A CONTRAST BETWEEN COMPLEX AND REAL-VALUED TAYLOR SERIES

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In this paper it is shown that if c is a point of the region of convergence of an analytic function $f(z) = \Sigma_n^\infty c_n z^n$ then in every neighborhood of c there exists a point e such that the value $f(c)$ of the function $f(z)$ is attained by some truncation $\Sigma_{n=0}^k c_n z^n$ of $f(z)$ at $z = e$, i.e., $\Sigma_{n=0}^k c_n e^n = \Sigma_{n=0}^\infty c_n c^n$. Also it is shown that the above does not hold in the case of real-valued functions of a real variable.

THEOREM. *Let c be a complex number inside the circle of convergence of an analytic function $f(z)$ given by*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

Then in every neighborhood of c there exists a point e such that

$$f(c) = \sum_{n=0}^{k} c_n e^n \quad \textit{for some } k < \infty$$

PROOF. If $f(z)$ is a constant function then the conclusion of the Theorem follows trivially. Thus, in what follows, we let $f(z) = \Sigma_{n=0}^\infty c_n z^n$ be nonconstant. But then there exists a circumference E of positive radius with center at c such that $f(z) \neq f(c)$ on E. Clearly, $|f(z) - f(c)|$ is a continuous function on E and therefore it has a positive minimum p. Hence,

$$(1) \qquad |f(z) - f(c)| \geqq p > 0 \quad \textit{for every } z \in E$$

Since $\Sigma_{n=0}^\infty c_n z^n$ has uniform convergence on E and since $p > 0$, we see that

$$(2) \qquad \left| \left(\sum_{n=0}^{k} c_n z^n \right) - f(z) \right| < p \textit{ for some } k < \infty \textit{ on } E$$

Consequently, from (1) and (2) we have

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(3) $$\left|f(z)-f(c)\right| > \left|\left(\sum_{n=0}^{k} c_n z^n\right) - f(z)\right| \quad \textit{for every } z \in E$$

But then from (3), by Rouché's theorem [1, p. 157] it follows that inside the circle E the function $g(z)$ given by

$$g(z) = (f(z)-f(c)) + \left(\left(\sum_{n=0}^{k} c_n z^n\right) - f(z)\right) = \left(\sum_{n=0}^{k} c_n z^n\right) - f(c)$$

has as many zeros as the function $f(z)-f(c)$ has. But clearly, $f(z)-f(c)$ has at least one zero, namely, $z = c$, inside E. Hence, $(\sum_{n=0}^{k} c_n z^n) - f(c)$ must have also at least one zero, say, $z = e$ inside E implying the conclusion of the Theorem.

REMARK. If in the hypothesis of the above Theorem it is also assumed that $c \neq 0$ and that $f(z)$ is not a polynomial (i.e., $f(z)$ is transcendental) then it can be shown that the conclusion of the Theorem holds for some $e \neq c$.

As mentioned earlier, the statement of the Theorem is not valid in the setup of real-valued functions of a real variable. In other words, in contrast with the above Theorem, we have:

PROPOSITION. *There exists a real-valued function $f(x)$ of a real variable x given by*

$$f(x) = \sum_{n=0}^{\infty} r_n x^n$$

and there exists a real number r inside the interval of convergence of $f(x)$ and there exists a neighborhood N of r such that

$$f(r) \neq \sum_{n=0}^{k} r_n x^n \quad \textit{for every } k < \infty \textit{ and every } x \in N$$

PROOF. Let us consider the function $f(x)$ defined on the real open interval $(-1, 1)$ by

$$f(x) = 1 - 10x + 24x^2 + \frac{1}{1-x}$$

Thus,

(4) $$f(x) = -11x + 23x^2 - x^3 - x^4 - x^5 - \cdots \quad (-1 < x < 1)$$

It can be readily verified that $f'(0) = -11$ and $f'(.05) = 10$. As a consequence, $f(x)$ has a unique minimum $f(r)$ for $x = r$ in the open interval $(0, 0.5)$ Also, it can be readily verified that $f(x)$ has no maximum in $(0, 0.5)$. But then since, after the second term, every term in (4) has a negative sign, it can be readily shown that there exists a neighborhood N of r such that $N \subseteq (0, 0.5)$ and such that

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$$f(r) \neq -11x + 23x^2 - x^3 - x^4 - x^5 - \cdots - x^k$$

for every $k < \infty$ and every $x \in N$.

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Reference

[1] S. Saks and A. Zygmund, *Analytic Functions*, (Warsaw, 1952).

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