On a recurrence relation

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It is proposed here to consider the sequence u_n determined by the relation

(*a*) $$u_n - a_n\, u_{n-1} = k_n\,(\theta_n - b_n\,\theta_{n-1})$$

where, in particular,

$$k_n = \frac{1 - a_n}{1 - b_n},$$

and initially $u_1 = \theta_1$. The following is the main result to be proved.

Theorem I. *Suppose that* u_n *satisfies* (*a*), *and that* $\lim\, \theta_n = l$. *Then* $\lim\, u_n = l$ *if the following conditions be fulfilled,*

(*b*) $$\lim \prod_{r=2}^{n} a_r = 0, \text{ as } n \to \infty,$$

(*c*) $$|\, a_n\, k_{n-1} - k_n\, b_n \,| \leqq \alpha_n - \alpha_{n-1}\, |\, a_n \,|,$$

where α_n, k_n *are bounded.*

To prove this, by means of (*a*) express u_n in the form

(1) $$u_n = A_n^n.\, \theta_n + A_n^{n-1}.\, \theta_{n-1} + \ldots . + A_n^1.\, \theta_1.$$

Then $$\sum_{r=1}^{n} A_n^r = 1,$$

since the expression on the left-hand side of this equation is the value of u_n when all the θ's are equal to unity. Furthermore

$$A_n^n = k_n, \quad |\, A_n^{n-1} \,| \leqq \alpha_n - \alpha_{n-1}\, |\, a_n \,|,$$
$$|\, A_n^{r-1} \,| \leqq |\, a_n\, a_{n-1} \ldots . a_{r+1} \,|\, (\alpha_r - \alpha_{r-1}\, |\, a_r \,|\,)$$

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where $r = 2, 3, \ldots . n - 1$. Hence, by addition

$$\sum_{r=1}^{n} | A_n^r | \leqq | k_n | + a_n - | a_n a_{n-1} \ldots . a_2 | a_1$$

which is bounded. Again, from (b),

$$A_n^r \to 0 \qquad \text{as } n \to \infty$$

for a fixed r. Thus (1) fulfils Toeplitz's conditions[1] for convergence, and so

$$\lim u_n = \lim \theta_n = l,$$

the desired result.

When $l = 0$, Theorem I is true for general k_n, *i.e.* without the restriction

$$k_n = \frac{1 - a_n}{1 - b_n}$$

provided (a), (b) and (c) are still satisfied. This is evident from (1), as the condition

$$\sum_{r=1}^{n} A_n^r = 1$$

is here unnecessary when applying Toeplitz's theorem.

Some familiar theorems are obtainable at once as special cases of the above. The following[2] is a case in point.

If Σp_n, Σq_n are two divergent series of positive terms, then

$$\lim \frac{p_0 s_0 + p_1 s_1 + \ldots . + p_n s_n}{p_0 + p_1 + \ldots . + p_n} = \lim \frac{q_0 s_0 + q_1 s_1 + \ldots . + q_n s_n}{q_0 + q_1 + \ldots . + q_n}$$

provided that the second limit exists, and that either (d) p_n/q_n steadily decreases, or (e) p_n/q_n steadily increases, subject to the condition

$$\frac{p_n}{p_0 + p_1 + \ldots . + p_n} < \lambda \frac{q_n}{q_0 + q_1 + \ldots . + q_n}, \ \lambda \textit{ being fixed.}$$

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[1] Knopp, "*Theory and Application of Infinite Series*" (1928), 72.

[2] Bromwich, "*Infinite Series*" (1926), 427.

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Make the following substitutions

$$\sum_{r=0}^{n-1} p_r = a_n \sum_{r=0}^{n} p_r$$

$$\sum_{r=0}^{n-1} q_r = b_n \sum_{r=0}^{n} q_r$$

$$\sum_{r=0}^{n} p_r s_r = u_n \sum_{r=0}^{n} p_r$$

$$\sum_{r=0}^{n} q_r s_r = \theta_n \sum_{r=0}^{n} q_r.$$

Then θ_n converges by hypothesis, say to l, and $u_0 = \theta_0$, while

$$u_n - a_n u_{n-1} = k_n (\theta_n - b_n \theta_{n-1});$$

also

$$\prod_{r=1}^{n} a_r \to 0, \text{ as } n \to \infty,$$

and

$$k_n = \frac{1 - a_n}{1 - b_n}.$$

Since

$$k_n = \frac{p_n}{q_n} \frac{\sum_0^n q_r}{\sum_0^n p_r},$$

we have either $0 < k_n < 1$ or $1 < k_n < \lambda$ for all n according as (d) or (e) is satisfied. Furthermore

$$a_n k_{n-1} - k_n b_n$$

is positive or negative according as (d) or (e) is true.

Thus when (d) is satisfied, writing $\alpha_n = 1 - k_n$

$$| a_n k_{n-1} - k_n b_n | = \alpha_n - \alpha_{n-1} a_n = \alpha_n - \alpha_{n-1} | a_n |.$$

Again when (e) is satisfied, writing $\alpha_n = k_n - 1$, we have

$$| a_n k_{n-1} - k_n b_n | = \alpha_n - \alpha_{n-1} | a_n |.$$

The conditions of Theorem I are now seen to be fulfilled; the required result $\lim u_n = l$ follows.

Now put $l = 0$ in Theorem I, and let $b_n = 0$. This yields at once a result previously obtained[1] for a sequence u_n determined by

$$u_n - a_n u_{n-1} = k_n \theta_n$$

[1] *Proc. Edinburgh Math. Soc.* (2), 3 (1932), 147-150.

in which lim $\theta_n = 0$. This combines the theorems of Copson and Ferrar[1], and of Stolz[2].

By means of a lemma used in my previous note[3] Theorem I may be extended to cover sequences defined by

$$u_n - \sum_{r=1}^{m} a_n^r \, u_{n-r} = k_n \left(\theta_n - \sum_{r=1}^{m} b_n^r \theta_{n-r}\right)$$

where m is any positive integer. These extensions are, however, clumsy in form, and for that reason need not be stated here.

[1] Copson and Ferrar, *Journal London Math. Soc.*, 4 (1929), 258-264.

[2] Bromwich, *ibid.*, 414.

[3] *Proc. Edinburgh Math. Soc.* (2), 3 (1932), 220-222.

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