-1}W^u_{\text{loc}}(x)$ is the graph of some

$$g_x : B_1^u\to B_1^{p-u}$$

with $g(0)=0$ and $\|Dg\|\le\frac{1}{1000}$. The analogous statement for $W^s_{\text{loc}}(x)$ also holds.

(3) If $x, y\in\Lambda_l$ and $d(x,y)<\delta_l$, then

$$\psi_x^{-1}W^u_{\text{loc}}(y)\cap B^u_{\frac12}\times B_1^{p-u}$$

is the graph of some $g: B^u_{\frac12}\to B_1^{p-u}$ with $\|Dg\|\le\frac{1}{100}$.

3. *Lemmas*

Let m denote Riemannian measure on M. If $L\subset M$ is an embedded disk then L inherits from M a Riemannian structure and hence a Riemannian measure. Let m_L denote this Riemannian measure on L. For $x\in\Lambda_l$ and small $\rho>0$, let

$$D^u_\rho(x)=\{y\in W^u_{\text{loc}}(x): d(x,y)<\rho\}$$

and

$$D^s_\rho(x)=\{y\in W^s_{\text{loc}}(x): d(x,y)<\rho\}.$$

LEMMA 1. *For each l, $\exists A_l>0$ s.t. for every $x\in\Lambda_l$ and $\rho<1/K_l$, we have*

$$m_{W^u_{\text{loc}}(x)}D^u_\rho(x)\le A_l\rho^u.$$

Proof. This follows from the fact that $\psi_x^{-1}D^u_\rho(x)$ is contained in the graph of $g_x|B^u_{K_l\rho}$, $\text{Vol}\,(B^u_{K_l\rho})\sim\rho^u$ and that Riemannian metric is uniformly equivalent to x-chart metrics for all $x\in\Lambda_l$. □

LEMMA 2. *For each l, $\exists B_l>0$ s.t. for $\rho>0$ and $x, y\in\Lambda_l$ with $d(x,y)<\delta_l$, if*

$$W^u_{\text{loc}}(y)\cap D^s_\rho(x)=\varnothing$$

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then

$$d(x, y) \geq B_l \rho.$$

Proof. Again this is clear in the x-chart from the slopes of $W^u_{\rm loc}(x)$ and $D^s_\rho(x)$ and the result follows from the equivalence of metric. □

LEMMA 3. *For every* $\varepsilon > 0$ *and* μ -a.e. $x \in M$, $\exists C(x)$, $\delta(x) > 0$ *s.t. for all* $y \in D^s_{\delta(x)}$,

$$d(f^n x, f^n y) \geq C(x) \exp\{(\lambda_p - \varepsilon)n\} d(x, y)$$

for all $n \geq 0$. (λ_p = *the smallest* μ*-exponent.*)

Proof. One way of seeing this is via theorem 4.1 (inequality (4.5) in particular) and the proof of theorem 5.1 of [**14**]. □

For small $\varepsilon > 0$, let

$$\Gamma(l, C, \delta) = \{x \in \Lambda_l : C(x) \geq C, \delta(x) \geq \delta \text{ and}$$
$$|\mathrm{Det}\,(Df^n|E^u_x)| \geq l^{-1} \exp\{(\lambda_1 + \cdots + \lambda_u - \varepsilon)n\}, \forall n \geq 0\}.$$

Choose l, C and δ so that $\Gamma_0 = \Gamma(l, C, \delta)$ has positive μ-measure. Let $\tilde{\Gamma}_0 \subset \Gamma_0$ be compact and have positive measure. Assume also that, for every $x \in \tilde{\Gamma}_0$,

$$(1/n) \sum_{i=0}^{n-1} \chi_{\Lambda_l} f^i x \to \mu \Lambda_l.$$

LEMMA 4. *For* $n = 1, 2, \cdots$, $\exists \Gamma_n \subset \tilde{\Gamma}_0$ *s.t.*

(1) $\forall x \in \Gamma_n$, $\exists n \leq m_n(x) < n(1+\varepsilon)$ *with*

$$f^{m_n(x)} x \in \Lambda_l;$$

(2) $\lim_{n\to\infty} \mu \Gamma_n = \mu \tilde{\Gamma}_0$.

Proof. This proof is borrowed from [**5**]. Let

$$\Gamma_n = \left\{ x \in \tilde{\Gamma}_0 : \sum_{k=0}^{n-1} \chi_{\Lambda_l}(f^k x) < n\mu\Lambda_l(1 + \tfrac{1}{3}\varepsilon) \right.$$
$$\left. \text{and} \sum_{k=0}^{n(1+\varepsilon)} \chi_{\Lambda_l}(f^k x) > n\mu\Lambda_l(1 + \tfrac{2}{3}\varepsilon) \right\}.$$ □

Next we consider a chart $\phi : B^u \times S \to M$, where B^u is a closed u-dimensional disk, $S \subset B^s$ is compact, ϕ is continuous, $\phi | B^u \times \{y\}$ is C^1 for each $y \in S$, this derivative in the first variable is continuous on $B^u \times S$, and ϕ takes each fibre $B^u \times \{y\}$ into $W^u_{\rm loc}(x)$ for some $x \in \tilde{\Gamma}_0$. We require also that $V = (\text{Image}\,\phi) \cap \tilde{\Gamma}_0$ has positive μ-measure.

LEMMA 5. $\exists D > 0$ *with the property that, for large enough* $n \in \mathbb{Z}^+$, $\exists x_n \in \tilde{\Gamma}_0$ *s.t. if* $L_n = W^u_{\rm loc}(x_n)$ *then*

$$m_{L_n}(\Gamma_n \cap L_n) \geq D.$$

Proof. Let $\tilde{\mu} = \phi^*(\mu | V)$. Decompose $\tilde{\mu}$ into a transverse measure $\tilde{\mu}_T$ on S and leaf measures $\tilde{\mu}_y$ on $B^u \times \{y\}$. Since μ has absolutely continuous conditional measures on W^u-leaves,

$$\tilde{\mu} \ll \tilde{\mu}_T \times \lambda,$$

where λ is Lebesgue measure on B^u. It follows from this and $\mu(\Gamma_n \cap V) \to \mu V$ that

$$(\tilde{\mu}_T \times \lambda)(\phi^{-1}(V \cap \Gamma_n))$$

is bounded away from zero. Hence the desired result. □

LEMMA 6. *$\exists N$ s.t. $\forall n \geq N$, $\exists n \leq m_n < n(1+\varepsilon)$, $D > 0$, $L_n = W^u_{\text{loc}}(x_n)$, some $x_n \in \tilde{\Gamma}_0$ and $B_n \subset L_n \cap \Gamma_n$ s.t.*

(1) $\forall x \in B_n$, $f^{m_n}x \in \Lambda_l$;

(2) $m_{L_n} B_n \geq (1/n\varepsilon)D$.

Proof. Choose D and L_n as in lemma 5. Partition $L_n \cap \Gamma_n$ by return time, i.e. for $n \leq j < n(1+\varepsilon)$ let

$$R_{nj} = \{x \in L_n \cap \Gamma_n : f^j x \in \Lambda_l\}.$$

Then for some j,

$$m_{L_n} R_{nj} \geq (1/n\varepsilon) m_{L_n}(L_n \cap \Gamma_n)$$
$$\geq (1/n\varepsilon)D.$$

Let this $j = m_n$ and $R_{nj} = B_n$. □

4. *Proof of the theorem*

Recall that

$$S(\alpha) = \text{maximum number of } \alpha\text{-separated points in supp}\,(\mu).$$

We fix some arbitrarily small $\varepsilon > 0$; choose $\Gamma_0 = \Gamma(l, C, \delta)$ and then $\tilde{\Gamma}_0$ and $\{\Gamma_n\}_{n=1,2,\cdots}$ as in lemma 4. Let $D > 0$, L_n and B_n be as in lemmas 5 and 6. Choose $\delta_0 \leq \delta$ small enough that $\forall x \in B_n$,

$$D^s_{\delta_0}(x) \cap \bigcup_{y \in B_n} W^u_{\text{loc}}(y) = \{x\}.$$

This is possible because $\tilde{\Gamma}_0$ is compact and uniformly hyperbolic and $W^u_{\text{loc}}(\tilde{\Gamma}_0)$ and $W^s_{\text{loc}}(\tilde{\Gamma}_0)$ are continuous families of disks.

For large n, we construct a set S_n as follows. Start with an arbitrary point $z_1 \in f^{m_n}B_n$. Let $\rho = \exp\{\lambda_p m_n\}$ and pick

$$z_i \in f^{m_n}B_n - (D^u_\rho(z_1) \cup \cdots \cup D^u_\rho(z_{i-1})) \quad \text{for } i = 2, 3, \cdots$$

until the process cannot be continued. Let S_n consist of these z_i's. We know that

$$\text{card } S_n \geq \frac{\exp\{(\lambda_1 + \cdots + \lambda_u - \varepsilon)m_n\}D}{lA_l \exp\{\lambda_p m_n u\} n\varepsilon}$$

because

$$m_{f^{m_n}L_n} D^u_\rho(z_i) \leq A_l \rho^u$$

by lemma 1, and

$$m_{f^{m_n}L_n} f^{m_n} B_n \geq l^{-1} \exp\{(\lambda_1 + \cdots + \lambda_u - \varepsilon)m_n\}(D/n\varepsilon),$$

since $B_n \subset \Gamma_n \subset \Gamma_0$.

Claim. For large n,

$$x, y \in S_n \Rightarrow d(x, y) \geq \min\{\exp\{\lambda_p m_n\}, B_l C\delta_0 \exp\{(\lambda_p - \varepsilon)m_n\}\}.$$

Let $x, y \in S_n$. Since

$$f^{m_n} D^s_{\delta_0}(f^{-m_n}x) \supset D^s_{C \exp\{(\lambda_p-\varepsilon)m_n\}\delta_0}(x)$$

we have

$$W^u_{\rm loc}(y) \cap D^s_{C \exp\{(\lambda_p-\varepsilon)m_n\}\delta_0}(x) = \{x\} \quad \text{or} \quad \varnothing.$$

Thus if y is very near x, then either $y \in W^u_{\rm loc}(x)$ in which case

$$d(x, y) \geq \exp\{\lambda_p m_n\}$$

by construction of S_n or the intersection above is empty and lemma 2 applies to give

$$d(x, y) \geq B_l C\delta_0 \exp\{(\lambda_p - \varepsilon)m_n\}.$$

To sum up, if

$$E = \min\{1, B_l C\delta_0\}$$

and

$$\varepsilon_n = E \exp\{(\lambda_p - \varepsilon)n(1+\varepsilon)\}$$

then

$$S(\varepsilon_n) \geq \frac{\exp\{(\lambda_1 + \cdots + \lambda_u - \varepsilon)m_n\}D}{lA_l \exp\{\lambda_p m_n u\}n\varepsilon},$$

where E, D, l and A_l are independent of n. Thus

$$\liminf_{n\to\infty} \frac{\log S(\varepsilon_n)}{\log(1/\varepsilon_n)} \geq \lim_{n\to\infty} \frac{(\lambda_1+\cdots+\lambda_u - \varepsilon - \lambda_p u) + (1/m_n)\log\{D/(lA_l\varepsilon n)\}}{-(1/m_n)\log E - (\lambda_p - \varepsilon)}$$

$$= \frac{\lambda_1 + \cdots + \lambda_u - \varepsilon - \lambda_p u}{-(\lambda_p - \varepsilon)}.$$

Now

$$\frac{\log \varepsilon_n}{\log \varepsilon_{n+1}} \to 1 \quad \text{as } n \to \infty$$

and for $\varepsilon_{n+1} < \tilde{\varepsilon} < \varepsilon_n$,

$$\frac{\log S(\tilde{\varepsilon})}{\log(1/\tilde{\varepsilon})} \geq \frac{\log S(\varepsilon_n)}{\log(1/\varepsilon_n)} \cdot \frac{\log(1/\varepsilon_n)}{\log(1/\varepsilon_{n+1})}.$$

Letting $\varepsilon \to 0$ we complete the proof. □

5. *Final remarks*

(1) The assumption that μ has absolutely continuous conditional measures on W^u-leaves implies that the μ-exponents are (theoretically) observable, that is, there is a set $A \subset M$ with $mA > 0$ such that for all x in A, forward iterates of f along the trajectory of x give the μ-exponents. This follows from the fact that

$$m\Big(\bigcup_{x\ \mu\text{-regular}} W^s(x)\Big) > 0$$

which in turn follows from the absolute continuity of the W^s-foliation **[12]**, **[13]**.

(2) Even in the case of attractors, Lyapunov exponents do not reflect the dimension of the attractor if the underlying measure is too singular. An extreme example in the direction is the well-known figure 8 (see for instance **[5]**).

(3) For a *general* lower bound such as we have here one cannot hope to include the other negative exponents. For example, consider $\Lambda = T^2$ in a 3-manifold with $f|\Lambda =$ a linear Anosov diffeomorphism with exponents λ and $-\lambda$. Suppose that normal to Λ we have a contraction with exponent γ where $|\gamma| < \lambda$. Clearly

$$C(\Lambda) = 1 + \frac{\lambda}{|-\lambda|},$$

and γ plays no role in the dimension of Λ. We do not know whether *generically* sharper estimates involving the other negative exponents can be given.

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