

ERROR ESTIMATES FOR THE SPECTRAL GALERKIN APPROXIMATIONS OF THE SOLUTIONS OF NAVIER-STOKES TYPE EQUATIONS

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Introduction. In [8], [9] R. Rautmann has given a systematic development of error estimates for the spectral Galerkin approximations of the solution of the Navier-Stokes equations (spectral in the sense that one chooses as basis functions the eigenfunctions of the Stokes operator).

Error estimates are presented locally in time, valid on a finite interval determined by certain norms of the data. If one assumes the solution to be approximated is uniformly regular for $t \geq 0$, the method gives an error estimate which grows exponentially with time. Without further assumptions this is the best estimate that one can expect. However, as pointed out in [2] by J. G. Heywood, if one assumes, additionally, that the solution to be approximated is stable, then one obtains an error estimate which is uniform in time.

Notice that the existence theory for the equations of viscous incompressible non-homogeneous fluids can be developed by Galerkin type approximation ([3], [5], [6]). Then one expects that some results in [2], [8], [9] hold for these equations. The paper deals with this problem.

In §1 we give preliminaries. In §2 we consider error estimates in the L^2 -norm and in the Dirichlet norm, locally in time. In §3 we consider a stability condition in the L^2 -norm and obtain the best rate of convergence, uniform in time, for the spectral Galerkin approximations of the Navier-Stokes solution. We note that the problem of the best estimate was raised in [8 page 438]. In §4 we consider error estimates, uniform in time, for the solution of the equations of viscous incompressible non-homogeneous fluids.

1. Preliminaries. Let $\Omega \subset R^3$ be a bounded domain with smooth boundary Γ (uniformly of class C^3 at least). The functions in this paper are either R or R^3 -valued and we will not distinguish them in our notations. The $L^2(\Omega)$ -product and norm are denoted by (\cdot, \cdot) and $\|\cdot\|$ respectively, the $L^p(\Omega)$ -norm by $\|\cdot\|_p$ and $H^m(\Omega)$ product and norm are denoted by $((\cdot, \cdot))_m$, $\|\cdot\|_m$. We set $((\cdot, \cdot))_1 = (\cdot, \cdot)$, $\|\cdot\|_1 = \|\cdot\|$. We shall consider the following spaces of divergence free functions;

$$D(\Omega) = \{\varphi : \varphi \in C_0^\infty(\Omega), \nabla \cdot \varphi = 0\};$$

$$H = \text{completion of } D(\Omega) \text{ in } L^2(\Omega);$$

$$V = \text{completion of } D(\Omega) \text{ in } H^1(\Omega).$$

Throughout the paper P denotes the orthogonal projection from $L^2(\Omega)$ onto H and $\tilde{\Delta} = -P\Delta$ (the Stokes operator). We shall denote by $\omega^k(x)$ and λ_k the eigenfunctions and the eigenvalues of the Stokes operator defined in $V \cap H^2(\Omega)$ ($H^m(\Omega)$ are defined as usual). It is well known (see [2], [8]) that $\omega^k(x)$ are orthogonal in the inner product (\cdot, \cdot) ,

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$((,)), (\bar{\Delta}, \bar{\Delta})$ and complete in the spaces $H, V, H^2(\Omega) \cap V$. Furthermore if $u = \sum_{k=1}^{\infty} a_k \omega^k(x) \in V \cap H^2(\Omega)$ then (see [2, Lemma 4])

$$\begin{aligned} \left| \sum_{k=n}^{\infty} a_k \omega^k(x) \right|^2 &\leq \lambda_n^{-1} \left| \nabla \sum_{k=n}^{\infty} a_k \omega^k(x) \right|^2 \\ &\leq \lambda_n^{-2} \left| \bar{\Delta} \sum_{k=n}^{\infty} a_k \omega^k(x) \right|^2. \end{aligned}$$

Now we consider the equations which govern the flow of a viscous incompressible non-homogeneous fluid (non-homogeneous in the sense that the density ρ is not constant):

$$\left. \begin{aligned} \rho \partial_t \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} &= \nabla p + \rho \mathbf{f} \\ \partial_t \rho + \mathbf{u} \cdot \nabla \rho &= 0 \quad \text{in } \Omega \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \right\} \tag{1.1}$$

Here $\partial_t = \partial/\partial t, \mathbf{u} = \mathbf{u}(t) = \mathbf{u}(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))$ is the velocity, $\rho = \rho(t) = \rho(t, x)$ the density, $p = p(t) = p(t, x)$ the pressure, $\mathbf{f} = \mathbf{f}(t) = \mathbf{f}(t, x) = (f_1(t, x), f_2(t, x), f_3(t, x))$ the external force, and $\mathbf{u} \cdot \nabla \mathbf{u} = \sum_{i=1}^3 u_i \partial_{x_i} \mathbf{u}, \nabla \cdot \mathbf{u} = \sum_{i=1}^3 \partial_{x_i} u_i$. We assume the viscosity $\mu = 1$. We consider the initial boundary conditions

$$\left. \begin{aligned} \mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma \\ \mathbf{u}(0) &= \mathbf{u}_0; \quad \rho(0) = \rho_0; \quad 0 < \alpha \leq \rho_0 \leq \beta \end{aligned} \right\} \tag{1.2}$$

with α, β positive constants.

Solutions of problem (1.1), (1.2) can be obtained using a semi-Galerkin approximation i.e. a Galerkin approximation $\mathbf{u}^n = \sum_{i=1}^n c_i^n(t) \omega^i(x)$ on the velocity and an infinite dimensional approximation ρ^n on the density, solution of the continuity equation

$$\partial_t \rho^n + \mathbf{u}^n \cdot \nabla \rho^n = 0$$

with $\rho^n(0) = \rho_0$.

For the n unknown coefficients

$$c_i^n(t) = \int_{\Omega} \mathbf{u}(t, x) \omega^i(x) dx$$

we have the system of n ordinary differential equations and initial conditions

$$\left. \begin{aligned} P_n \rho^n \partial_t \mathbf{u}^n - P \Delta \mathbf{u}^n + P_n \rho^n \mathbf{u}^n \cdot \nabla \mathbf{u}^n &= P_n \rho^n \mathbf{f} \\ P_n \mathbf{u}(0) &= P_n \mathbf{u}_0 \end{aligned} \right\} \tag{1.3}$$

($P_n =$ projection operator of $L^2(\Omega)$ onto the space spanned by $(\omega^1(x), \omega^2(x), \dots, \omega^n(x))$).

Error estimates result from a differential inequality for the respective norm of the difference of two Galerkin approximations and a subsequent limiting process (see [8]) or of the difference of Galerkin approximation and the partial sum of the series for \mathbf{u} (see [2]); therefore we need uniform regularity of the approximations \mathbf{u}^n . For this we consider the following theorem (see [10]); we assume $\mathbf{f} = \mathbf{0}$ for simplicity.

THEOREM 1. *Assume $\mathbf{u}_0 \in V \cap H^2(\Omega)$, $\rho_0 \in C^1(\Omega)$. Then there exists T such that there exists a unique strong solution \mathbf{u} of problem (1.1), (1.2), i.e. a function \mathbf{u} such that*

$$\left. \begin{aligned} \mathbf{u} \in L^2(0, T; V \cap H^2(\Omega)); \quad \rho \in C^1(\Omega \times [0, T]) \\ \partial_t \mathbf{u} \in L^2((0, T), V) \end{aligned} \right\} \tag{1.4}$$

and

$$P(\rho \partial_t \mathbf{u} - \Delta \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u}) = \mathbf{0}$$

holds a.e. in $\Omega \times [0, T]$.

Moreover the approximations \mathbf{u}^n, ρ^n satisfy the estimates

$$\left. \begin{aligned} |\nabla \mathbf{u}^n|^2 + \int_0^t |\partial_\tau \mathbf{u}^n|^2 d\tau \leq F(t); \\ \int_0^t |\bar{\Delta} \mathbf{u}^n|^2 d\tau \leq h(t); \quad |\bar{\Delta} \mathbf{u}^n(t)|^2 \leq g(t); \\ \int_0^t \|\nabla \mathbf{u}^n\|_{C(\Omega)} d\tau \leq H(t); \\ |\partial_t \mathbf{u}^n(t)|^2 + \int_0^t |\nabla \partial_\tau \mathbf{u}^n|^2 d\tau \leq G(t); \\ \alpha \leq \rho^n \leq \beta; \quad \|\nabla \rho^n\|_{L^\infty(\Omega)} \leq c \|\nabla \rho_0\|_{L^\infty(\Omega)} \exp H(t). \end{aligned} \right\} \tag{1.5}$$

The functions on the right hand sides depend on their argument t , and in addition on $T, \alpha > 0, \Gamma$ and the norm $\|\mathbf{u}_0\|_2$. On the interval in question these functions are continuous in the variable t , the functions $F(t), G(t), h(t)$ being continuously differentiable with respect to t .

We notice O. A. Ladyzhenskaya and V. A. Solonnikov proved in [5] the regularity of the solution of problem (1.1), (1.2) in L^p -theory by linearization and potential theory. Instead below we need estimates as in (1.5).

Theorem 1 holds with $\alpha > 0$. The existence of a weak solution of problem (1.1), (1.2) was proved by J. Simon in [12] with $\alpha = 0$. With a different method, the author proved in [11] the existence of a weak solution for the equations of non-homogeneous fluids in presence of diffusion with $\alpha = 0$. For other results on this problem see [4], [7].

2. Error estimates for the approximations \mathbf{u}^n, ρ^n . *Let $[0, T]$ be an interval as in Theorem 1, \mathbf{u}^n, ρ^n the n -th approximations of \mathbf{u} and ρ .*

THEOREM 2. *Suppose the assumptions of Theorem 1 hold. Then the approximations \mathbf{u}^n, ρ^n satisfy*

- (i) $|\rho(t) - \rho^n(t)|^2 \leq c\lambda_{n+1}^{-1}\bar{F}(t),$
- (ii) $|\mathbf{u}(t) - \mathbf{u}^n(t)|^2 + \int_0^t |\nabla(\mathbf{u}(\tau) - \mathbf{u}^n(\tau))|^2 d\tau \leq c\lambda_{n+1}^{-1}\bar{F}(t)$

for any $t \in [0, T]$. The continuous functions $\bar{F}(t), \tilde{F}(t)$ depend on t and on the functions $F(t), g(t), G(t)$ and $H(t)$ in (1.5).

Proof. First we suppose (ii) true. The difference $\rho^l - \rho^n$ with $l > n$ satisfies

$$\partial_t(\rho^l - \rho^n) + \mathbf{u}^l \cdot \nabla(\rho^l - \rho^n) = -(\mathbf{u}^l - \mathbf{u}^n) \cdot \nabla\rho^n$$

and

$$\rho_0^l - \rho_0^n = 0.$$

Let $\mathbf{y}^l(t, \tau, \mathbf{x})$ be the solution of the Cauchy problem

$$\begin{aligned} \partial_\tau \mathbf{y}^l &= \mathbf{u}^l(\mathbf{y}^l, \tau) \\ \mathbf{y}^l &= \mathbf{x} \quad \text{for } t = \tau. \end{aligned}$$

(the hydrodynamic meaning of the above system is well known). Then the following relation holds (see [5, p. 707]):

$$\rho^l(\mathbf{x}, t) - \rho^n(\mathbf{x}, t) = - \int_0^t \mathbf{f}_{l,n}(\mathbf{y}^l(\tau, t, \mathbf{x}), \tau) d\tau \tag{2.1}$$

where $\mathbf{f}_{l,n}(\mathbf{y}^l, t) = (\mathbf{u}^l(\mathbf{y}^l, t) - \mathbf{u}^n(\mathbf{y}^l, t)) \cdot \nabla\rho^n(\mathbf{y}^l, t)$. Bearing in mind (ii), the properties of \mathbf{y}^l (see [1, pp. 93–96]) and the last estimate in (1.5) we have

$$\begin{aligned} |\rho^l - \rho^n| &\leq c \int_0^t |\mathbf{u}^l - \mathbf{u}^n| \|\nabla\rho^n\|_{L^\infty(\Omega)} d\sigma \\ &\leq c\lambda_{n+1}^{-1/2} \|\nabla\rho_0\|_{L^\infty(\Omega)} (t \exp H(t) (\sup_{0 \leq s < t} \tilde{F}(s))) \leq c\bar{F}(t)\lambda_{n+1}^{-1/2}. \end{aligned}$$

Consequently we obtain (i).

Now we prove (ii). We consider the following equations ($l > n$):

$$P_l \rho^l \partial_t \mathbf{u}^l + P_l \rho^l \mathbf{u}^l \cdot \nabla \mathbf{u}^l - P \Delta \mathbf{u}^l = \mathbf{0}; \tag{2.2}$$

$$P_n \rho^n \partial_t \mathbf{u}^n + P_n \rho^n \mathbf{u}^n \cdot \nabla \mathbf{u}^n - P \Delta \mathbf{u}^n = \mathbf{0}. \tag{2.3}$$

(ρ^l, ρ^n satisfy the continuity equation with $\mathbf{u}^l, \mathbf{u}^n$ respectively). Subtracting (2.3) from (2.2), the difference $\omega = \mathbf{u}^l - \mathbf{u}^n$ satisfies

$$\begin{aligned} P_l \rho^l \partial_t \mathbf{u}^l - P_n \rho^n \mathbf{u}^n \cdot \nabla \mathbf{u}^n + P_l \rho^l \mathbf{u}^l \cdot \nabla \mathbf{u}^l - P_n \rho^n \partial_t \mathbf{u}^n - P \Delta \omega &= \mathbf{0}, \\ \omega(0) &= (P_l - P_n)\mathbf{u}_0. \end{aligned} \tag{2.4}$$

We take the inner product in $L^2(\Omega)$ of (2.4) with ω and after some calculations we obtain (writing $d_t = d/dt$)

$$\begin{aligned} \frac{1}{2} d_t |\rho^l \omega|^2 + |\nabla \cdot \omega|^2 + (P_n(\rho^l - \rho^n) \mathbf{u}^n \cdot \nabla \mathbf{u}^n, \omega) + (P_n \rho^l \omega \cdot \nabla \mathbf{u}^n, \omega) \\ = -P_n(\rho^l - \rho^n) \partial_t \mathbf{u}^n, \omega - ((P_l - P_n)(\rho^l \partial_t \mathbf{u}^n + \rho^l \mathbf{u}^l \cdot \nabla \mathbf{u}^n), \omega). \end{aligned} \tag{2.5}$$

By virtue of (1.5), we have the following estimates:

$$\begin{aligned} |((P_l - P_n)(\rho^l \partial_t \mathbf{u}^n + \rho^l \mathbf{u}^l \cdot \nabla \mathbf{u}^n), \omega)| \leq c((g(t)G(t))^{1/2} + g(t)(F(t)^{1/2})\lambda_{n+1}^{-1}; \\ |(P_n \rho^l \omega \cdot \nabla \mathbf{u}^n, \omega)| \leq \delta |\nabla \omega|^2 + c_\delta |\bar{\Delta} \mathbf{u}^n|^2 |\omega|^2. \end{aligned} \tag{2.6}$$

Moreover, bearing in mind (2.1),

$$\begin{aligned} |(P_n(\rho^l - \rho^n) \partial_t \mathbf{u}^n, \omega)| = \left| \left(\int_0^t ((\omega \cdot \nabla \rho^n) d\tau) \partial_t \mathbf{u}^n, \omega \right) \right| \\ \leq c_\delta \left(t \int_0^t |\omega|^2 \|\nabla \rho^n\|_{L^\infty(\Omega)}^2 d\tau \right) |\nabla \partial_t \mathbf{u}^n|^2 + \delta |\nabla \omega|^2. \end{aligned} \tag{2.7}$$

Analogously, (2.1), (1.5), imply

$$|(P_n(\rho^l - \rho^n) \mathbf{u}^n \cdot \nabla \mathbf{u}^n, \omega)| \leq c_\delta \left(t \int_0^t |\omega|^2 \|\nabla \rho^n\|_{L^\infty(\Omega)}^2 d\tau \right) ((F(t))^{1/2}(g(t))^{3/2} + (g(t))^2 + \delta |\nabla \omega|^2) \tag{2.8}$$

(the above constants c, c_δ depend on the boundedness of ρ_0). Bearing in mind (2.6), (2.7), (2.8), and assuming $\delta \in (0, 1/4)$, the differential equality (2.5) yields the integral inequality

$$|\omega|^2 + \int_0^t |\nabla \omega|^2 d\tau \leq a(t)\lambda_{n+1}^{-1} + \int_0^t b(\tau) |\omega|^2 d\tau + \int_0^t f(\tau) d\tau \int_0^t |\omega|^2 d\tau$$

where the functions $a(t), b(t),$ and $f(t)$ depend on $g(t), F(t), G(t), H(t), |\nabla \partial_t \mathbf{u}^n|,$ and $\alpha > 0$. This inequality can be written in the following form:

$$|\omega|^2 + \int_0^t |\nabla \omega|^2 d\tau \leq a(t)\lambda_{n+1}^{-1} + \int_0^t \bar{b}(\tau) |\omega|^2 d\tau.$$

Now applying a variant of Gronwall's lemma proved in [8], we get

$$|\mathbf{u}^l - \mathbf{u}^n|^2 + \int_0^t |\nabla(\mathbf{u}^l - \mathbf{u}^n)|^2 d\tau \leq c\bar{F}(t)\lambda_{n+1}^{-1}.$$

Now passing to the limit $l \rightarrow \infty$ on the left side we obtain (ii). Thus the theorem is proved.

THEOREM 3. *Under the assumptions of Theorem 1, the approximations \mathbf{u}^n satisfy*

$$|\nabla(\mathbf{u} - \mathbf{u}^n)|^2 + \int_0^t |\partial_\tau(\mathbf{u} - \mathbf{u}^n)|^2 d\tau \leq c\lambda_{n+1}^{-1}K(t) \tag{2.9}$$

for any $t \in (0, T)$. The continuous function $K(t)$ of the variable t depends on the functions in (1.5).

Proof. We take the inner product of (2.4) with $\partial_t \omega$ in $L^2(\Omega)$ to get

$$\begin{aligned} & (P_l \rho^l \partial_t \omega, \partial_t \omega) + (P_l \rho^l \mathbf{u}^l \cdot \nabla \omega, \partial_t \omega) + (P_n \rho^l \omega \cdot \nabla \mathbf{u}^n, \partial_t \omega) + \frac{1}{2} d_t |\nabla \cdot \omega|^2 \\ &= -((P_l - P_n) \rho^l \partial_t \mathbf{u}^n, \partial_t \omega) - (P_n (\rho^l - \rho^n) \partial_t \mathbf{u}^n, \partial_t \omega) - ((P_l - P_n) \rho^l \mathbf{u}^l \cdot \nabla \mathbf{u}^n, \partial_t \omega) \\ &\quad - (P_n (\rho^l - \rho^n) \mathbf{u}^n \cdot \nabla \mathbf{u}^n, \partial_t \omega); \end{aligned} \tag{2.10}$$

$$\omega(0) = (P_l - P_n) \mathbf{u}_0.$$

From the estimates (1.5), and (i) of Theorem 2, we get

$$\begin{aligned} |(P_n \rho^l \omega \cdot \nabla \mathbf{u}^n, \partial_t \omega)| &\leq \delta |\partial_t \omega|^2 + c_\delta |g(t)| |\nabla \omega|^2; \\ |P_n (\rho^l - \rho^n) (\partial_t \mathbf{u}^n + \mathbf{u}^n \cdot \nabla \mathbf{u}^n, \partial_t \omega)| &\leq c \lambda_{n+1}^{-1/2} \bar{F}(t) (\|\partial_t \mathbf{u}^l\|^2 + \|\partial_t \mathbf{u}^n\|^2 + g^2(t)); \\ |(P_l - P_n) \rho^l \mathbf{u}^l \cdot \nabla \mathbf{u}^n, \partial_t \omega)| &\leq c \lambda_{n+1}^{-1/2} g(t) (\|\partial_t \mathbf{u}^l\| + \|\partial_t \mathbf{u}^n\|); \\ |(P_l - P_n) \rho^l \mathbf{u}^l \cdot \nabla \omega, \partial_t \omega)| &\leq \delta |\partial_t \omega|^2 + c_\delta |\Delta \mathbf{u}^l|^2 |\nabla \omega|^2 \end{aligned}$$

(the above constants c, c_δ depend on the boundedness of ρ_0). If $\delta \in (0, \alpha/4)$ (consequently c_δ depends on $\alpha > 0$) these inequalities and (2.10) lead to the integral inequality

$$|\nabla \cdot \omega|^2 + \int_0^t |\partial_\tau \omega|^2 d\tau \leq a(t) \lambda_{n+1}^{-1/2} + \int_0^t b(\tau) |\nabla \cdot \omega|^2 d\tau.$$

Applying the variant of Gronwall’s lemma proved in [8] and passing to the limit $l \rightarrow \infty$ we obtain (2.9).

3. $L^2(\Omega)$ -error estimates uniform in time for the spectral Galerkin approximations of the Navier–Stokes solution. Let u, p be the solution of the Navier–Stokes problem

$$\left. \begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} &= -\nabla p, \\ \nabla \cdot \mathbf{u} &= 0, \\ \mathbf{u}(0) &= \mathbf{u}_0; \quad \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma. \end{aligned} \right\} \tag{3.1}$$

ASSUMPTION 1. $\mathbf{u}_0 \in V \cap H^2(\Omega)$, and the solution \mathbf{u} is strong in the sense that

$$\mathbf{u} \in L^\infty(0, \infty; V \cap H^2(\Omega)); \quad \partial_t \mathbf{u} \in L^\infty(0, \infty; H) \tag{3.2}$$

and

$$P(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u}) = \mathbf{0}$$

is satisfied a.e. in Ω .

We assume \mathbf{u} is conditionally stable in the following sense (see [2]). First, a function ζ defined on some interval $t \geq t_0$ is called a perturbation of \mathbf{u} if $\zeta(t, x) + \mathbf{u}$ is a solution of the Navier–Stokes equations with $\zeta = \mathbf{0}$ on Γ . Thus, setting $\zeta_0 = \zeta(t_0, x)$, $\zeta(t)$ is a solution

of the initial-boundary value problem

$$\left. \begin{aligned} \partial_t \zeta - \Delta \zeta + \zeta \cdot \nabla \mathbf{u} + \zeta \cdot \nabla \zeta + \nabla q + \mathbf{u} \cdot \nabla \zeta &= \mathbf{0} \\ \nabla \cdot \zeta &= 0 \\ \zeta(t_0) = \zeta_0; \quad \zeta &= \mathbf{0} \quad \text{on } \Gamma. \end{aligned} \right\} \tag{3.3}$$

The assumption, then, is as follows.

ASSUMPTION 2. *There exist positive numbers A, C, ε such that for every $t_0 \geq 0$ and $\zeta_0 \in V \cap H^2(\Omega)$ with $|\zeta_0| \leq \varepsilon$ the perturbation problem (3.3) is uniquely solvable and its solution is strong in the sense of assumption 1 and satisfies*

$$|\zeta(t)| \leq C |\zeta_0| \exp(-A(t - t_0)). \tag{3.4}$$

We observe the n -th spectral Galerkin approximation

$$\mathbf{u}^n = \sum_{i=1}^n c_i^n(t) \omega^i(x) \tag{3.5}$$

to the solution of the problem (3.1) is uniquely determined by the equations

$$\begin{aligned} (\partial_t \mathbf{u}^n, \varphi^n) + (\nabla \mathbf{u}^n, \nabla \varphi^n) + (\mathbf{u}^n \cdot \nabla \mathbf{u}^n, \varphi^n) &= \mathbf{0}; \\ (\mathbf{u}^n(0) - \mathbf{u}_0, \varphi^n) &= 0 \end{aligned} \tag{3.6}$$

for all φ^n of the form $\varphi^n = \sum_{i=1}^n d_i \omega^i(x)$. We shall prove the following theorem.

THEOREM 4. *Under the assumptions 1, 2, there exist constants N and K depending on the domain Ω , the norms of the data and the constants in (3.4), such that*

$$|\mathbf{u} - \mathbf{u}^n|^2 \leq K \lambda_{n+1}^{-2} \tag{3.7}$$

for all $t \geq 0$ and $n > N$.

Proof. Let $\mathbf{u} = \sum_{i=1}^{\infty} g_i(t) \omega^i(x)$ be the eigenfunction expansion of the solution of the problem (3.1). Let $\mathbf{v}^n = \sum_{i=1}^n g_i(t) \omega^i(x)$ be the n -th partial sum of the series for \mathbf{u} . Let $\mathbf{e}^n = \mathbf{u} - \mathbf{v}^n$ and $\psi^n = \mathbf{u}^n - \mathbf{v}^n$ where \mathbf{u}^n is the n -th Galerkin approximation of \mathbf{u} . Then $\mathbf{u} - \mathbf{u}^n = \mathbf{e}^n - \psi^n$. Now ψ^n satisfies the equation (see [2, p. 337])

$$\begin{aligned} (\partial_t \psi^n, \varphi^n) + (\nabla \psi^n, \nabla \varphi^n) &= -(\mathbf{u} \cdot \nabla \psi^n, \varphi^n) - (\psi^n \cdot \nabla \mathbf{u}, \varphi^n) - (\psi^n \cdot \nabla \psi^n, \varphi^n) \\ &+ (\psi^n \cdot \nabla \mathbf{e}^n, \varphi^n) + (\mathbf{e}^n \cdot \nabla \mathbf{v}^n, \varphi^n) + (\mathbf{e}^n \cdot \nabla \psi^n, \varphi^n) + (\mathbf{u} \cdot \nabla \mathbf{e}^n, \varphi^n). \end{aligned} \tag{3.8}$$

Let P_n and Q_n be the orthogonal projection of $L^2(\Omega)$ onto span of $(\omega^1(x), \dots, \omega^n(x))$ and of $(\omega^{n+1}(x), \omega^{n+2}(x), \dots)$ respectively. For $\varphi \in H$ let us write $\varphi = P_n \varphi + Q_n \varphi = \varphi^n + Q_n \varphi$. Also, for $\varphi \in V$

$$(\nabla \psi^n, \nabla \varphi^n) = (\nabla \psi^n, \nabla \varphi).$$

Using such identities, we rewrite (3.8) as

$$\begin{aligned}
 & (\partial_t \psi^n, \varphi) + (\nabla \psi^n, \nabla \varphi) + (\mathbf{u} \cdot \nabla \psi^n, \varphi) + (\psi^n \cdot \nabla \mathbf{u}, \varphi) + (\psi^n \cdot \nabla \psi^n, \varphi) \\
 &= (Q_n(\mathbf{u} \cdot \nabla \psi^n), \varphi) + (Q_n(\psi^n \cdot \nabla \mathbf{u}), \varphi) + (Q_n(\psi^n \cdot \nabla \psi^n), \varphi) + (P_n(\psi^n \cdot \nabla \mathbf{e}^n), \varphi) \\
 &\quad + (P_n(\mathbf{e}^n \cdot \nabla \psi^n), \varphi) + (P_n(\mathbf{u} \cdot \nabla \mathbf{e}^n), \varphi) + (P_n(\mathbf{e}^n \cdot \nabla \mathbf{v}^n), \varphi) \quad (3.9)
 \end{aligned}$$

which is valid for all $\varphi \in V$ and for all $t \geq 0$. Then $\theta = \psi^n - \zeta$ satisfies

$$\begin{aligned}
 & (\partial_t \theta, \varphi) + (\nabla \cdot \theta, \nabla \cdot \varphi) + (\mathbf{u} \cdot \nabla \theta, \varphi) + (\psi^n \cdot \nabla \theta, \varphi) + (\theta \cdot \nabla \zeta, \varphi) - (\theta \cdot \nabla \mathbf{u}, \varphi) \\
 &= (Q_n(\mathbf{u} \cdot \nabla \psi^n), \varphi) + (Q_n(\psi^n \cdot \nabla \mathbf{u}), \varphi) + (Q_n(\psi^n \cdot \nabla \psi^n), \varphi) + (P_n(\psi^n \cdot \nabla \mathbf{e}^n), \varphi) \\
 &\quad + (P_n(\mathbf{e}^n \cdot \nabla \psi^n), \varphi) + (P_n(\mathbf{u} \cdot \nabla \mathbf{e}^n), \varphi) + (P_n(\mathbf{e}^n \cdot \nabla \mathbf{v}^n), \varphi). \quad (3.10)
 \end{aligned}$$

Setting $\varphi = \theta$ in (3.10) we get

$$\begin{aligned}
 \frac{1}{2} d_t |\theta|^2 + |\nabla \theta|^2 &= -(\theta \cdot \nabla \mathbf{u}, \theta) - (\theta \cdot \nabla \zeta, \theta) + (Q_n(\mathbf{u} \cdot \nabla \psi^n), \theta) + (Q_n(\psi^n \cdot \nabla \psi^n), \theta) \\
 &\quad + (Q_n(\psi^n \cdot \nabla \mathbf{u}), \theta) + (Q_n(\psi^n \cdot \nabla \psi^n), \theta) + (P_n(\psi^n \cdot \nabla \mathbf{e}^n), \theta) \\
 &\quad + (P_n(\mathbf{e}^n \cdot \nabla \psi^n), \theta) + (P_n(\mathbf{e}^n \cdot \nabla \mathbf{v}^n), \theta) + (P_n(\mathbf{u} \cdot \nabla \mathbf{e}^n), \theta). \quad (3.11)
 \end{aligned}$$

Furthermore we have

$$\begin{aligned}
 |(\theta \cdot \nabla \mathbf{u}, \theta)| &\leq c_\delta |\tilde{\Delta} \mathbf{u}|^2 |\theta|^2 + \delta |\nabla \theta|^2; \\
 |(\theta \cdot \nabla \zeta, \theta)| &\leq c_\delta |\tilde{\Delta} \zeta|^2 |\theta|^2 + \delta |\nabla \theta|^2; \\
 |(Q_n(\psi^n \cdot \nabla \psi^n), \theta)| &\leq c_\delta |\psi^n|^2 |\tilde{\Delta} \psi^n|^2 \lambda_{n+1}^{-1/2} + \delta |\nabla \theta|^2; \\
 |(Q_n(\psi^n \cdot \nabla \mathbf{u}), \theta)| &\leq c_\delta |\psi^n|^2 |\tilde{\Delta} \mathbf{u}|^2 \lambda_{n+1}^{-1/2} + \delta |\nabla \theta|^2; \\
 |(Q_n(\mathbf{u} \cdot \nabla \psi^n), \theta)| &\leq c_\delta |\tilde{\Delta} \mathbf{u}|^2 |\nabla \psi^n|^2 \lambda_{n+1}^{-1} + \delta |\nabla \theta|^2; \\
 |(P_n(\psi^n \cdot \nabla \mathbf{e}^n), \theta)| &\leq c_\delta |\tilde{\Delta} \psi^n| |\tilde{\Delta} \mathbf{u}|^2 \lambda_{n+1}^{-2} + \delta |\nabla \theta|^2; \\
 |(P_n(\mathbf{e}^n \cdot \nabla \psi^n), \theta)| &\leq c_\delta |\tilde{\Delta} \mathbf{u}|^2 |\tilde{\Delta} \psi^n|^2 \lambda_{n+1}^{-2} + \delta |\nabla \theta|^2; \\
 |(P_n(\mathbf{e}^n \cdot \nabla \mathbf{v}^n), \theta)| &\leq c_\delta |\tilde{\Delta} \mathbf{u}|^4 \lambda_{n+1}^{-2} + \delta |\nabla \theta|^2; \\
 |(P_n(\mathbf{u} \cdot \nabla \mathbf{e}^n), \theta)| &\leq c_\delta |\tilde{\Delta} \mathbf{u}|^4 \lambda_{n+1}^{-2} + \delta |\nabla \theta|^2.
 \end{aligned} \quad (3.12)$$

(In (3.12) c_δ denotes different constants independent of n .) If $\delta \in (0, 1/10)$ these inequalities and (3.11) lead to the integral inequality

$$\begin{aligned}
 |\theta|^2 &\leq \exp \int_{t_0}^t c(|\tilde{\Delta} \mathbf{u}|^2 + |\tilde{\Delta} \zeta|^2) d\tau \left(|\theta(t_0)|^2 + c \int_{t_0}^t (\lambda_{n+1}^{-2} |\tilde{\Delta} \mathbf{u}|^2 (|\tilde{\Delta} \mathbf{u}|^2 + |\tilde{\Delta} \psi^n|^2) \right. \\
 &\quad \left. + \lambda_{n+1}^{-1} |\nabla \psi^n|^2 |\tilde{\Delta} \mathbf{u}^n|^2) d\tau \right. \\
 &\quad \left. + \int_{t_0}^t \lambda_{n+1}^{-1/2} |\psi^n|^2 (|\tilde{\Delta} \psi^n|^2 + |\tilde{\Delta} \mathbf{u}|^2) d\tau \right). \quad (3.13)
 \end{aligned}$$

Now we need to estimate $|\nabla \psi^n|, |\Delta \psi^n|$. For this we shall prove the following lemma.

LEMMA 1. Suppose $|\psi^n|^2 \leq c_1 \lambda_{n+1}^{-1}$ holds on some interval $(0, t^*)$. Then there hold on $(0, t^*)$ also $|\nabla \psi^n|^2 \leq c_2 \lambda_{n+1}^{-1}$, $|\bar{\Delta} \psi^n| \leq c_3 \lambda_{n+1}^{-1}$ provided that $n \geq N$ (where c_1, c_2, c_3, N depend on $\Omega, |\bar{\Delta} \mathbf{u}|$).

Proof. Setting $\varphi^n = \psi^n$ in (3.8) we obtain

$$\frac{1}{2} d_t |\psi^n|^2 + |\nabla \cdot \psi^n|^2 \leq c |\bar{\Delta} \mathbf{u}| |\psi^n| |\nabla \psi^n| + c \lambda_{n+1}^{-1} |\bar{\Delta} \mathbf{u}|^2 |\nabla \psi^n|$$

whence

$$d_t |\psi^n|^2 + |\nabla \psi^n|^2 \leq c (|\bar{\Delta} \mathbf{u}|^2 + |\bar{\Delta} \mathbf{u}|^4) \lambda_{n+1}^{-1}. \tag{3.14}$$

Multiplying (3.14) by e^t and integrating over $Q(t) = \Omega \times [0, t]$ one has

$$e^{-t} \int_0^t e^\tau |\nabla \psi^n|^2 d\tau \leq c \lambda_{n+1}^{-1}. \tag{3.15}$$

Setting $\varphi^n = \bar{\Delta} \psi^n$ in (3.8) we obtain

$$\frac{1}{2} d_t |\nabla \psi^n|^2 + |\bar{\Delta} \psi^n|^2 \leq c |\bar{\Delta} \mathbf{u}| |\nabla \psi^n| |\bar{\Delta} \psi^n| + c |\nabla \psi^n|^{3/2} |\bar{\Delta} \psi^n|^{3/2} + c |\bar{\Delta} \mathbf{u}|^2 |\bar{\Delta} \psi^n| \lambda_{n+1}^{-1/2}$$

whence

$$d_t |\nabla \psi^n|^2 + |\bar{\Delta} \psi^n|^2 \leq c |\bar{\Delta} \mathbf{u}|^2 |\nabla \psi^n|^2 + c |\nabla \psi^n|^6 + c |\bar{\Delta} \mathbf{u}|^4 \lambda_{n+1}^{-1}. \tag{3.16}$$

Multiplying (3.16) by e^t and integrating over $Q(t)$ one has

$$|\nabla \psi^n(t)|^2 \leq c \lambda_{n+1}^{-1} + c e^{-t} \int_0^t e^\tau (|\nabla \psi^n|^6 + c \lambda_{n+1}^{-1}) d\tau. \tag{3.17}$$

Now there exists N such that for $n > N, t < t^*$

$$|\nabla \psi^n|^2 < \lambda_{n+1}^{-1/2}. \tag{3.18}$$

If not, that is, if (3.18) fails for some $n > N$, let \bar{t} be the first value of t for which $|\nabla \psi^n(\bar{t})|^2 = \lambda_{n+1}^{-1/2}$. From (3.15), (3.17) we have

$$|\nabla \psi^n(t)|^2 \leq c \lambda_{n+1}^{-1} + c \lambda_{n+1}^{-1/2} e^{-t} \int_0^t e^\tau |\nabla \psi^n|^2 d\tau + c \lambda_{n+1}^{-1} < c \lambda_{n+1}^{-1}.$$

If we choose N such that for $n > N, \lambda_{n+1}^{-1/2} < \frac{1}{2}c$ (c depends on $\Omega, |\bar{\Delta} \mathbf{u}|$ only) we have

$$|\nabla \psi^n(\bar{t})|^2 \leq \frac{1}{2} \lambda_{n+1}^{-1/2} < \lambda_{n+1}^{-1/2}$$

contradicting our supposition on \bar{t} . So (3.18) must hold on $(0, t^*)$.

Now we reconsider (3.8) with $\varphi^n = \bar{\Delta} \psi^n$; we get

$$\frac{1}{2} d_t |\nabla \psi^n|^2 + |\bar{\Delta} \psi^n|^2 \leq c |\bar{\Delta} \mathbf{u}| |\nabla \psi^n| |\bar{\Delta} \psi^n| + c |\nabla \psi^n| |\bar{\Delta} \psi^n|^2 + c |\bar{\Delta} \mathbf{u}|^2 |\bar{\Delta} \psi^n| \lambda_{n+1}^{-1/2}. \tag{3.19}$$

If we re-choose N (if it is necessary) such that $\lambda_{n+1}^{-1/2} < \frac{1}{4}c$ then on multiplying (3.18) by e^t

and integrating over $Q(t)$ we obtain

$$|\nabla\psi^n(t)|^2 \leq e^{-t} \int_0^t e^\tau |\nabla\psi^n|^2 d\tau + c\lambda_{n+1}^{-1}.$$

From (3.15) we obtain

$$|\nabla\psi^n(t)|^2 \leq c_2\lambda_{n+1}^{-1} \tag{3.20}$$

provided that $n > N$. From (3.20) and [2, Lemma 8] we have

$$|\bar{\Delta}\psi^n|^2 < c_3\lambda_{n+1}^{-1}.$$

Now we are in the position to prove $|\psi^n(t)|^2 \leq c\lambda_{n+1}^{-2}$. We choose $\bar{N} (> N)$ and T sufficiently large such that

$$\beta e^{-\alpha T} \leq \frac{1}{2}; \quad Tbe^{BT}\lambda_{n+1}^{-1} \leq \frac{1}{8}; \quad Te^{bT}(c_3\lambda_{n+1}^{-1} + \sup |\bar{\Delta}\mathbf{u}|^2)\lambda_{n+1}^{-1/2} < \frac{1}{8} \tag{3.21}$$

where

$$B > \sup c(|\bar{\Delta}\mathbf{u}|^2 + |\bar{\Delta}\zeta|^2); \quad b > \sup(c|\bar{\Delta}\mathbf{u}|^4 + c_2|\bar{\Delta}\mathbf{u}|^2).$$

We set

$$\gamma_n = 8e^{BT}bT\lambda_{n+1}^{-2}.$$

Now we are in the situation of [2, p. 344]. We shall give the proof for the reader's convenience.

For $n > \bar{N}$ we claim

$$|\psi^n|^2 < \gamma_n \quad \text{for all } t > 0. \tag{3.22}$$

If not, that is, if (3.22) fails for some $n > \bar{N}$, let \bar{t} be the first value of t for which $|\psi^n(\bar{t})|^2 = \gamma_n$. To show it is impossible that $\bar{t} \leq T$, we consider (3.13) with $t_0 = 0$, $\zeta = \mathbf{0}$ and Lemma 1 and we obtain

$$|\psi^n(\bar{t})|^2 \leq \frac{1}{4}\gamma_n,$$

which contradicts our supposition about \bar{t} . On the other hand, if $\bar{t} > T$, then $|\psi^n(t)|^2 < \gamma_n$ in $[\bar{t} - T, \bar{t})$ and from Lemma 1, $|\nabla\psi^n(t)|^2 \leq c_2\lambda_{n+1}^{-1}$ and $|\bar{\Delta}\psi^n(t)|^2 \leq c_3\lambda_{n+1}^{-1}$ for $t \in [\bar{t} - T, \bar{t})$. So considering (3.13) with $t_0 = \bar{t} - T$ and $\zeta(\bar{t} - T) = \psi^n(\bar{t} - T)$ we find

$$|\psi^n(t) - \zeta(t)|^2 \leq \frac{1}{4}\gamma_n. \tag{3.23}$$

In view of (3.21), the stability condition implies

$$|\zeta(t)|^2 \leq \frac{1}{4}\gamma_n. \tag{3.24}$$

Together (3.23) and (3.24) imply $|\psi^n(\bar{t})|^2 < \gamma_n$ again contradicting our supposition about \bar{t} . So (3.21) must hold.

4. $L^2(\Omega)$ -Error estimates uniform in time for the approximations \mathbf{u}^n , ρ^n of the solution of the system (1.1). Let (\mathbf{u}, p, ρ) be a solution of the problem (1.1), (1.2). We make the following assumptions.

ASSUMPTION 3. $\mathbf{u}_0 \in V \cap H^2(\Omega)$; $\mathbf{f} = \mathbf{0}$ and the solution (\mathbf{u}, ρ) is unique and satisfies

$$\mathbf{u} \in L^\infty(0, \infty; H^3(\Omega) \cap V); \quad \rho \in C^1(\Omega \times (0, \infty)); \quad \partial_t \mathbf{u} \in L^\infty(0, \infty; H). \quad (4.1)$$

Now we give the definition of the perturbation of the system (1.1), (1.2).

Functions ζ, ρ defined on some interval $t \geq t_0$ are called a *perturbation* of \mathbf{u}, ρ if $(\zeta + \mathbf{u}, \hat{\rho} = \bar{\rho} + \rho)$ is the solution of the system (1.1) with $\zeta = \mathbf{0}$ on Γ . Setting $\zeta_0 = \zeta(t_0)$; $\bar{\rho}_0 = \bar{\rho}(t_0)$, $(\zeta, \hat{\rho})$ is the solution of the initial boundary value problem

$$\left. \begin{aligned} \hat{\rho} \partial_t \zeta + \hat{\rho} \mathbf{u} \cdot \nabla \zeta + \hat{\rho} \zeta \cdot \nabla \mathbf{u} + \hat{\rho} \zeta \cdot \nabla \zeta - \Delta \zeta + \nabla q &= (\hat{\rho} - \rho)(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}); \\ \partial_t \hat{\rho} + (\mathbf{u} + \zeta) \cdot \nabla \hat{\rho} &= 0; \quad \nabla \cdot \zeta = 0; \\ \zeta(0) = \zeta_0; \quad \bar{\rho}(0) = \bar{\rho}_0; \quad \zeta &= \mathbf{0} \quad \text{on } \Gamma. \end{aligned} \right\} \quad (4.2)$$

ASSUMPTION 4. (\mathbf{u}, ρ) is “partially” conditionally exponentially stable if there exist positive numbers D, L, r such that for every $t > 0$ and every $\zeta_0 \in V \cap H^2(\Omega)$ with $|\zeta_0| < r$ the perturbation problem (4.2) is uniquely solvable and the solution $\zeta(t)$ is strong in the sense of (4.1) and

$$|\zeta(t)| < |\zeta_0|^2 L \exp(-D(t - t_0)). \quad (4.3)$$

From section 1 we know that the approximations ρ^n and $\mathbf{u}^n = \sum_{i=1}^n c_i^n(t) \omega^i(x)$ to the solution of the problem (1.1), (1.2) are uniquely determined by the following system:

$$(\rho^n \partial_t \mathbf{u}^n, \boldsymbol{\varphi}^n) + (\nabla \mathbf{u}^n, \nabla \boldsymbol{\varphi}^n) + (\rho^n \mathbf{u}^n \cdot \nabla \mathbf{u}^n, \boldsymbol{\varphi}^n) = 0, \quad (4.4)$$

$$\partial_t \rho^n + \mathbf{u}^n \cdot \nabla \rho^n = 0; \quad (\mathbf{u}^n(0) - \mathbf{u}_0, \boldsymbol{\varphi}^n) = 0; \quad \rho^n(0) = \rho_0 \quad (4.5)$$

for all $\boldsymbol{\varphi}^n$ of the form $\boldsymbol{\varphi}^n = \sum_{i=1}^n d_i^n \omega^i(x)$.

We use the notation of the section 3. Then \mathbf{v}^n satisfies

$$(\rho \partial_t \mathbf{u}, \boldsymbol{\varphi}^n) + (\nabla \mathbf{v}^n, \nabla \boldsymbol{\varphi}^n) = (\rho \mathbf{u} \cdot \nabla \mathbf{u}, \boldsymbol{\varphi}^n) = 0 \quad (4.6)$$

for all $\boldsymbol{\varphi}^n = \sum_{i=1}^n d_i^n \omega^i(x)$. Subtracting (4.6) from (4.4) one has

$$\begin{aligned} (\rho^n \partial_t \boldsymbol{\psi}^n, \boldsymbol{\varphi}^n) + (\nabla \boldsymbol{\psi}^n, \nabla \boldsymbol{\varphi}^n) &= (\rho \mathbf{u} \cdot \nabla \mathbf{u}, \boldsymbol{\varphi}^n) - (\rho^n \mathbf{u}^n \cdot \nabla \mathbf{u}^n, \boldsymbol{\varphi}^n) \\ &\quad - (P_n((\rho - \rho^n)(\mathbf{u} \cdot \nabla \mathbf{u} + \partial_t \mathbf{u}), \boldsymbol{\varphi}^n). \end{aligned} \quad (4.7)$$

Now we re-write (4.7) for all test functions $\boldsymbol{\varphi} \in V \cap H^2(\Omega)$ as (3.9); hence we have

$$\begin{aligned} &(\rho^n \partial_t \boldsymbol{\psi}^n, \boldsymbol{\varphi}) + (\nabla \boldsymbol{\psi}^n, \nabla \boldsymbol{\varphi}) + (\rho^n \mathbf{u} \cdot \nabla \boldsymbol{\psi}^n, \boldsymbol{\varphi}) + (\rho^n \boldsymbol{\psi}^n \cdot \nabla \mathbf{u}, \boldsymbol{\varphi}) + (\rho^n \boldsymbol{\psi}^n \cdot \nabla \boldsymbol{\psi}^n, \boldsymbol{\varphi}) \\ &\quad - (P_n(\rho - \rho^n)(\mathbf{u} \cdot \nabla \mathbf{u} + \partial_t \mathbf{u}), \boldsymbol{\varphi}) \\ &= (Q_n(\rho^n \mathbf{u} \cdot \nabla \boldsymbol{\psi}^n), \boldsymbol{\varphi}) + (Q_n(\rho^n \boldsymbol{\psi}^n \cdot \nabla \mathbf{u}), \boldsymbol{\varphi}) + (Q_n(\rho^n \boldsymbol{\psi}^n \cdot \nabla \boldsymbol{\psi}^n), \boldsymbol{\varphi}) + (P_n(\rho^n \boldsymbol{\psi}^n \cdot \nabla \mathbf{e}^n), \boldsymbol{\varphi}) \\ &\quad + (P_n(\rho^n \mathbf{e}^n \cdot \nabla \boldsymbol{\psi}^n), \boldsymbol{\varphi}) + (P_n(\rho^n \mathbf{u} \cdot \nabla \mathbf{e}^n), \boldsymbol{\varphi}) + (P_n(\rho^n \mathbf{e}^n \cdot \nabla \mathbf{v}^n), \boldsymbol{\varphi}) + (Q_n(\rho^n \partial_t \boldsymbol{\psi}^n), \boldsymbol{\varphi}). \end{aligned} \quad (4.8)$$

Now we denote the right side of (4.8) by (g^n, φ) and we set

$$\theta = \psi^n - \zeta; \quad \mathbf{h} = \partial_t \zeta + \mathbf{u} \cdot \nabla \zeta + \zeta \cdot \nabla \zeta + \zeta \cdot \nabla \mathbf{u}; \quad \mathbf{b} = \mathbf{u} \cdot \nabla \mathbf{u} + \partial_t \mathbf{u}.$$

Subtracting the weak form of the first equation of (4.2) from (4.8) gives

$$\begin{aligned} (\rho^n \partial_t \theta, \varphi) + (\nabla \theta, \nabla \varphi) + (\rho^n \mathbf{u} \cdot \nabla \theta, \varphi) + (\rho^n \theta \cdot \nabla \mathbf{u}, \varphi) + (\rho^n \psi^n \cdot \nabla \theta, \varphi) + (\rho^n \theta \cdot \nabla \zeta, \varphi) \\ = ((\hat{\rho} - \rho^n)(\mathbf{h} + \mathbf{b}), \varphi) + (g^n, \varphi). \end{aligned}$$

Furthermore, $\hat{\rho} - \rho^n$ satisfies

$$\partial_t(\hat{\rho} - \rho^n) + (\mathbf{u} + \zeta) \cdot \nabla(\hat{\rho} - \rho^n) + (\mathbf{u} + \zeta - \mathbf{u}^n) \cdot \nabla \rho^n = 0.$$

Now using the procedures of sections 2, 3 one can prove the following theorem.

THEOREM 4. *Let the assumptions 3, 4 be satisfied. Then there exist constants N , c , and a function $q(t)$, monotonously increasing in t , depending only on the domain Ω , the norms of data, $\sup |\Delta \mathbf{u}|^2$ and the constants in (4.3) such that*

$$|\rho - \rho^n|^2 \leq q(t) \lambda_{n+1}^{-1} \quad \text{and} \quad |\mathbf{u} - \mathbf{u}^n|^2 \leq c \lambda_{n+1}^{-1}$$

provided $n > N$.

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