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On the invariance of certain estimators

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In this note, L_p estimators for the parameters in the linear model $y = X\beta$ are considered. In particular, it is shown that these estimators are invariant under scale transformations on the dependent variable; that is, if $\hat{\beta}(y, X)$ is an L_p estimator for β, then $a\hat{\beta}(y, X) = \hat{\beta}(ay, X)$ for any nonzero scalar a. It is shown that this result does not extend to more general transformations on y, and the invariance property does not hold for general nonlinear models.

1. Invariance

Computing L_p estimators by linear functions on discrete data has received much attention in the literature in the past few years; recent articles have appeared for determining L_p estimators when $p = 1, 2$, and ∞ (see Appa and Smith [1], Barrodale and Roberts [2], and Wagner [9]). The L_p-criteria for various values of p have been discussed by Barrodale and Roberts [3], Barrodale, Roberts, and Hunt [4], Ekblom and Henriksson [5], Forsythe [6], and Rice and White [8]. Several properties of L_1 and L_∞ estimators are given in [1]; in [3] some properties of L_p estimators are established. This note will address itself to another property of any L_p estimator (for any $p \neq 0$). In particular, it is shown that such estimators are invariant under scale transformations on the dependent

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variable. The concept of invariance is discussed by Fraser [7, p. 67] and is defined for scale transformations as:

DEFINITION. $\hat{\beta}(y, X)$ is an invariant estimator for β if

$$a\hat{\beta}(y, X) = \hat{\beta}(ay, X) .$$

The following lemma shows that under the L_p-criteria for any p ($\neq 0$) the L_p estimator, $\hat{\beta}(y, X)$, is invariant in the case of the linear model.

LEMMA. *If* $\min_{\beta} \|y-X\beta\|^p = \|y-X\hat{\beta}\|^p$, *then for any scalar* $a \neq 0$,

(i) $\min_{\delta} \|ay-X\delta\|^p = \|a\|^p \|y-X\hat{\beta}\|^p$, *and*

(ii) $\hat{\delta} = a\hat{\beta}$.

Proof. If $\min_{\beta} \|y-X\beta\|^p = \|y-X\hat{\beta}\|^p$, then

$$\min_{\delta} \|ay-X\delta\|^p = \|a\|^p \min_{\delta} \left\|y - \frac{1}{a} X\delta\right\|^p$$

$$= \|a\|^p \min_{\beta} \|y-X\beta\|^p ,$$

where

$$\beta = \frac{1}{a} \delta$$

$$= \|a\|^p \|y-X\hat{\beta}\|^p ,$$

and the result follows.

2. Summary and examples

When $p = 1, 2$, and ∞, the best L_p estimators of the location parameter in the model $y = \beta_0$ correspond to the median, mean, and midrange, respectively ([1], [5], [7], and [8]). Hence, appealing to the lemma, for any scalar a ($\neq 0$),

$$ay_M, \ \overline{ay}, \text{ and } ay_{MR}$$

are optimal under L_1, L_2, and L_∞, respectively.

Consider the model $y = \beta^3$. This model is nonlinear with respect to the unknown parameter, β. Under the L_2-criterion

$$\hat{\beta} = (\bar{y})^{1/3},$$

and for any scalar a ($\neq 0$),

$$\hat{\beta}* = (a\bar{y})^{1/3}.$$

Hence, the estimator is not invariant, but is invariant under the transformation $\delta = \beta^3$. In this case,

$$\hat{\delta} = \bar{y} \quad \text{and} \quad a\hat{\delta} = a\bar{y}.$$

When the nonlinear model is separable with respect to the unknown parameters, then the corresponding L_p estimator is invariant in view of the above transformation. The invariance property of the L_p estimators does not hold for general nonlinear models. For example, consider the model

$$y = \beta_1 e^{-\beta_2 x},$$

then for any optimal L_p estimator

$$\tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2)$$

and any a ($\neq 0$),

$$\tilde{\tilde{\beta}} = (a\tilde{\beta}_1, \tilde{\beta}_2).$$

In [4], models nonlinear in one parameter are discussed; for these models, the estimators are not invariant. Examples of these models are:

$$y = (\beta_0+\beta_1 x)/(1+\beta_2 x),$$

$$y = \beta_0 + \beta_1/(1+x)^{\beta_2},$$

and

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$$y = e^{\beta_2 x}(\beta_0+\beta_1 x) .$$

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