For the *converse*, suppose

$$\left.\begin{aligned} a_1X + b_1Y + c_1Z = 0, \\ a_2X + b_2Y + c_2Z = 0, \\ a_3X + b_3Y + c_3Z = 0, \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

where X, Y, Z are not all zero, say $Z \neq 0$.

As in *Ex.* 1, take λ, μ, ν, ρ so that

$$\lambda(a_1x + b_1y + c_1z) + \mu(a_2x + b_2y + c_2z) + \nu(a_3x + b_3y + c_3z) + \rho z \equiv 0. \quad (20)$$

Put X, Y, Z for x, y, z in (20) and we find $\rho = 0$.

Example 3. *Theorem.—In Cartesian coordinates, at least one curve whose equation is of the form*

$$a(x^2 + y^2) + 2gx + 2fy + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(21)$$

passes through any three points; where a, g, f are not all zero.

Observe that we cannot say without qualification that there is only one such curve, even if we premise that the three given points are distinct; *e.g.* the three points $(-iy_1, y_1)$, $(-iy_2, y_2)$, $(-iy_3, y_3)$ lie on the curve $x^2 + y^2 + \lambda(x + iy) = 0$, where the coefficient λ is arbitrary.

(*To be continued.*) See P. 137

JOHN DOUGALL

Notes on Algebraic Inequalities.—

1. It is worthy of remark that there is a method of proving algebraic inequalities which is very generally applicable, and which furnishes proofs of great directness and completeness.

The method consists in *expressing the difference $A - B$ in a manifestly positive form*, when we have to prove $A > B$.

The first inequality occurring in school Algebra is usually proved in this way. We have to show that if a, b, and x are positive quantities $\frac{a+x}{b+x}$ is nearer to 1 than $\frac{a}{b}$. We have

$$\frac{a+x}{b+x} - \frac{a}{b} = \frac{x(b-a)}{b(b+x)}.$$

The two cases $a > b$ and $b > a$ are then considered separately, and the conclusion drawn.

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With regard to this inequality, it may be interesting to note a method which obviates the necessity of considering the two cases separately.

$$\left(\frac{a}{b}-1\right)^2-\left(\frac{a+x}{b+x}-1\right)^2=\left(\frac{a}{b}-\frac{a+x}{b+x}\right)\left(\frac{a}{b}+\frac{a+x}{b+x}-2\right)$$

$$=\frac{x(x+2b)(a-b)^2}{b^2(b+x)^2}>0, \text{ unless } a=b.$$

Hence $\frac{a}{b}-1$ is numerically $>\frac{a+x}{b+x}-1$.

Note the artifice of squaring the quantities to facilitate réasoning about their numerical values.

Amongst other applications of the general method above described, we may take these:

To prove that if a be the greatest of four quantities in proportion, so that $a:b=c:d$, then $a+d>b+c$. We have

$$(a+d)-(b+c)=a+\frac{bc}{a}-b-c=\frac{1}{a}(a-b)(a-c)>0.$$

Again, to prove that the arithmetic mean of two positive quantities a, b is greater than their geometric mean, and that greater than their harmonic mean:—

$$\frac{a+b}{2}-\sqrt{ab}=\frac{(\sqrt{a}-\sqrt{b})^2}{2}>0$$

$$\sqrt{ab}-\frac{2ab}{a+b}=\frac{\sqrt{ab}}{a+b}(\sqrt{a}-\sqrt{b})^2>0.$$

In C. Smith's Treatise on Algebra, in the chapter on Inequalities are enumerated five "elementary principles" and six "Theorems," which are to be used in proving other inequalities. But most of the examples at the end of the chapter can be proved directly by the method of this article. The following are specimens taken at random; the numbers refer to "Examples XXXV." in Smith's Treatise, and the letters are supposed to denote positive quantities:

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1. To prove $y^2z^2+z^2x^2+x^2y^2 \nless xyz(x+y+z)$

$$y^2z^2+z^2x^2+x^2y^2-xyz(x+y+z)=\tfrac{1}{2}\{(yz-zx)^2+(zx-xy)^2+(xy-yz)^2\}$$

3. To prove $(a_1b_1+a_2b_2+\ldots)\left(\dfrac{a_1}{b_1}+\dfrac{a_2}{b_2}+\ldots\right) \nless (a_1+a_2+\ldots)^2$,

$$\begin{aligned}(a_1b_1+a_2b_2+\ldots)\left(\frac{a_1}{b_1}+\frac{a_2}{b_2}+\ldots\right)&-(a_1+a_2+\ldots)^2\\ &=a_1a_2\left(\frac{b_1}{b_2}+\frac{b_2}{b_1}-2\right)+\ldots+\ldots\\ &=\frac{a_1a_2}{b_1b_2}(b_1-b_2)^2+\ldots+\ldots\end{aligned}$$

4. To prove $a^6+b^6>a^5b+ab^5$.

$$\begin{aligned}a^6+b^6-(a^5b+ab^5)&=a^5(a-b)+b^5(b-a)=(a-b)(a^5-b^5)\\ &=(a-b)^2(a^4+a^3b+a^2b^2+ab^3+b^4)\end{aligned}$$

6. To prove $(a+b+c)(a^2+b^2+c^2) \nless 9abc$.

$$(a+b+c)(a^2+b^2+c^2)-9abc=\Sigma a^3+\Sigma a^2b-9abc=\Sigma a^3-3abc+\Sigma a^2b-6abc$$

$$=\frac{a+b+c}{2}\{(b-c)^2+(c-a)^2+(a-b)^2\}+a(b-c)^2+b(c-a)^2+c(a-b)^2$$

11. To prove $\dfrac{a-x}{a+x}<\dfrac{a^2-x^2}{a^2+x^2}$ if $x<a$

$$\frac{a-x}{a+x}-\frac{a^2-x^2}{a^2+x^2}=\frac{(a-x)\{a^2+x^2-(a+x)^2\}}{(a+x)(a^2+x^2)}=\frac{2ax(x-a)}{(a+x)(a^2+x^2)}$$

which is of the same sign as $x-a$.

Note that when a rational algebraic inequality holds good without any restriction as to the value of the letters involved, the *difference* is usually expressible as a sum of squares, or as a fraction whose numerator and denominator are sums of squares. If the letters are restricted to denote positive quantities, we can substitute squared letters for them, and then with confidence seek the expression of the *difference* in terms of squares.

Thus in C. Smith's Example 16 "to prove that if a, b, c are positive $\dfrac{2}{b+c}+\dfrac{2}{c+a}+\dfrac{2}{a+b} \nless \dfrac{9}{a+b+c}$", writing x^2, y^2, z^2 for a, b, c the *difference* is

$$\frac{2}{y^2+z^2}+\frac{2}{z^2+x^2}+\frac{2}{x^2+y^2}-\frac{9}{x^2+y^2+z^2}.$$

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Reducing this to a single fraction with denominator

$$(y^2+z^2)(z^2+x^2)(x^2+y^2)(x^2+y^2+z^2),$$

the numerator is

$$\begin{aligned} &2\Sigma x^2\Sigma(y^2+z^2)(z^2+x^2)-9(y^2+z^2)(z^2+x^2)(x^2+y^2)\\ &=2\Sigma x^2(\Sigma x^4+3\Sigma x^2y^2)-9(\Sigma x^4y^2+2x^2y^2z^2)\\ &=2(\Sigma x^6+\Sigma x^4y^2+3\Sigma x^4y^2+9x^2y^2z^2)-9(\Sigma x^4y^2+2x^2y^2z^2)\\ &=2\Sigma x^6-\Sigma x^4y^2=\Sigma(x^2-y^2)(x^4-y^4)\\ &\qquad\qquad=\Sigma(x^2+y^2)(x^2-y^2)^2 \end{aligned}$$

which is a sum of squares.

R. F. MUIRHEAD

Cauchy's Condensation Test.—

Theorem.—If the terms of the series $\Sigma f(n)$ never increase as n increases, and if $\underset{n\to\infty}{\mathrm{L}} f(n)=0$ then $\Sigma f(n)$ converges or diverges with $\Sigma a^n f(a^n)$, $a>1$.

I. Suppose a an integer $\geqq 2$.

Then

$$\begin{aligned} \Sigma f(n)=&[f(1)+f(2)+\quad+f(a-1)]\\ &+[f(a)+f(a+1)+\quad+f(a^2-1)]\\ &+[f(a^2)+\quad+f(a^3-1)]\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ &+[f(a^m)+\ldots+f(a^{m+1}-1)]\\ &+\text{ etc.} \end{aligned}$$

Hence $\Sigma f(n)$ converges or diverges with

$$\Sigma[f(a^m)+\ldots+f(a^{m+1}-1)].$$

Now $f(a^m)>f(a^m+1)>\ldots>f(a^{m+1}-1)>f(a^{m+1})$.

$\therefore\ (a^{m+1}-a^m)f(a^m)>f(a^m)+\ldots+f(a^{m+1}-1)>(a^{m+1}-a^m)f(a^{m+1})$.

i.e. $(a-1)a^m f(a^m)>f(a^m)+\ldots+f(a^{m+1}-1)>\dfrac{a-1}{a}a^{m+1}f(a^{m+1})$.

$\therefore\ \Sigma f(n)$ converges with $\Sigma a^m f(a^m)$ or $\Sigma a^n f(a^n)$

or diverges with $\Sigma a^{m+1}f(a^{m+1})$ or $\Sigma a^n f(a^n)$.

This proves the theorem for a integral.

II. Assume $a>1$, but not necessarily an integer

(1°) Suppose $\underset{x\to\infty}{\mathrm{L}} xf(x)\pm 0$ then ultimately $xf(x)>\mathrm{A}$.

i.e. $f(x)>\dfrac{\mathrm{A}}{x}$.

i.e. $f(n)>\dfrac{\mathrm{A}}{n}$.