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THE MAHLER MEASURE OF TRINOMIALS OF HEIGHT 1

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In memory of my mother

Abstract

We study the Mahler measure of the trinomials $z^n \pm z^k \pm 1$. We give two criteria to identify those whose Mahler measure is less than $1.381\,356 \cdots = M(1 + z_1 + z_2)$. We prove that these criteria are true for n sufficiently large.

1. Introduction

The *Mahler measure* of a polynomial

$$P(z) = a_0 z^n + \cdots + a_n = a_0 \prod_{j=1}^{n} (z - \alpha_j) \in \mathbb{C}[z], \qquad a_0 \neq 0,$$

as defined by Lehmer [Le] in 1933, is

$$M(P) = |a_0| \prod_{j=1}^{n} \max(1, |\alpha_j|).$$

In 1962, Mahler [Ma] gave the definition

$$M(P) = \exp\Big(\int_0^1 \log |P(e^{2\pi i t})| \, dt \Big),$$

which is equivalent to Lehmer's definition by Jensen's formula [Je],

$$\int_0^1 \log |e^{2\pi i t} - \alpha| \, dt = \log \max(1, |\alpha|).$$

The polynomial P is *reciprocal* if $z^n P(1/z) = P(z)$, and an algebraic number is *reciprocal* if its minimal polynomial is reciprocal. Smyth [Sm1] has proved that, if the algebraic number $\alpha \neq 0, 1$ is nonreciprocal, then $M(\alpha) \geq \theta_0$, where $\theta_0 = 1.324\,717\ldots$ is the smallest Pisot number which is the real root of the polynomial $z^3 - z - 1$.

Concerning the Mahler measures of reciprocal polynomials, Boyd [Bo1, Bo3] computed all irreducible, noncyclotomic integer polynomials P with degree $D \leq 20$ having $M(P) < 1.3$. Mossinghoff [Mo], using the same algorithm, extended the computation to $D \leq 24$. The author, Rhin and Sac-Epée [FRSE] employed a new method that uses a large family of explicit auxiliary functions to produce improved bounds on the coefficients of polynomials with small Mahler measure and determined all irreducible polynomials P with $M(P) < \theta_0$ and $D \leq 36$ and polynomials P with $M(P) < 1.31$ and $D = 38$ or 40. More recently, Mossinghoff *et al.* [MRW] computed all primitive, irreducible, noncyclotomic polynomials P with degree at most 44 and $M(P) < 1.3$.

Smyth [Sm2] has also shown that θ_0 is an isolated point in the spectrum of Mahler measures of nonreciprocal algebraic integers. We have done an exhaustive search of nonreciprocal polynomials of height 1 and Mahler measure less than 1.381 356 . . . up to degree 12. We observed that the smallest points of the spectrum are either trinomials of the type $P(z) = z^n \pm z^k \pm 1$ or their irreducible factors. As the number of nonreciprocal polynomials grows quickly, we have studied the trinomials of height 1 with Mahler measure less than 1.381 356 . . . from degree 13 up to degree 20. They are either irreducible and give a new point of the spectrum or are divisible by $z^2 + z + 1$. In the latter case, their quotient gives a new point of the spectrum. The results of these computations are in the Appendix A. For example, the six smallest known points are as follows:

$$
\begin{aligned}
1.324\,717\cdots &= M(z^3 - z - 1),\\
1.349\,716\cdots &= M(z^5 - z^4 + z^2 - z + 1) = M\Big(\frac{z^7 + z^2 + 1}{z^2 + z + 1}\Big),\\
1.359\,914\cdots &= M(z^6 - z^5 + z^3 - z^2 + 1) = M\Big(\frac{z^8 + z + 1}{z^2 + z + 1}\Big),\\
1.364\,199\cdots &= M(z^5 - z^2 + 1),\\
1.367\,854\cdots &= M(z^9 - z^8 + z^6 - z^5 + z^3 - z + 1) = M\Big(\frac{z^{11} + z^4 + 1}{z^2 + z + 1}\Big),\\
1.370\,226\cdots &= M(z^9 - z^8 + z^6 - z^5 + z^3 - z^2 + 1) = M\Big(\frac{z^{11} + z + 1}{z^2 + z + 1}\Big)
\end{aligned}
$$

(more points of the spectrum are given in the Appendix A). The four first points were found by Boyd.

Boyd and Mossinghoff [BM] have computed a set of 48 small Mahler measures of two-variable polynomials less than 1.37. These measures are limit points of Mahler measures of one-variable polynomials. This set contains the five first points of our list.

For any trinomial $P(z)$, we have $M(P(z)) = M(\pm P(-z)) = M(z^n P(1/z)) = M(P(z^l))$ for all integers l so we can assume that $\gcd(n, k) = 1$, $k < n/2$ and we can restrict the trinomials $z^n \pm z^k \pm 1$ to three families:

(1) $z^n + z^k + 1$
(2) $z^n - z^k + 1$ with n odd
(3) $z^n - z^k - 1$ with n even.

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Put $\lambda = M(z_1 + z_2 + 1)$. Smyth proved that $\lambda = 1.381\,356\ldots$ (for more details, see [Bo2]). Smyth (private communication) claims that the following conjecture is true.

Conjecture 1.1.

(1) $M(z^n + z^k + 1) < \lambda$ if and only if 3 divides $n + k$.
(2) $M(z^n - z^k + 1) < \lambda$ with n odd if and only if 3 does not divide $n + k$.
(3) $M(z^n - z^k - 1) < \lambda$ with n even if and only if 3 does not divide $n + k$.

This criterion generalizes a result of Boyd. In [Bo2], he has shown the criterion for the first family when k is equal to 1 and n is sufficiently large. A first generalization for the first family was done by Duke in [Du] in 2007. He showed, for $0 < k < n$ with $(n, k) = 1$, that

$$\log M(x^n + x^k + 1) = \log M(x + y + 1) + \frac{c(n,k)}{n^2} + O\Big(\frac{k}{n^3}\Big)$$

where $c(n, k) = -\pi\sqrt{3}/6$ if 3 divides $n + k$ and $c(n, k) = \pi\sqrt{3}/18$ otherwise.

Then, in 2012, Condon [Co] studied the quantities $\mu_n(P) = M(P(x, x^n)) - M(P(x, y))$ for a large set of bivariate polynomials that are uniquely irreducible. He obtained some considerably more general results. However, some polynomials of our families are reducible. So the results that we need are easier to obtain by following Duke's proof because it does not depend on the factorization of the polynomial. Although Condon's formula is more precise, it is not feasible in practice to use it to obtain a general formula for polynomials of our families whose factorization depends on n and k. It makes sense to recall here a result of Ljunggren [Lj] on the irreducibility of trinomials of height 1. He proved that if $n = n_1 d$, $k = k_1 d$, $(n_1, k_1) = 1$, $n \geq 2k$ then the polynomial $g(x) = x^n + \epsilon x^k + \epsilon'$, $\epsilon = \pm 1$, $\epsilon' = \pm 1$, is irreducible, apart from the following three cases, where $n_1 + k_1 \equiv 0 \bmod 3$: n_1, k_1 both odd, $\epsilon = 1$; n_1 even, $\epsilon' = 1$; k_1 even, $\epsilon = \epsilon'$, $g(x)$ then being a product of the polynomial

$$x^{2d} + \epsilon^k \epsilon'^n x^d + 1$$

and a second irreducible polynomial.

In Section 2 we establish Conjecture 1.1 when n is sufficiently large relative to k for the second family of trinomials $z^n - z^k + 1$. Section 3 deals with the third family. We give the main elements of the proof that differ from those of Section 2. In Section 4 we give a second criterion equivalent to the first one and involving resultants of trinomials of the three families with some cyclotomic polynomials.

2. Proof of Conjecture 1 for n large

We prove the following result.

Theorem 2.1.

(1) *For the second family of trinomials,*

$$\log M(z^n - z^k + 1) = \log M(z_1 + z_2 + 1) + \frac{c(n,k)}{n^2} + O\Big(\frac{k}{n^3}\Big),$$

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where $c(n,k) = -\pi\sqrt{3}/36$ *if* 3 *does not divide* $n+k$ *and* $c(n,k) = \pi\sqrt{3}/12$ *otherwise.*

(2) *For the third family of trinomials,*

$$\log M(z^n - z^k - 1) = \log M(z_1 + z_2 + 1) + \frac{c(n,k)}{n^2} + O\Big(\frac{k}{n^3}\Big),$$

where $c(n,k) = -\pi\sqrt{3}/18$ *if* 3 *does not divide* $n+k$ *and* $c(n,k) = \pi\sqrt{3}/6$ *otherwise.*

The constants involved in O *are effective.*

The proof follows the same scheme as Duke's one in [Du].

Corollary 2.2. *There exists a computable constant* $c_0 \geq 2$ *such that if* $n > c_0 k$ *then* $M(z^n - z^k \pm 1) - M(z_1 + z_2 + 1) < 0$ *if* 3 *does not divide* $n+k$ *and* >0 *otherwise.*

We choose to present first the proof in detail for the second family of trinomials $z^n - z^k + 1$. Let $z = e^{it}$ for $0 \leq t \leq 2\pi$. When t belongs to

$$\bigcup_{l=0}^{k-1}\Big(\frac{\pi + 6\pi l}{3k}, \frac{5\pi + 6\pi l}{3k}\Big),$$

that is, $|1 - z^k| > 1$,

$$\log(z^n - z^k + 1) = \log(1 - z^k) + \sum_{m\geq 1} \frac{(-1)^{m-1}}{m}\Big(\frac{z^n}{1 - z^k}\Big)^m;$$

and when t belongs to

$$\Big(0, \frac{\pi}{3k}\Big) \cup \Big(\frac{5\pi + 6\pi(k-1)}{3k}, 2\pi\Big) \cup \bigcup_{l=0}^{k-2}\Big(\frac{5\pi + 6\pi l}{3k}, \frac{7\pi + 6\pi l}{3k}\Big),$$

that is, $|1 - z^k| < 1$,

$$\log(z^n - z^k + 1) = \log(z^n) + \sum_{m\geq 1} \frac{(-1)^{m-1}}{m}\Big(\frac{1 - z^k}{z^n}\Big)^m.$$

Put

$$\lambda_{n,k} = \log M(z^n - z^k + 1) = \frac{1}{2\pi}\int_0^{2\pi} \log|z^n - z^k + 1|\,dt$$

and

$$\lambda = \log M(z_1 + z_2 + 1) = \frac{1}{2\pi}\int_0^{2\pi} \log_+|z + 1|\,dt.$$

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Putting $u = tk$,

$$\sum_{l=0}^{k-1} \int_{(\pi+6\pi l)/3k}^{(5\pi+6\pi l)/3k} \log|1-e^{itk}|\,dt = \frac{1}{k}\sum_{l=0}^{k-1}\int_{(\pi/3)+2\pi l}^{(5\pi/3)+2\pi l} \log|1-e^{iu}|\,du$$
$$= \int_{\pi/3}^{5\pi/3} \log|1-e^{iu}|\,du$$
$$= 2\int_{\pi/3}^{\pi} \log|1-e^{iu}|\,du$$
$$= 2\int_{\pi/3}^{\pi} \log_+|1-e^{iu}|\,du = \int_0^{2\pi} \log_+|1+e^{iu}|\,du.$$

Hence

$$\lambda_{n,k} - \lambda = \frac{1}{2\pi}\mathrm{Re}\sum_{m\geq 1}\frac{(-1)^{m-1}}{m}(c_1(m)+c_2(m)),$$

where

$$c_1(m) = \sum_{l=0}^{k-1}\int_{(\pi+6\pi l)/3k}^{(5\pi+6\pi l)/3k} e^{inmt}(1-e^{itk})^{-m}\,dt$$

and

$$c_2(m) = \int_0^{\pi/3k} e^{-inmt}(1-e^{itk})^m\,dt + \sum_{l=0}^{k-2}\int_{(5\pi+6\pi l)/3k}^{(7\pi+6\pi l)/3k} e^{-inmt}(1-e^{itk})^m\,dt$$
$$+ \int_{(5\pi+6\pi(k-1))/3k}^{2\pi} e^{-inmt}(1-e^{itk})^m\,dt.$$

Put $u = tk$; then

$$c_1(m) = \frac{1}{k}\sum_{l=0}^{k-1}\int_{(\pi/3)+2\pi l}^{(5\pi/3)+2\pi l} e^{inmu/k}(1-e^{iu})^{-m}\,du$$
$$= \frac{1}{k}\sum_{l=0}^{k-1}\bigg(\int_{(\pi/3)+2\pi l}^{\pi(1+2l)} e^{inmu/k}(1-e^{iu})^{-m}\,du + \int_{\pi(1+2l)}^{(5\pi/3)+2\pi l} e^{inmu/k}(1-e^{iu})^{-m}\,du\bigg)$$
$$= \frac{1}{k}\bigg(\sum_{l=0}^{k-1} e^{(2\pi lnm/k)}\bigg)\bigg(2\int_{\pi/3}^{\pi} e^{inmu/k}(1-e^{iu})^{-m}\,du\bigg).$$

Thus,

$$\mathrm{Re}\ c_1(m) = 2\ \mathrm{Re}\int_{\pi/3}^{\pi} e^{inmt/k}(1-e^{it})^{-m}\,dt$$

if k divides m and $\mathrm{Re}\ c_1(m) = 0$ otherwise.

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By the same argument,

$$\mathrm{Re}\, c_2(m) = 2\,\mathrm{Re} \int_0^{\pi/3} e^{-inmt/k}(1 - e^{it})^m\, dt$$

if k divides m and $\mathrm{Re}\, c_2(m) = 0$ otherwise.

Put $m = kq$. In order to estimate $c_1(kq) + c_2(kq)$, we need to integrate by parts three times the integrals

$$\int_{\pi/3}^{\pi} e^{inqt}(1 - e^{it})^{-kq}\, dt \quad \text{and} \quad \int_0^{\pi/3} e^{-inqt}(1 - e^{it})^{kq}\, dt.$$

We obtain for each integral four types of terms that we have to study.

The first type of term is

$$\Big[\frac{e^{inqt}(1 - e^{it})^{-kq}}{inq}\Big]_{\pi/3}^{\pi} \quad \text{and} \quad \Big[-\frac{e^{-inqt}(1 - e^{it})^{kq}}{inq}\Big]_0^{\pi/3}.$$

It is easy to see that the sum of such terms is not real and thus does not occur in $\mathrm{Re}(c_1(kq) + c_2(kq))$.

The second type of term is

$$\Big[\frac{kq(kq+1)e^{it(nq+2)}(1 - e^{it})^{-kq-2}}{inq(nq+1)(nq+2)}\Big]_{\pi/3}^{\pi}$$

and

$$\Big[\frac{kq(kq-1)e^{-it(nq-2)}(1 - e^{it})^{kq-2}}{inq(nq-1)(nq-2)}\Big]_0^{\pi/3}.$$

These terms are less than or equal to $K_1 k/n^3$ in modulus.

Now we have to estimate the modulus of the third type of term

$$I = -\frac{kq(kq+1)(kq+2)}{nq(nq+1)(nq+2)} \int_{\pi/3}^{\pi} e^{it(nq+3)}(1 - e^{it})^{-kq-3}\, dt,$$

coming from the integration of $c_1(kq)$.

In the integral $I_1 = \int_{\pi/3}^{\pi} |1 - e^{it}|^{-kq-3}\, dt$, put $v = t/2$. Thus $I_1 = 2\int_{\pi/6}^{\pi/2} dv/(2\sin v)^{kq+3}$. For any $v \in [\pi/6, \pi/2]$, $1/(2\sin v) \le -(3/2\pi)v + (5/4)$ so that

$$I_1 \le 2\int_{\pi/6}^{\pi/2} \Big(-\frac{3}{2\pi}v + \frac{5}{4}\Big)^{kq+3} dv \le \frac{4\pi}{3(kq+4)}.$$

Finally, we get $|I| \le K_2 k^2/n^3$.

In the same way, we have to estimate the modulus of

$$J = -\frac{kq(kq-1)(kq-2)}{nq(nq-1)(nq-2)} \int_0^{\pi/3} e^{-it(nq-3)}(1 - e^{it})^{kq-3}\, dt,$$

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due to the integration of $c_2(kq)$. For $q > 3$, in the integral $J_1 = \int_0^{\pi/3} |1 - e^{it}|^{kq-3}\,dt$, put $v = t/2$. Thus $J_1 = 2\int_0^{\pi/6} (2\sin v)^{kq-3}\,dv$. For any $v \in [0, \pi/6]$, $2\sin v \le \sqrt{3}v + 1 - (\sqrt{3}\pi/6)$ so that

$$J_1 \le 2\int_0^{\pi/6} \left(\sqrt{3}\,v + 1 - \frac{\sqrt{3}\pi}{6}\right)^{kq-3} dv \le \frac{2}{\sqrt{3}(kq-2)}.$$

Thus, we get $|J| \le K_3 k^2/n^3$ for $m \ge 1$.

• Finally, the only terms that occur in $\mathrm{Re}(c_1(kq) + c_2(kq))$ are those which contain $kq/inq(nq+1)$ in $c_1(kq)$ and $kq/inq(nq-1)$ in $c_2(kq)$.

We obtain

$$\begin{aligned}
&\mathrm{Re}(c_1(kq) + c_2(kq)) \\
&\quad = \mathrm{Re}\frac{kq}{in^2q^2}\big(e^{i(\pi/3)(nq+1)}(1 - e^{i\pi/3})^{-kq-1} + e^{-i(\pi/3)(nq-1)}(1 - e^{i\pi/3})^{kq-1}\big) + O\Big(\frac{k^2}{n^3}\Big) \\
&\quad = \frac{2kq\sqrt{3}}{n^2q^2}\cos\Big(\frac{\pi}{3}q(n+k)\Big) + O\Big(\frac{k^2}{n^3}\Big).
\end{aligned}$$

Hence

$$\lambda_{n,k} - \lambda = \frac{\sqrt{3}}{\pi n^2}\sum_{q\ge1} \frac{(-1)^{qk-1}\cos(\frac{\pi}{3}q(n+k))}{q^2} + O\Big(\frac{k}{n^3}\Big),$$

that is,

$$\lambda_{n,k} - \lambda = \begin{cases} -\dfrac{\pi\sqrt{3}}{36n^2} + O\Big(\dfrac{k}{n^3}\Big) & \text{if 3 does not divide } n+k, \\[2ex] \dfrac{\pi\sqrt{3}}{12n^2} + O\Big(\dfrac{k}{n^3}\Big) & \text{if 3 divides } n+k. \end{cases}$$

3. The family $z^n - z^k - 1$

Here,

$$\begin{aligned}
\lambda_{n,k} = \log M(z^n - z^k - 1) &= \frac{1}{2\pi}\int_0^{2\pi} \log|z^n - z^k - 1|\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log|-z^n + z^k + 1|\,dt = \log M(-z^n + z^k + 1)
\end{aligned}$$

so we work with $\log(-z^n + z^k + 1)$.

If t belongs to

$$\Big(0, \frac{2\pi}{3k}\Big) \cup \Big(\frac{4\pi + 6(k-1)\pi l}{3k}, 2\pi\Big) \cup \bigcup_{l=0}^{k-2}\Big(\frac{4\pi + 6\pi l}{3k}, \frac{8\pi + 6\pi l}{3k}\Big)$$

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then $|1+e^{itk}|>1$, and if t belongs to

$$\bigcup_{l=0}^{k-1}\Big(\frac{2\pi+6\pi l}{3k},\frac{4\pi+6\pi l}{3k}\Big)$$

then $|1+e^{itk}|<1$.

Hence

$$\lambda_{n,k}-\lambda=-\frac{1}{2\pi}\mathrm{Re}\sum_{m\geq 1}\frac{(-1)^{m-1}}{m}(c_1(m)+c_2(m)),$$

where

$$c_1(m)=\int_0^{2\pi/3k} e^{inmt}(1+e^{itk})^{-m}\,dt+\sum_{l=0}^{k-2}\int_{(4\pi+6\pi l)/3k}^{(8\pi+6\pi l)/3k} e^{inmt}(1+e^{itk})^{-m}\,dt$$
$$+\int_{(4\pi+6(k-1)\pi)/3k}^{2\pi} e^{inmt}(1+e^{itk})^{-m}\,dt$$

and

$$c_2(m)=\sum_{l=0}^{k-1}\int_{(2\pi+6\pi l)/3k}^{(4\pi+6\pi l)/3k} e^{-inmt}(1+e^{itk})^{m}\,dt.$$

By the same argument as in Section 2,

$$\mathrm{Re}\ c_1(m)=2\ \mathrm{Re}\int_0^{2\pi/3} e^{inmt/k}(1+e^{it})^{-m}$$

if k divides m and $\mathrm{Re}\ c_1(m)=0$ otherwise, and

$$\mathrm{Re}\ c_2(m)=2\ \mathrm{Re}\int_{2\pi/3}^{\pi} e^{-inmt/k}(1+e^{it})^{m}\,dt$$

if k divides m and $\mathrm{Re}\ c_2(m)=0$ otherwise.

As before, we put $m=kq$. We integrate three times by parts and keep only the terms with m/in^2q^2. We get

$$\mathrm{Re}(c_1(kq)+c_2(kq))=2\frac{(-1)^{kq}kq\sqrt{3}}{n^2q^2}\cos\Big(\frac{2\pi}{3}q(n-2k)\Big)+O\Big(\frac{k^2}{n^3}\Big),$$

that is,

$$\lambda_{n,k}-\lambda=\frac{\sqrt{3}}{\pi n^2}\sum_{q\geq 1}\frac{\cos(\frac{2\pi}{3}q(n-2k))}{q^2}+O\Big(\frac{k}{n^3}\Big).$$

Therefore,

$$\lambda_{n,k}-\lambda=\begin{cases}-\dfrac{\pi\sqrt{3}}{18n^2}+O\Big(\dfrac{k}{n^3}\Big) & \text{if 3 does not divide } n+k,\\[2ex] \dfrac{\pi\sqrt{3}}{6n^2}+O\Big(\dfrac{k}{n^3}\Big) & \text{if 3 divides } n+k.\end{cases}$$

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4. An equivalent criterion

We claim that the following conjecture is true.

CONJECTURE 4.1. Let ϵ, η be equal to ± 1. Put r_1= resultant$(z^n + \epsilon z^k + \eta, z^2 + z + 1)$ and r_2= resultant$(z^n + \epsilon z^k + \eta, z^2 - z + 1)$.

(1) $M(z^n + z^k + 1) < \lambda$ if and only if $z^2 + z + 1$ divides $z^n + z^k + 1$.
(2) $M(z^n - z^k + 1) < \lambda$ with n odd if and only if $\{r_1, r_2\} = \{1, 1\}$ or $\{1, 7\}$.
(3) $M(z^n - z^k - 1) < \lambda$ with n even if and only if $\{r_1, r_2\} = \{1, 1\}$ or $\{1, 7\}$.

In this section, we prove the following theorem.

THEOREM 4.2. *Conjectures 1.1 and 4.1 are equivalent.*

PROOF. Put $j = e^{2i\pi/3}$.

(1) 3 divides $n + k \iff (n \equiv 1 \bmod 3$ and $k \equiv 2 \bmod 3)$ or $(n \equiv 2 \bmod 3$ and $k \equiv 1 \bmod 3) \iff j^n + j^k + 1 = j^2 + j + 1 = 0$.
(2) We give the proof for the family $z^n - z^k + 1$. The argument is the same for the family $z^n - z^k - 1$.

Suppose that 3 divides $n + k$.

If $n \equiv 1 \bmod 3$ and $k \equiv 2 \bmod 3$ then $r_1 = |j^n - j^k + 1|^2 = |j - j^2 + 1|^2 = 4$ and $r_2 = |(-j)^n - (-j)^k + 1|^2 = |-j - j^2 + 1|^2 = 4$.

If $n \equiv 2 \bmod 3$ and $k \equiv 1 \bmod 3$ then $r_1 = |j^n - j^k + 1|^2 = |j^2 - j + 1|^2 = 4$ and $r_2 = |(-j)^n - (-j)^k + 1|^2 = |-j^2 - j + 1|^2 = 4$.

Thus, the situations $\{r_1, r_2\} = \{1,1\}$ or $\{1,7\}$ are not possible.

Suppose that 3 does not divide $n + k$. It is easy to see that this is equivalent to 3 dividing $nk(n - k)$.

If 3 divides n then $r_1 = (2 - j^k)(2 - \bar{j}^k) = 5 - 2(j^k + \bar{j}^k) = 7$ and $r_2 = j^k \bar{j}^k = 1$.

If 3 divides k and 3 does not divide n then $r_1 = j^n \bar{j}^n = 1$. When k is even, $r_2 = (-j^n)(-\bar{j}^n) = 1$, and when k is odd, $r_2 = (-j^n + 2)(-\bar{j}^n + 2) = 5 - 2(j^n + \bar{j}^n) = 7$.

If 3 divides $n - k$ and 3 does not divide nk then $r_1 = (j^k(j^{n-k} - 1) + 1)(\bar{j}^k(\bar{j}^{n-k} - 1) + 1) = 1$. When k is even, $r_2 = (-j^k(j^{n-k} + 1) + 1)(-\bar{j}^k(\bar{j}^{n-k} + 1) + 1) = (-2j^k + 1)(-2\bar{j}^k + 1) = 5 - 2(j^k + \bar{j}^k) = 7$, and when k is odd, $r_2 = (-j^k(j^{n-k} - 1) + 1)(-\bar{j}^k(\bar{j}^{n-k} - 1) + 1) = 1$. Thus, we have $\{r_1, r_2\} = \{1, 1\}$ or $\{1, 7\}$. □

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Appendix A. Smallest known Mahler measures of nonreciprocal polynomials of height 1 up to degree 20

$$1.324\,717 = M(z^3 - z^2 + 1)$$
$$1.349\,716 = M(z^5 - z^4 + z^2 - z + 1) = M\Big(\frac{z^7 + z^2 + 1}{z^2 + z + 1}\Big)$$

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$$1.359\,914 = M(z^6 - z^5 + z^3 - z^2 + 1) = M\Big(\frac{z^8 + z + 1}{z^2 + z + 1}\Big)$$
$$1.364\,199 = M(z^5 - z^2 + 1)$$
$$1.367\,854 = M(z^9 - z^8 + z^6 - z^5 + z^3 - z + 1) = M\Big(\frac{z^{11} + z^4 + 1}{z^2 + z + 1}\Big)$$
$$1.370\,226 = M(z^9 - z^8 + z^6 - z^5 + z^3 - z^2 + 1) = M\Big(\frac{z^{11} + z + 1}{z^2 + z + 1}\Big)$$
$$1.370\,957 = M(z^6 - z^5 - 1)$$
$$1.371\,612 = M(z^{11} - z^{10} + z^8 - z^7 + z^6 - z^4 + z^3 - z + 1) = M\Big(\frac{z^{13} + z^8 + 1}{z^2 + z + 1}\Big)$$
$$1.372\,910 = M(z^{11} - z^{10} + z^8 - z^7 + z^5 - z^4 + z^2 - z + 1) = M\Big(\frac{z^{13} + z^2 + 1}{z^2 + z + 1)}\Big)$$
$$1.373\,895 = M(z^7 - z^4 + 1)$$
$$1.374\,571 = M(z^{12} - z^{10} + z^9 - z^7 + z^6 - z^4 + z^3 - z + 1) = M\Big(\frac{z^{14} + z^{13} + 1}{z^2 + z + 1}\Big)$$
$$1.375\,128 = M(z^{14} - z^{13} + z^{11} - z^{10} + z^9 - z^7 + z^6 - z^4 + z^3 - z + 1)$$
$$= M\Big(\frac{z^{16} + z^{11} + 1}{z^2 + z + 1}\Big)$$
$$1.375\,619 = M(z^{15} - z^{14} + z^{12} - z^{11} + z^9 - z^7 + z^6 - z^4 + z^3 - z + 1)$$
$$= M\Big(\frac{z^{17} + z^{10} + 1}{z^2 + z + 1}\Big)$$
$$1.376\,087 = M(z^{15} - z^{14} + z^{12} - z^{10} + z^9 - z^7 + z^6 - z^4 + z^3 - z + 1)$$
$$= M\Big(\frac{z^{17} + z^{13} + 1}{z^2 + z + 1}\Big)$$
$$1.376\,755 = M(z^{17} - z^{16} + z^{14} - z^{13} + z^{11} - z^{10} + z^9 - z^7 + z^6 - z^4 + z^3 - z + 1)$$
$$= M\Big(\frac{z^{19} + z^{11} + 1}{z^2 + z + 1}\Big)$$
$$1.376\,795 = M(z^{15} - z^{14} + z^{12} - z^{11} + z^9 - z^8 + z^6 - z^5 + z^3 - z^2 + 1)$$
$$= M\Big(\frac{z^{17} + z + 1}{z^2 + z + 1}\Big)$$
$$1.377\,059 = M(z^{17} - z^{16} + z^{14} - z^{13} + z^{12} - z^{10} + z^9 - z^7 + z^6 - z^4 + z^3 - z + 1)$$
$$= M\Big(\frac{z^{19} + z^{14} + 1}{z^2 + z + 1}\Big)$$
$$1.377\,280 = M(z^9 - z^5 - 1)$$
$$1.377\,299 = M(z^{18} - z^{17} + z^{15} - z^{14} + z^{12} - z^{10} + z^9 - z^7 + z^6 - z^4 + z^3 - z + 1)$$
$$= M\Big(\frac{z^{20} + z^{13} + 1}{z^2 + z + 1}\Big)$$
$$1.377\,364 = M(z^9 - z^8 + 1)$$

$$\begin{aligned}
1.377\,543 &= M(z^{17} - z^{16} + z^{15} - z^{13} + z^{12} - z^{10} + z^9 - z^7 + z^6 - z^4 + z^3 - z + 1) \\
&= M\Big(\frac{z^{19} + z^{17} + 1}{z^2 + z + 1}\Big) \\
1.377\,904 &= M(z^{10} - z^7 - 1) \\
1.378\,234 &= M(z^8 - z^5 - 1) \\
1.378\,672 &= M(z^{11} - z^9 - 1) \\
1.378\,783 &= M(z^{11} - z^6 + 1) \\
1.379\,300 &= M(z^{12} - z^{11} - 1) \\
1.379\,367 &= M(z^7 - z^6 + 1) \\
1.379\,458 &= M(z^{13} - z^{10} + 1) \\
1.379\,545 &= M(z^9 - z^7 - 1) \\
1.375\,619 &= M(z^{15} - z^{14} + z^{12} - z^{11} + z^9 - z^8 + z^6 - z^4 + z^3 - z + 1) \\
&= M\Big(\frac{z^{17} + z^7 + 1}{z^2 + z + 1}\Big) \\
1.379\,576 &= M(z^{13} - z^7 - 1) \\
1.379\,633 &= M(z^{12} - z^7 - 1) \\
1.379\,730 &= M(z^{14} - z^9 - 1) \\
1.379\,849 &= M(z^{11} - z^8 + 1) \\
1.379\,954 &= M(z^{15} - z^{11} - 1) \\
1.378\,082 &= M(z^{18} - z^{16} + z^{15} - z^{13} + z^{12} - z^{10} + z^9 - z^7 + z^6 - z^4 + z^3 - z + 1) \\
&= M\Big(\frac{z^{20} + z^{19} + 1}{z^2 + z + 1}\Big) \\
1.379\,849 &= M(z^{22} + z^{16} - 1) \\
1.380\,031 &= M(z^{10} - z^9 - 1) \\
1.380\,046 &= M(z^{15} - z^8 + 1) \\
1.380\,116 &= M(z^{15} - z^{14} + 1) \\
1.380\,131 &= M(z^{13} - z^9 - 1) \\
1.380\,175 &= M(z^{16} - z^{13} - 1) \\
1.380\,277 &= M(z^4 - z^3 - 1) \\
1.380\,283 &= M(z^{17} - z^{12} + 1) \\
1.380\,031 &= M(z^{20} - z^{18} - 1) \\
1.380\,330 &= M(z^{16} - z^9 - 1) \\
1.380\,350 &= M(z^{17} - z^9 - 1) \\
1.380\,369 &= M(z^{17} - z^{15} - 1) \\
1.380\,418 &= M(z^{14} - z^{11} - 1)
\end{aligned}$$

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$$
\begin{aligned}
1.380\,418 &= M(z^{18} - z^{11} - 1)\\
1.380\,460 &= M(z^{13} - z^{12} + 1)\\
1.380\,509\,8 &= M(z^{19} - z^{13} - 1)\\
1.380\,530\,7 &= M(z^{18} - z^{17} - 1)\\
1.380\,541 &= M(z^{17} - z^{11} - 1)\\
1.380\,557 &= M(z^{19} - z^{16} + 1)\\
1.380\,558 &= M(z^{19} - z^{10} + 1)\\
1.380\,596 &= M(z^{15} - z^{13} - 1)\\
1.380\,680 &= M(z^{19} - z^{12} + 1)\\
1.380\,684 &= M(z^{20} - z^{11} - 1)\\
1.380\,691 &= M(z^{18} - z^{13} - 1)\\
1.380\,707 &= M(z^{17} - z^{14} + 1)\\
1.380\,719 &= M(z^{16} - z^{15} - 1)\\
1.380\,799 &= M(z^{19} - z^{15} - 1)\\
1.380\,879 &= M(z^{20} - z^{17} - 1)\\
1.380\,882 &= M(z^{19} - z^{18} + 1).
\end{aligned}
$$

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