A THEOREM ON PURE SUBMODULES

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1. Introduction. In **(1)** Baer studied the following problem: If a torsion-free abelian group G is a direct sum of groups of rank one, is every direct summand of G also a direct sum of groups of rank one? For groups satisfying a certain chain condition, Baer gave a solution. Kulikov, in **(3)**, supplied an affirmative answer, assuming only that G is countable. In a recent paper **(2)**, Kaplansky settles the issue by reducing the general case to the countable case where Kulikov's solution is applicable. As usual, the result extends to modules over a principal ideal ring R (commutative with unit, no divisors of zero, every ideal principal).

The object of this paper is to carry out a similar investigation for *pure* submodules, a somewhat larger class of submodules than the class of direct summands. We ask: if the torsion-free R-module M is a direct sum of modules of rank one, is every pure submodule N of M also a direct sum of modules of rank one? Unlike the situation for direct summands, here the answer depends heavily on the ring R. If R is a field, there is no problem, and if R is a discrete valuation ring (one prime up to unit factors), it is easy to see that the answer is still yes. On the other hand, for abelian groups, or generally whenever R has an infinite number of primes, the question has a negative answer.

We fill in the gap by showing that if R has exactly two primes, an affirmative answer is obtained provided N has finite rank. If N has infinite rank or if R has three or more primes, examples are given showing that N need not be a direct sum of modules of rank one. In contrast to the large number of theorems on principal ideal rings with one prime, this appears to be the first result true specifically for rings with two primes.

2. Preliminaries. Let R be a principal ideal ring and K its quotient field. The unit of R is always assumed to act as unit operator on every R-module. We recall that a submodule N of an R-module M is *pure* if $aN = N \cap aM$ for every a in R. M is *torsion-free* if for a in R, x in M, and $ax = 0$, we have either $a = 0$ or $x = 0$. In this case, the intersection of pure submodules is pure, and so every subset of M generates a unique pure submodule. The *rank* of M is the cardinal number of a maximal set of linearly independent (over R) elements of M, or equivalently, the dimension of the K-vector space $K \otimes_R M$.

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The torsion-free R-modules of rank one are (up to isomorphism) the submodules of the R-module K. Two such submodules M_1 and M_2 are isomorphic if and only if $M_1 = \alpha M_2$ for some α in K. In particular, M_1 is free precisely when $M_1 = \alpha R$ for some α in K. For each prime p in R, we denote by R_p the submodule of K consisting of those elements which can be written with a denominator prime to p.

Let the torsion-free module M be a direct sum $M = \Sigma M_i$, i ranging over an index set, each M_i of rank one. Let N be a pure submodule of M. We note that we can for our purpose confine ourselves to the case where none of the summands M_i is free or divisible.

Indeed, write $M = M' \oplus F$, where F is the sum of the free M_i's and M' the sum of the remaining M_i's. $N/(M' \cap N) \cong (M' + N)/M'$ is a submodule of M/M', which is free. Thus $M' \cap N$ is a direct summand of N whose complementary summand is free. It follows that N is a direct sum of modules of rank one whenever $M' \cap N$, a pure submodule of M', is a direct sum of modules of rank one.

Next, write $M = D \oplus M''$, where D is the sum of all the divisible M_i's and M'' the sum of the remaining M_i's. The purity of N and the divisibility of D combine to yield the divisibility of $N \cap D$. Thus $N \cap D$ is a direct summand of N. The complementary summand $N/(N \cap D) \cong (N + D)/D$ is a submodule of $M/D \cong M''$. For any a in R, by the divisibility of D, we have $D = aD \subset aM$. The modular law then gives $aM \cap (N + D) = (aM \cap N) + D$, which, since N is pure, is just $aN + D$. Modulo D this becomes $(aM/D) \cap ((N + D)/D) = a(N + D)/D$, which is exactly the assertion that $(N + D)/D$ is pure in M/D. So N is a direct sum of modules of rank one whenever $N/(N \cap D)$, which can be regarded as a pure submodule of M'', is a direct sum of modules of rank one.

In conclusion, we remark that if R has just one prime, every rank one module is either free or divisible, and the above reductions are all that are needed to show that N is a direct sum of modules of rank one.

3. R with two primes. Throughout this section, we assume that R has exactly two primes (up to unit factors). Denote them by p and q. The quotient field K is the set of all fractions $a/(p^m q^n)$, a in R, m, $n \geqq 0$. The submodules of K fall into four classes according as they do or do not contain unbounded powers of p, and of q, in the denominators of their elements (when these are written in "lowest terms"). Using this classification it is easily seen that every submodule of K is isomorphic to one of R, R_p, R_q, or K. Thus these are the modules of rank one.

Now for the theorem:

THEOREM. *Let the torsion-free module M be a direct sum of modules of rank one. Then every pure submodule of finite rank is also a direct sum of modules of rank one.*

Proof. We may suppose that M has finite rank and that each rank one summand is either a copy of R_p or of R_q. Write $M = P \oplus Q$ where P is a direct sum of copies of R_p, and Q of copies of R_q. Choose elements $u_1, \ldots, u_t$ in M so that $P = R_p u_1 \oplus \ldots \oplus R_p u_s$ and $Q = R_q u_{s+1} \oplus \ldots \oplus R_q u_t$ where $0 \leqq s \leqq t$.

Assume that every pure submodule of rank $n - 1$ ($n \geqq 2$) is a direct sum of modules of rank one, and let N be a pure submodule of rank n. For every k with $1 \leqq k \leqq t$, let N_k be the intersection of N with the direct sum of all the rank one summands except the kth one. Then N_k is a pure submodule of M whose rank is $n - 1$ or n depending on whether or not there is an element of N having a non-zero kth component. It will be sufficient to show that at least one of the N_k of rank $n - 1$ is a direct summand of N, for such an N_k is a direct sum of modules of rank one whose complementary summand is of rank one.

There is no loss in generality in assuming that $Q \neq 0$ and that $N_t \neq N$. We consider two cases:

Case I. $N \cap Q = 0$.

Since $N_t \neq N$, N_t is of rank $n - 1$. We will prove that N_t is a direct summand of N by showing that N/N_t is free. To do this, we need only show that when the elements of N are expressed in terms of the u_i's there is an upper bound to the powers of p that can occur as denominators in the coefficients of u_t.

Let $x_1, \ldots, x_n$ be a maximal independent subset of N. If $x \neq 0$ is in N, some non-zero multiple of x, say rx, lies in the module generated by the x_j's. If $rx = r_1 x_1 + \ldots + r_n x_n$, we can clearly suppose that not every one of $r, r_1, \ldots, r_n$ is a multiple of p in R.

Assume that N/N_t is not free, and let m be a given positive integer. Then we can choose the element x so that, in the expressions for x and the x_j's in terms of the u_i's, the coefficient of u_t for x has a power of p in its denominator so large in comparison to those for the x_j's so as to require r to be a multiple of p^m in R.

Using primes to denote images in $M/Q \cong P$, we observe that since $N \cap Q = 0$, the elements $x_1', \ldots, x_n'$ are independent. Say $x' = c_1 u_1' + \ldots + c_s u_s'$ and $x_j' = c_{j1} u_1' + \ldots + c_{js} u_s'$ where all the c's are in R_p. The $n \times s$ matrix (c_{ji}) thus obtained has all its rows independent. Say the first n columns are also independent, and let $D \neq 0$ be the determinant of the $n \times n$ submatrix (c_{ji}), $1 \leqq i, j \leqq n$.

We have the following system of s equations:

$$rc_i = r_1 c_{1i} + \ldots + r_n c_{ni}.$$

From the first n of these, and the fact that r is in $p^m R$, we see that $r_j D$ is in $p^m R_p$ for each j. Hence D is in $p^m R_p$.

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Thus the assumption that N/N_t is not free requires D to be in $\bigcap_m p^m R_p = 0$, a contradiction.

Case II. $N \cap Q \neq 0$.

Let $x \neq 0$ be an element of $N \cap Q$, say $x = b_{s+1}u_{s+1} + \ldots + b_t u_t$, each b_i in R_q. The result of dividing x by the largest power of q common to all of the b_i's will again be an element of N (N is pure), and so we may as well assume that at least one of the b_i's, say b_k, is not in qR_q.

The submodule $R_q x$ is contained in N since N is pure. We have $N = N_k \oplus R_q x$. For since $b_k \neq 0$, $N \cap R_q x = 0$. On the other hand suppose w in N has $b^*_k u_k$ as its kth component where b^*_k is in R_q. Since b_k is not in qR_q, b_k^{-1} is in R_q. Hence $w - b_k^* b_k^{-1} x$ is in N_k. This shows that $N = N_k + R_q x$.

4. Examples. Let R be an arbitrary principal ideal ring and M a torsion-free R-module. One readily verifies that for every prime p in R,

$$\bigcap_{j=1}^{\infty} p^j M$$

is a *pure* submodule of M. Since it is clear that a torsion-free module of rank one has no proper *pure* submodules, we see that if M is a direct sum ΣM_i of modules of rank one, the submodule

$$\bigcap_{j=1}^{\infty} p^j M$$

is the sum of those rank one summands M_i for which $pM_i = M_i$ and is therefore a direct summand of M. This gives a necessary condition for a module to be a direct sum of modules of rank one.

Using this condition, we give two examples. The first example shows that in the theorem the hypothesis of finite rank is indispensable. The second example shows that the theorem cannot survive the presence of three primes.

Example 1. We assume again that R has just two primes, p and q. Let $M = P \oplus Q$, where $P = R_p u_0$ is a copy of R_p and Q is a direct sum

$$\sum_{i=1}^{\infty} R_q u_i$$

of an infinite number of copies of R_q. Let N be the pure submodule generated by all the elements $(1/q^i)u_0 - u_i$. We will show that N is not a direct sum of modules of rank one.

First, we note that $N \cap P = 0$. Indeed, an element of P will lie in N only if some non-zero multiple of it lies in the module generated by the elements $(1/q^i)u_0 - u_i$. Clearly, for a_i in R, a sum

$$\sum_{i=1}^{m} a_i\left(\frac{1}{q^i} u_0 - u_i\right)$$

can only be in P if each $a_i = 0$.

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Next, we note that since every element $(1/q^i)u_0$ lies in $N + Q$, we have $M = N + Q$.

Since N is pure,

$$\bigcap_{j=1}^{\infty} p^j N$$

is $N \cap Q$. Now

$$N/(N \cap Q) \cong (N + Q)/Q \cong R_p = qR_p.$$

Any submodule L of M for which $qL = L$ must be contained in P. Thus a complementary summand for $N \cap Q$ in N must be contained in $N \cap P = 0$. Since it is clear that $N \cap Q \neq N$, we conclude that $N \cap Q$ is not a direct summand of N and that N is not a direct sum of modules of rank one.

Example 2. Let R have at least three non-associated primes. Say p, q, and r are three of them. Let $M = R_p u_1 \oplus R_q u_2 \oplus R_r u_3$ be the direct sum of a copy each of R_p, R_q, and R_r. Let N be the pure submodule generated by $u_1 - u_2$ and $u_2 - u_3$. It is immediate that $N \cap R_p u_1 = 0$.

N contains all the elements $(1/p^k)(u_2 - u_3)$, $(1/q^m)(u_1 - u_3)$, and $(1/r^n)(u_1 - u_2)$. If a and b are elements of R for which $aq^m + br^n = 1$,

$$\frac{b}{q^m}(u_1 - u_3) + \frac{a}{r^n}(u_1 - u_2) - \frac{1}{q^m r^n} u_1$$

lies in $R_q u_2 + R_r u_3$. This shows that all elements of the form $(1/q^m r^n)u_1$ lie in $N + R_q u_2 + R_r u_3$. It follows that $M = N + R_q u_2 + R_r u_3$.

As in the first example,

$$\bigcap_{j=1}^{\infty} p^j N = N \cap (R_q u_2 + R_r u_3)$$

is not a direct summand of N. For

$$N/(N \cap (R_q u_2 + R_r u_3)) \cong R_p = qrR_p,$$

and a submodule L of N for which $qrL = L$ must be contained in $R_p u_1 \cap N = 0$.

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