On the necessary and sufficient condition for the degeneracy of a quadratic function of a number of variables.—The following method for three variables can be extended at once to the general case.

Let $\phi(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ ....(1)

$$\left.\begin{aligned}&\text{Then}\quad \phi(x+\lambda X,\ y+\lambda Y,\ z+\lambda Z) \equiv \phi(x, y, z)\\ &+2\lambda\{x(aX+hY+gZ)+y(hX+bY+fZ)+z(gX+fY+cZ)\}\\ &+\lambda^2\{X(aX+hY+gZ)+Y(hX+bY+fZ)+Z(gX+fY+cZ)\}.\end{aligned}\right\}\text{(2)}$$

THEOREM.—*A necessary and sufficient condition that $\phi(x, y, z)$ should be expressible as a homogeneous quadratic function of two variables which are themselves homogeneous linear functions of x, y, and z, is that the three equations*

$$\left.\begin{aligned}ax+hy+gz&=0,\\ hx+by+fz&=0,\\ gx+fy+cz&=0,\end{aligned}\right\}\text{.........................(3)}$$

should have a non-null solution.

(i) *Necessity of the condition.*

Let $\phi(x, y, z) \equiv f(u, v)$, ...(4)

where f is a quadratic function, and u, v are homogeneous linear functions of x, y, z.

Let U, V be what u, v become when X, Y, Z are put for x, y, z.

Then u, v become $u+\lambda U$, $v+\lambda V$ when x, y, z become $x+\lambda X, y+\lambda Y, z+\lambda Z$.

Hence $\phi(x+\lambda X,\ y+\lambda Y,\ z+\lambda Z) \equiv f(u+\lambda U,\ v+\lambda V)$........(5)

Now values of X, Y, Z, not all zero, can always be found to make

$$U=0,\ V=0 \text{ ....(6)}$$

We then have $\phi(x+\lambda X,\ y+\lambda Y,\ z+\lambda Z) \equiv \phi(x, y, z)$.......(7)

With these values of X, Y, Z the coefficients of λx, λy, λz on the right of (2) must therefore vanish.

(ii) *Sufficiency of the condition.*

Suppose that (X, Y, Z) is a non-null solution of (3).

Then (2) gives $\phi(x, y, z) \equiv \phi(x+\lambda X,\ y+\lambda Y,\ z+\lambda Z)$ .... (8)

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Say $Z \neq 0$, and take $\lambda = -z/Z$.

Thus $$\phi(x, y, z) \equiv \phi\left(x - \frac{Xz}{Z}, y - \frac{Yz}{Z}, 0\right),$$

that is to say, $\phi(x, y, z)$ is a quadratic function of the two linear functions $x - Xz/Z$ and $y - Yz/Z$.

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On the sufficiency of the condition for a limit.—Let $z_1, z_2, z_3, \ldots$ be a sequence of quantities, real or complex, such that, corresponding to any arbitrary small positive quantity ϵ, we can find a positive integer n such that $|z_{n+p} - z_n| < \epsilon$ for all positive integral values of p.

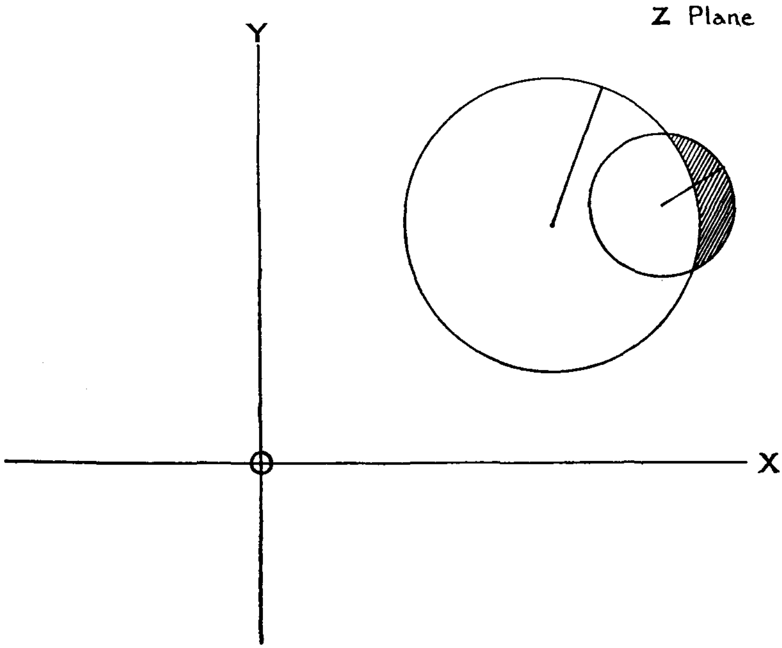
Take a sequence of ϵ's,

$$\epsilon_1, \epsilon_2, \epsilon_3, \ldots$$

which steadily decreases to the limit zero.

Let $n_1, n_2, n_3, \ldots$ be the smallest n's corresponding to $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ respectively.

We have $n_1 \leqq n_2 \leqq n_3 \leqq \ldots$.



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