Can. J. Math., Vol. XXIX, No. 2, 1977, pp. 441–448

LEBESGUE CONSTANTS FOR CARDINAL $\mathscr{L}$-SPLINE INTERPOLATION

J. TZIMBALARIO

1. Introduction. Recently the theory of cardinal polynomial spline interpolation was extended to cardinal $\mathscr{L}$-splines [**3**]. Let

(1) $\quad P_n(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + x^n$

be a polynomial with only real zeros. Denote the set of zeros by $T \equiv T_n = \{t_0, t_1, \ldots, t_{n-1}\}$. If $\mathscr{L}_n(D) = p_n(D)$ is the associated differential operator, the null-space of $\mathscr{L}_n(D)$

(2) $\quad \pi_{n-1}(T) = \{y | \mathscr{L}_n(D)y = 0\}$

is a natural generalization of the usual polynomials.

The space of *cardinal* $\mathscr{L}$*-splines* is defined as the class of functions

(3) $\quad \mathscr{S}_n(T) \equiv \{S(x) | S \in C^{n-2}(\mathbf{R}), S|_{(\nu,\nu+1)} \in \pi_{n-1}(T), \nu \in Z\}.$

Sometimes it is convenient to place the knots of the splines half way between the integers. Accordingly we define

(4) $\quad \mathscr{S}_n^*(T) \equiv \{S(x) | S(x + \tfrac{1}{2}) \in \mathscr{S}_n(T)\}.$

The special case of cardinal polynomial splines was studied systematically by I. J. Schoenberg in his monograph [**6**]. The first systematic study of interpolation problems connected to cardinal $\mathscr{L}$-splines was done by C. Micchelli [**3**]. Another approach to the problem was given by I. J. Schoenberg in one of his recent papers [**7**]. Some extremal properties were given by A. Sharma and the author in [**8**].

The interpolation problem mentioned above consists in checking existence and uniqueness of a spline in some subspaces of $\mathscr{L}_n(T)$ (or $\mathscr{L}_n^*(T)$) which satisfies

(5) $\quad S(\nu + \alpha) = y_\nu, \quad \nu \in Z,$

where $y = \{y_\nu\}$ is any set of given data and $0 \leqq \alpha < 1$.

For the sake of commodity we shall assume that for n even the knots are the integers and if n is odd the knots are at the integers shifted by $\frac{1}{2}$.

It is known [**3**] that for every α in $[0, 1)$ with only one exception, and any data of powergrowth $y = \{y_\nu\}$, there exists a unique function $S \in \mathscr{L}_n(T)$ (or $\mathscr{L}_n^*(T)$) of the same power growth which satisfies (5). In this paper we

Received June 14, 1976 and in revised form, August 12, 1976.

shall restrict ourselves to the space of bounded data with the norm $||y||_\infty = \sup_{\nu \in Z} |y_\nu|$ and we shall assume $\alpha = 0$.

The existence and uniqueness of bounded interpolating cardinal $\mathscr{L}$-spline follows from [**3**].

The corresponding operator $P_n : l^\infty(Z) \to \mathscr{S}_n(T) \cap L^\infty(\mathbf{R})$ (or $\mathscr{S}_n^*(T) \cap L^\infty(\mathbf{R})$) is called the *cardinal $\mathscr{L}$-spline interpolation operator of order n* and its norm is the $n - 1$th Lebesgue constant.

$$(6) \quad ||\mathscr{L}_{n-1}{}^T|| = ||P_n||_\infty = \sup_{||y||_\infty = 1} ||P_n y||_\infty.$$

We propose to study the behaviour of $\mathscr{L}_n(T)$ as $n \to \infty$. For the special case $T = 0$, Richards [**5**] has shown that $\mathscr{L}_n{}^0$ is asymptotic to $(2/\pi) \log n$.

We shall use mainly the notations introduced by Micchelli [**3**].

2. Auxiliary results. It is known [**3**] that the solution $S(x)$ to the interpolation problem can be given explicitely as a cardinal series

$$(26) \quad S(x) = \sum_{-\infty}^{+\infty} y_\nu L_{n-1}(x - \nu),$$

where

$$(7) \quad \begin{cases} \text{(i) } L_{n-1}(x) \in \begin{cases} \mathscr{S}_{n-1}(T) & n \text{ even} \\ \mathscr{S}_{n-1}{}^*(T) & n \text{ odd,} \end{cases} \\ \text{(ii) } L_{n-1}(\nu) = \delta_{0\nu}, \quad \nu \in Z, \\ \text{(iii) } |L_{n-1}(x)| \leqq Ae^{-B|x|}, \quad x \in \mathbf{R}. \end{cases}$$

The function $L_{n-1}(x)$ is called the *fundamental function* for the interpoiation problem. In order to evaluate the Lebesgue constants later we need a proper representation of the fundamental functions.

LEMMA 1.

$$(8) \quad L_{n-1}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\psi_n{}^T(u)}{\phi_n{}^T(u)} e^{ixu} du,$$

where

$$(9) \quad \phi_n{}^T(u) = \sum_{j=-\infty}^{\infty} \psi_n{}^T(u + 2\pi j),$$

and

$$(10) \quad \psi_n{}^T(u) = \prod_{\mu=0}^{n-1} \frac{(-t_\mu)(e^{-t_\mu} - e^{-iu})e^{iu/2}}{(-t_\mu + iu)(e^{-t_\mu} - 1)}.$$

Proof. The proof follows easily on using the properties of B-splines obtained in [**3**] and following similar lines as those of the special case $T = 0$ [**5**].

Considering the problem of studying the behaviour of the Lebesgue constants when $n \to \infty$, we restrict our discussion to the case $T = -T$, i.e. when the associated differential operator is formally self-adjoint. However we do allow the zeros of $p_n(x)$ to change with n.

Now we shall find an estimate from below for the Lebesgue constant $||\mathscr{L}_{n-1}{}^T||$.

LEMMA 2. *If* $T = -T$, *we have*

$$(11) \quad ||\mathscr{L}_{n-1}{}^T|| \geqq \frac{1}{\pi}\int_0^\pi \frac{\gamma_n{}^T(u)}{\phi_n\ (u)} \sec u/2\, du,$$

where

$$(12) \quad \gamma_n{}^T(u) = \sum_{j=-\infty}^{\infty} (-1)^j \psi_n{}^T(u + 2\pi j).$$

Proof. Since

$$(13) \quad ||\mathscr{L}_n{}^T|| = \sup_{||y||_\infty=1} ||P_n y||_\infty \geqq \sum_{\nu=-\infty}^{\infty} \tilde{y}_\nu L_{n-1}(\tfrac{1}{2} - \nu),$$

where

$$(14) \quad \tilde{y}_\nu = \begin{cases} (-1)^{\nu+1} & \nu = 1, 2, \dots \\ (-1)^\nu & \nu = 0, -1, -2, \dots, \end{cases}$$

the result now follows on using the integral representation (8), and performing some quite standard transformations.

The next lemma will prove that in the representation (11) we can replace in some sense the functions $\gamma_n{}^T(u)$ and $\phi_n{}^T(u)$ by the dominant terms respectively.

LEMMA 3. *Let* $0 < u < \pi$, $n \geqq 3$ *and assume that* $\sup_n \sup_{t \in T_n} |t| \leqq d < \infty$. *Then*

$$(15) \quad \left| \frac{\gamma_n{}^T(u)}{\psi_n{}^T(u) - \psi_n{}^T(u - 2\pi)} - 1 \right| < K \left(\frac{\frac{1}{u} + d^2}{\frac{5}{u} + d^2} \right)^{n/2}$$

$$\left| \frac{\phi_n{}^T(u)}{\psi_n{}^T(u) + \psi_n{}^T(u - 2\pi)} - 1 \right| < K \left(\frac{\frac{1}{u} + d^2}{\frac{5}{u} + d^2} \right)^{n/2}$$

for some positive K.

Proof. We shall use the following notations:

$$(16) \quad p_n{}^*(u) = \prod_{\mu=0}^{n-1} \left(u + \frac{i t_\mu}{2\pi} \right)$$

$$(17) \quad \tilde{\phi}_n{}^T(u) = \sum_j \frac{(-1)^{jn}}{p_n{}^*(u + j)}$$

and

$$(18)\quad \tilde{\gamma}_n{}^T(u) = \sum_j \frac{(-1)^{j(n+1)}}{p_n{}^*(u+j)}.$$

Observe that $p_n{}^*(u)$ is a polynomial with positive coefficients and satisfying $p_n{}^*(u) = (-1)^n p_n{}^*(-u)$.

The proof will be similar to the proof [**5**] in the special case $T = 0$, but we will present it here for sake of completeness (only for n even).

Clearly

$$(19)\quad \max_{0<u<\pi} \left| \frac{\gamma_n{}^T(u)}{\psi_n{}^T(u) - \psi_n{}^T(u-2\pi)} - 1 \right| = \max_{0<u<\frac{1}{2}} \left| \frac{\tilde{\gamma}_n{}^T(u)}{\dfrac{1}{p_n{}^*(u)} - \dfrac{1}{p_n{}^*(1-u)}} - 1 \right|$$

and

$$(20)\quad \max_{0<u<\pi} \left| \frac{\phi_n{}^T(u)}{\psi_n{}^T(u) + \psi_n{}^T(u-2\pi)} - 1 \right| = \max_{0<u<\frac{1}{2}} \left| \frac{\tilde{\phi}_n{}^T(u)}{\dfrac{1}{p_n{}^*(u)} + \dfrac{1}{p_n{}^*(1-u)}} - 1 \right|.$$

We shall use the fact that $0 < u < \frac{1}{2}$. By (12) we get

$$\left| \frac{\tilde{\gamma}_n{}^T(u)}{\dfrac{1}{p_n{}^*(u)} - \dfrac{1}{p_n{}^*(1-u)}} - 1 \right| \leqq \sum_{j=1}^{\infty} \alpha_i(u, n)$$

where

$$0 \leqq \alpha_j(u,n) = \frac{\dfrac{1}{p_n{}^*(j+u)} - \dfrac{1}{p_n{}^*(j+1-u)}}{\dfrac{1}{p_n{}^*(u)} - \dfrac{1}{p_n{}^*(1-u)}}$$

$$= \frac{p_n{}^*(u)}{p_n{}^*(j+u)p_n{}^*(j+1-u)} \frac{[p_n{}^*(j+1-u) - p_n{}^*(j+u)]}{1 - \dfrac{p_n{}^*(u)}{p_n{}^*(1-u)}}.$$

By the mean value theorem and by Markov's inequality we get for some $u < \xi < 1 - u$,

$$p^*(j+1-u) - p^*(j+u) = (1-2u)p^*(j+\xi)$$
$$\leqq n^2 \max_{u<\xi<1-u} p^*(j+\xi)(1-2u) = n^2 p^*(j+1-u)(1-2u).$$

It can be shown easily that

$$\frac{1}{1-\dfrac{p^*(u)}{p^*(1-u)}} \leqq \frac{(1-u)^2+t_0^2}{1-2u} \quad 0 < u < \tfrac{1}{2},$$

where t_0 is the root of $p_n(x)$ with the smallest absolute value. Then

$$\alpha_j(u, n) \leqq \frac{p^*(u)}{p^*(j+u)} n^2[(1-u)^2+t_0^2].$$

Since $|t_\mu| \leqq d < \infty$ uniformly in n, we get

$$\alpha_j(u, n) \leqq \left| \prod_{\mu=0}^{n-1} \frac{\left(u+\dfrac{it_\mu}{2\pi}\right)}{\left(j+u+\dfrac{it_\mu}{2\pi}\right)} \right| n^2(1+t_0^2)$$

$$\leqq \left(\frac{u^2+d^2}{(j+u)\ +d^2}\right)^{n/2} n^2(1+d^2).$$

Then

$$\sum_{i=1}^{\infty} \alpha_j(u, n) \leqq \sum_{j=1}^{n} \left(\frac{u^2+d^2}{(j+u)^2+d^2}\right)^{n/2} \cdot n^2(1+d^2)$$

$$\leqq n^2(1+d^2) \cdot \sum_{j=1}^{n} \left(\frac{\frac{1}{4}+d^2}{j^2+\frac{1}{4}+d^2}\right)^{n/2} \leqq K\left(\frac{\frac{1}{4}+d^2}{\frac{5}{4}+d^2}\right)^{n/2}$$

for some constant K depending only on d. In a similar way we prove also the second inequality in (15), and also the case n odd.

COROLLARY 1. *Suppose that* $\sup_n \sup_{t \in T_n} |t| \leqq d < \infty$. *Then the Lebesgue constant* $||\mathscr{L}_{n-1}{}^T||$ *satisfies the following inequality:*

$$(21) \quad ||\mathscr{L}_n{}^T|| \geqq \frac{R_n(\xi_n)}{\pi} \int_0^\pi \frac{\tilde{p}_n(2\pi-u)-\tilde{p}_n(u)}{\tilde{p}_n(2\pi-u)+\tilde{p}_n(u)} \sec u/2\, du$$

for some $0 \leqq \xi_n \leqq \pi$ *and*

$$(22) \quad |R_n(\xi_n)-1| \leqq K\left(\frac{\frac{1}{4}+d^2}{\frac{1}{4}+d^2}\right)^{n/2}.$$

Here

$$\tilde{p}_n(u) \equiv \prod_{\mu=0}^{n-1} (u+it_\mu).$$

From Corollary 1 we get

COROLLARY 2. *If the sequence*

$$(23) \quad ||\mathscr{L}_n{}^T||^* = \frac{1}{\pi} \int_0^\pi \frac{\tilde{p}_n(2\pi-u)-\tilde{p}_n(u)}{\tilde{p}_n(2\pi-u)+\tilde{p}_n(u)} \sec u/2\, du$$

increases no faster than

$$\left(\left(\frac{\frac{1}{4}+d^2}{\frac{1}{4}+d^2}\right)^{n/2}\right)$$

then

$$(24)\quad ||\mathscr{L}_n{}^T|| \geqq ||\mathscr{L}_n{}^T||^* + O(1) \quad \textit{as } n \to \infty.$$

3. The main result. Now we can state the main theorem.

THEOREM. *Assume that* $T = -T$ *and* $\sup_n \sup_{t \in T_n} |t| \leqq d < \infty$. *Then the sequence of the Lebesgue constants for the cardinal* $\mathscr{L}$*-spline interpolation is not bounded.*

Proof. As in [**5**], we write

$$\sec \frac{u}{2} = \frac{2}{\pi - u} + h(u), \quad 0 < u < \pi$$

where $h(u)$ is continued in $[0, 2\pi]$. Then

$$||\mathscr{L}_n{}^T||^* = \frac{2}{\pi} \int_0^\pi \frac{\tilde{p}_n(2\pi - u) - \tilde{p}_n(u)}{\tilde{p}_n(2\pi - u) + \tilde{p}_n(u)} \frac{du}{\pi - u} + \frac{1}{\pi} \int_0^\pi \frac{\tilde{p}_n(2\pi - u) - \tilde{p}_n(u)}{\tilde{p}_n(2\pi - u) + \tilde{p}_n(u)} h(u)du.$$

The second integral is certainly uniformly bounded with respect to n.

Let us denote the first integral by I_n. Using the mean value theorem for the numerator of the integrand in I_n and the Markov's inequality we get

$$I_1 = \frac{2}{\pi} \int_0^\pi \frac{(2\pi - 2u)\tilde{p}_n'(\xi)}{\tilde{p}_n(2\pi - u) + \tilde{p}_n(u)} \frac{du}{\pi - u}$$

(for some $u < \xi < 2\pi - u$)

$$I_1 \leqq K_1 n^2 \int_0^\pi \frac{\sup_{u<\xi<2\pi-u} |\tilde{p}_n(\xi)|}{\tilde{p}_n(2\pi - u) + \tilde{p}_n(u)} du \leqq K_2 n^2$$

for some K_2 independent of n.

Then $||\mathscr{L}_n{}^T||^*$ does not increase faster than $O(n^2)$, which is certainly

$$o\left(\left(\frac{\frac{1}{4}+d^2}{\frac{1}{4}+d^2}\right)^{n/2}\right).$$

This means that $||\mathscr{L}_n{}^T||$ is not bounded if $||\mathscr{L}_n{}^T||^*$ is not bounded, and we intend to prove now that $||\mathscr{L}_n{}^T||^*$ is not bounded. Changing the variable to

https://doi.org/10.4153/CJM-1977-047-4 Published online by Cambridge University Press

$x = (2\pi/u) - 1$, the integral I_n is transformed into

$$I_n = \frac{4}{\pi}\int_1^\infty \frac{\tilde{p}_n\left(\frac{2\pi x}{1+x}\right) - \tilde{p}_n\left(\frac{2\pi}{1+x}\right)}{\tilde{p}_n\left(\frac{2\pi x}{1+x}\right) + \tilde{p}_n\left(\frac{2\pi}{1+x}\right)} \frac{dx}{x^2-1}$$

$$\geqq \frac{4}{\pi}\int_{1+\epsilon}^\infty \frac{1 - \dfrac{\tilde{p}_n\left(\frac{2\pi}{1+x}\right)}{\tilde{p}_n\left(\frac{2\pi x}{1+x}\right)}}{1 + \dfrac{\tilde{p}_n\left(\frac{2\pi}{1+x}\right)}{\tilde{p}_n\left(\frac{2\pi x}{1+x}\right)}} \frac{dx}{x^2-1}$$

for $\epsilon > 0$ arbitrarily small. Now, if $x \geqq 1 + \epsilon$,

$$\left|\frac{\tilde{p}_n\left(\frac{2\pi}{1+x}\right)}{\tilde{p}_n\left(\frac{2\pi x}{1+x}\right)}\right|^2 = \prod_{\mu=0}^{n-1}\left|\frac{\dfrac{4\pi^2}{(1+x)^2} + t_\mu^2}{\dfrac{4\pi^2 x^2}{(1+x)^2} + t_\mu^2}\right|$$

$$\leqq \exp\left[-(x^2-1)\cdot\frac{n}{x^2 + \dfrac{d^2(1+x)^2}{4\pi^2}}\right]$$

$$\leqq \exp\left[-(x^2-1)\frac{n}{x^2 + \dfrac{d^2(1+x^2)}{2\pi^2}}\right].$$

Then

$$\lim_{n\to\infty}\left|\frac{\tilde{p}_n\left(\frac{2\pi}{1+x}\right)}{\tilde{p}_n\left(\frac{2\pi x}{1+x}\right)}\right| = 0,$$

uniformly in $1 + \epsilon \leqq x < \infty$.
Thus

$$\varliminf_{n\to\infty} I_n \geqq \frac{4}{\pi}\int_{1+\epsilon}^\infty \frac{dx}{x^2-1} = \frac{2}{\pi}\ln\left(\frac{2+\epsilon}{\epsilon}\right) \geqq -\frac{2}{\pi}\ln\epsilon.$$

But ϵ was arbitrarily small. Thus

$$\lim_{n\to\infty} \mathrm{I}_n = \infty.$$

Remark 1. It is interesting to mention that if $f(x)$ is entire of exponential type less than π, the cardinal $\mathscr{L}$-spline interpolation converges to the function [**3**]. Our result shows that this is not so in general.

Remark 2. Using the procedure of [**2**] it is possible to prove that under the same assumptions as Theorem 5, the sequence of Lebesgue constants for data in l_1 is also not bounded, but if the data is in l_p $(1 < p < \infty)$ and we use the L_p norm for the $\mathscr{L}$-spline, we get a bounded sequence.

References

1. S. Karlin, *Total positivity, Vol. 1*, Stanford, California, 1968.

2. M. J. Marsden, F. B. Richards and S. D. Riemenschneider, *Cardinal spline interpolation operators on l^p data*, Indiana Journal of Mathematics *24* (1975), 677–690.

3. C. A. Micchelli, *Cardinal $\mathscr{L}$-splines*, in Studies in splines and approximation theory (Academic Press, to appear).

4. F. B. Richards, *Best bounds for the uniform periodic spline interpolation operator*, J. Approx. Theory *7* (1973), 302–17.

5. ——— *The Lebesgue constants for cardinal spline interpolation*, J. Approx. Theory *19* (1975), 83–92.

6. I. J. Schoenberg, *Cardinal spline interpolation*, BMS Regional Conf. Monograph No. 12 (S.I.A.M., Philadelphia, 1973).

7. ——— *On Charles Micchelli's theory of cardinal $\mathscr{L}$-splines*, in Studies in splines and approximation theory, (Academic Press, to appear).

8. A. Sharma and J. Tzimbalario, *Cardinal t-perfect splines*, S.I.A.M. Journal of Numerical Analysis, to appear.

University of Alberta,
Edmonton, Alberta