ON THE IRREDUCIBILITY OF CONVEX BODIES

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1. Introduction. We select a Cartesian co-ordinate system in n-dimensional Euclidean space R_n with origin O and employ the usual point-vector notation.

By a lattice Λ in R_n we mean the set of all rational integral combinations of n linearly independent points $X_1, X_2, \ldots, X_n$ of R_n. The points $X_1, X_2, \ldots, X_n$ are said to form a basis of Λ. Let $\{X_1, X_2, \ldots, X_n\}$ denote the determinant formed when the co-ordinates of X_i are taken in order as the ith row of the determinant for $i = 1, 2, \ldots, n$. The absolute value of this determinant is called the determinant $d(\Lambda)$ of Λ. It is well known that $d(\Lambda)$ is independent of the particular basis one takes for Λ.

A star body in R_n is a closed set of points K such that if $X \in K$ then every point of the form tX where $-1 < t < 1$ is an inner point of K. A star body K is called a convex body if it is bounded and satisfies the convex property: if $X \in K$, $Y \in K$ then $tX + (1 - t)Y \in K$ provided $0 \leqq t \leqq 1$. It is further called strictly convex if $X \in K$, $Y \in K$ implies that $tX + (1 - t)Y$ is an inner point of K when $0 < t < 1$ and $X \neq Y$.

Let Λ be a lattice and K a star body in R_n. We say that Λ is K-admissible if no point of Λ other than 0 is an inner point of K. If K is such that no K-admissible lattice exists then K is said to be of the infinite type, otherwise K is said to be of the finite type. If K is of the finite type the number inf $d(\Lambda)$ extended over all K-admissible lattices Λ is called the critical determinant $\Delta(K)$ of K and any K-admissible lattice Λ of determinant $d(\Lambda) = \Delta(K)$ is called a critical lattice of K. It is well known that if K is of the finite type then at least one critical lattice of K exists.

Let K be a star body of the finite type in R_n. If K is such that any star body properly contained in K has a smaller critical determinant than K has we say that K is S-irreducible; otherwise K is said to be S-reducible.

Let K be a convex body in R_n. If K is such that any convex body properly contained in K has a smaller critical determinant than K has then we say that K is C-irreducible; otherwise we say that K is C-reducible.

The property of S-irreducibility was first studied by Mahler **(1)** who gave necessary but insufficient conditions for a star body to be S-irreducible. Later **(2)** he considered the property of C-irreducibility and showed that if $n = 2$ then any C-irreducible convex body is also S-irreducible. Rogers **(5)** then gave a set of necessary and sufficient conditions for S-irreducibility which will be stated later.

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The purpose here is to give an example of a convex body in R_3 that is C-irreducible but not S-irreducible. The proof that the example has these properties relies to a large extent on the work of Whitworth **(6)**. To clarify the picture regarding C-irreducibility we formulate a set of necessary and sufficient conditions for C-rreducibility analogous to the set given by Rogers for S-irreducibility, the proof following similar lines.

2. The set $L(K)$. The results stated in this section are classical.

Let K be a convex body in R_n. We define $L(K)$ to be the set of all points X of the boundary of K such that if X is contained in any line segment of the boundary of K then X is an endpoint of the line segment. Such points are sometimes called extremal points of K so that $L(K)$ constitutes the set of all extremal points of K. As K is symmetric in 0 it is evident that $L(K)$ is also symmetric in 0. Further:

LEMMA 1. *The convex hull of $L(K)$ is K.*

LEMMA 2. *Given $X \in L(K)$ and $\epsilon > 0$ there exists a convex body $K(\epsilon) \subset K$ such that $X \notin K(\epsilon)$ and such that any point of $K - K(\epsilon)$ lies within a distance ϵ of one of the two points $\pm X$.*

3. C-irreducibility. Let K be a star body in R_n. Further let Λ be a critical lattice of K. Let X be a point of Λ on the boundary of K. We say that Λ is free at the point X if, given $\epsilon > 0$, there exists a lattice $\Lambda(\epsilon)$ of determinant $d(\Lambda(\epsilon)) < d(\Lambda) = \Delta(K)$ such that the interior of K contains no point of $\Lambda(\epsilon)$ apart from 0 and any that are within a distance ϵ from one of the two points $\pm X$. Rogers' criterion for S-irreducibility is then as follows:

LEMMA 3. *K is S-irreducible if, and only if, to each point of the boundary of K there corresponds a critical lattice of K that is free at this point.*

We now give an analogous criterion for C-irreducibility.

THEOREM 1. *If K is a convex body then K is C-irreducible if, and only if, to each point of $L(K)$ there corresponds a critical lattice of K that is free at this point.*

Proof. (i) Only if: Assume that K is C-irreducible and let X be an arbitrary point of $L(K)$. By Lemma 2 given $\epsilon > 0$ there exists a convex body $K(\epsilon) \subset K$ such that $X \in K - K(\epsilon)$ and such that any point of $K - K(\epsilon)$ is within a distance ϵ from one of the two points $\pm X$. Since $K(\epsilon)$ is properly contained in K it follows that $\Delta(K(\epsilon)) < \Delta(K)$. Hence there exists a critical lattice $\Lambda(\epsilon)$ of $K(\epsilon)$ of determinant $d(\Lambda(\epsilon)) < d(\Lambda)$. It is evident that K contains no point of $\Lambda(\epsilon)$ in its interior other than 0 and any that may lie within a distance ϵ from one of the two points $\pm X$. Moreover $\Lambda(\epsilon)$ is certainly not K-admissible and therefore taking into account the fact that K is symmetric in 0 we conclude that there must be a point of $\Lambda(\epsilon)$ in the interior of K and within a distance ϵ from the point X. The sequence $\Lambda(n^{-1})$ of lattices is compact in the sense of Mahler **(3)** and so contains a convergent subsequence with the limit Λ' say. But $\lim_{n\to\infty} K(n^{-1}) = K$ and $\Lambda(n^{-1})$ is a critical lattice of $K(n^{-1})$ for each n, hence Λ' is a critical lattice of K. Further each $\Lambda(n^{-1})$ contains a point within a distance n^{-1} from the point X. Thus Λ' contains X which implies that Λ' is free at X. As X was chosen an arbitrary point of $L(K)$ this proves (i).

(ii) If: Assume that to each point of $L(K)$ there corresponds a critical lattice of K that is free at this point. Take an arbitrary convex body $K' \subset K$ such that $K' \neq K$. There exists a point $X \in L(K) - K'$ for otherwise $L(K) \subset K'$ and so by Lemma 1 $K' = K$ contrary to hypothesis. Let $X \in L(K) - K'$ be fixed. As K' is closed there exists $\epsilon > 0$ such that no point within a distance ϵ from either of the two points $\pm X$ is in K'. By hypothesis there exists a critical lattice Λ of K such that Λ is free at the point X. In particular this implies that there exists a lattice of determinant $d(\Lambda(\epsilon)) < d(\Lambda) = \Delta(K)$ such that no point of $\Lambda(\epsilon)$ apart from 0 and any that may lie within a distance ϵ from one of the two points $\pm X$ is an inner point of K. Hence $\Lambda(\epsilon)$ is K'-admissible from which it follows that $\Delta(K') \leqq d(\Lambda(\epsilon)) < \Delta(K)$. Whence K is C-irreducible. This completes the proof of the theorem.

4. An Example. In looking for a convex body that is C-irreducible and S-reducible we may by Mahler's result confine our attention to dimensions $n \geqq 3$. Further if K is a strictly convex body it is obvious that $L(K)$ is the whole boundary of K. Hence using the previous results K is C-irreducible if, and only if, it is S-irreducible. Again, Dr. Kathleen Ollerenshaw has obtained the following two results **(4)**:

(a) The n-dimensional parallelopiped is S-irreducible for every n.

(b) If K is a two-dimensional S-irreducible convex body then the three-dimensional cylinder on the base K is also S-irreducible.

A more suitable candidate for our purpose has proved to be a sawn-off three-dimensional cube. Whitworth **(6)** has shown that the convex body K in R_3 defined by the inequalities

$$|x_1| \leqq 1, \qquad |x_2| \leqq 1, \qquad |x_3| \leqq 1, \qquad |x_1 + x_2 + x_3| \leqq \tfrac{1}{2}$$

has the critical determinant $\Delta(K) = 3/8$. He has further determined all the critical lattices of K. It is necessary to give a table of these here but before doing so we remark that K has the six automorphisms obtained by permuting the co-ordinates together with the reflections in 0. Thus given any critical lattice of K we obtain six when we apply these transformations. In the following table the only critical lattices of K not included are those obtainable from the ones stated by applying the above automorphisms of K. There are three classes:

Class I: $\Lambda(\rho, \sigma, \beta)$ of basis $X_1 = (\rho - \frac{1}{2}, \sigma - 1, \beta)$, $X_2 = (\rho, \sigma - \frac{1}{2}, \beta - 1)$, $X_3 = (\rho - 1, \sigma, \beta - \frac{1}{2})$ where $\rho + \sigma + \beta = 2$. Another basis for $\Lambda(\rho, \sigma, \beta)$

would be $X_2, X_2 - X_1 = (\frac{1}{2}, \frac{1}{2}, -1)$, $X_3 - X_2 = (-1, \frac{1}{2}, \frac{1}{2},)$. The points $X_2 - X_1, X_3 - X_1$ lie in the plane $x_1 + x_2 + x_3 = 0$ while X_2 lies in the plane $x_1 + x_2 + x_3 = \frac{1}{2}$. Hence all points of $\Lambda(\rho, \sigma, \beta)$ that lie on the boundary of K are confined to the three planes $x_1 + x_2 + x_3 = 0$ or $\pm 1/2$. It follows that the same is true of the automorphic images of $\Lambda(\rho, \sigma, \beta)$.

Class II: $\Lambda(\lambda, \mu, \beta)$ of basis $X_1 = (1, -\frac{1}{2}, -\frac{1}{2})$, $X_2 = (-\frac{1}{2}, 1, -\frac{1}{2})$, $X_3 = (-\lambda, -\mu, \beta)$ where $\lambda + \mu - \beta = \frac{1}{2}$, $0 < -\beta \leqq \frac{1}{2}$, $0 \leqq \mu \leqq \frac{1}{2}$, $0 \leqq \lambda \leqq \frac{1}{2}$. The points X_1, X_2 lie in the plane $x_1 + x_2 + x_3 = 0$ while the point X_3 lies in the plane $x_1 + x_2 + x_3 = -\frac{1}{2}$. Hence all points of $\Lambda(\lambda, \mu, \beta)$ that lie on the boundary of K are confined to the three planes $x_1 + x_2 + x_3 = 0$ or $\pm \frac{1}{2}$ and the same is true of the automorphic images of $\Lambda(\lambda, \mu, \beta)$.

Class III: (i) $\Lambda(\nu_1, \nu_2, \chi_1, \chi_2, \beta)$ of basis $X_1 = (-\nu_1, \beta, -\chi_1)$, $X_2 = (-\nu_2, 1 - \beta, -\chi_2)$, $X_3 = (1, -\frac{1}{2}, -\frac{1}{2})$ where $\nu_1 + \nu_2 = \frac{1}{2}$, $\chi_1 + \chi_2 = \frac{1}{2}$, $\beta - \nu_1 - \chi_1 = \pm \frac{1}{2}$. The points X_1, X_2 lie in one of the planes $x_1 + x_2 + x_3 = \pm \frac{1}{2}$ while the point X_3 lies in the plane $x_1 + x_2 + x_3 = 0$ and hence all points of $\Lambda(\nu_1, \nu_2, \chi_1, \chi_2, \beta)$ that are on the boundary of K are confined to the planes $x_1 + x_2 + x_3 = 0$ or $\pm \frac{1}{2}$.

(ii) $\Lambda(\lambda)$ of basis $X_1 = (1, -\frac{1}{2}, -\frac{1}{2})$, $X_2 = (-\lambda, -\frac{1}{2}, 1)$, $X_3 = (\frac{1}{2}, 0, 0)$. Evidently the points of $\Lambda(\lambda)$ that are on the boundary of K are confined to the lines given by $(t, -\frac{1}{2}u_1 - \frac{1}{2}u_2, -\frac{1}{2}u_1 + u_2)$ where u_1, u_2 have one of the following pairs of values: $(0, 0)$, $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1)$, $(1, 1)$, $(-1, -1)$, $(2, 0)$, $(-2, 0)$. Hence the points of all the automorphic images of $\Lambda(\lambda)$ on the boundary of K are confined to the lines given above together with those obtained from them by permuting the co-ordinates.

(iii) Λ of basis $X_1 = (-\frac{1}{2}, 1, -\frac{1}{2})$, $X_2 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, $X_3 = (\frac{1}{2}, 0, 0)$. The point X_1 lies in the plane $x_1 + x_2 + x_3 = 0$, X_2 in $x_1 + x_2 + x_3 = -\frac{1}{2}$, X_3 in $x_1 + x_2 + x_3 = \frac{1}{2}$; hence all points of Λ that are on the boundary of K are confined to the planes $x_1 + x_2 + x_3 = 0$ or $\pm \frac{1}{2}$. It follows that the same is true of the automorphic images of Λ.

This completes the table of the critical lattices of K. We are now in a position to prove:

THEOREM 2. *K is C-irreducible and S-reducible.*

Proof. We show first that K is S-reducible. From the table given above we see that the only critical lattices of K with points on the boundary of K that do not lie in one of the three planes $x_1 + x_2 + x_3 = 0$ or $\pm \frac{1}{2}$ are those in Class III (ii). The point $(1, -\frac{1}{3}, -\frac{1}{3})$ is on the boundary of K and in the plane $x_1 + x_2 + x_3 = \frac{1}{3}$. Therefore if it is a point of some critical lattice of K it must be in Class III (ii). However, it is obvious that no lattice of this class can contain $(1, -\frac{1}{3}, -\frac{1}{3})$ nor can any lattice which is derived from one of those stated by permuting the co-ordinates. Therefore $(1, -\frac{1}{3}, -\frac{1}{3})$ belongs to no critical lattice of K. By Lemma 3, K is S-reducible.

We now show that K is C-irreducible. The set $L(K)$ consists of the twelve points obtained by permuting the co-ordinates of the point $(1, \frac{1}{2}, -1)$ and taking the six points thus obtained together with their reflections in 0. Hence, by virtue of Theorem 1, K is C-irreducible if we can show that there exists a critical lattice of K which is free at the point $(1, \frac{1}{2}, -1)$. Take the lattice $\Lambda(\frac{1}{2}, 0, 3/2)$ in Class I of the table above. A basis of this lattice is $X_1 = (1, -1, \frac{1}{2})$, $X_2 = (3/2, -\frac{1}{2}, -\frac{1}{2})$, $X_3 = (\frac{1}{2}, 0, 0)$. Another basis would be $Y_1 = X_2 - X_1 = (\frac{1}{2}, \frac{1}{2}, -1)$, $Y_2 = X_1 - X_3 = (\frac{1}{2}, -1, \frac{1}{2})$, $Y_3 = X_3$. The points of $\Lambda(\frac{1}{2}, 0, 3/2)$ on the boundary of K are Y_1, Y_2, Y_3, $Y_1 + Y_2 = (1, -\frac{1}{2}, -\frac{1}{2})$, $Y_1 + Y_3 = (1, \frac{1}{2}, -1)$, $Y_1 - Y_3 = (0, \frac{1}{2}, -1)$, $Y_2 - Y_3 = (0, -1, \frac{1}{2})$, $Y_2 + Y_3 = (1, -1, \frac{1}{2})$, $Y_1 + Y_2 - Y_3 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ together with their reflections in 0. In particular we see that $Y_1 + Y_3 = (1, \frac{1}{2}, -1)$ is a point of the lattice. For a given $\delta > 0$ denote by $\Lambda(\delta)$ the lattice of basis $Y_1' = (\frac{1}{2} - \delta, \frac{1}{2}, -1)$, $Y_2' = (\frac{1}{2} + \delta, -1, \frac{1}{2})$, $Y_3' = (\frac{1}{2} - \delta, 0, \delta)$. Evidently as $\delta \to 0$ so $Y_1' \to Y_1$, $Y_2' \to Y_2$, $Y_3' \to Y_3$ and therefore also $\Lambda(\delta) \to \Lambda(\frac{1}{2}, 0, 3/2)$. Moreover,

$$d(\Lambda(\delta)) = \left\| \begin{matrix} \frac{1}{2} - \delta & \frac{1}{2} & -1 \\ \frac{1}{2} + \delta & -1 & \frac{1}{2} \\ \frac{1}{2} - \delta & 0 & \delta \end{matrix} \right\| = \tfrac{3}{8} - \tfrac{1}{2}\delta^2 < \tfrac{3}{8}$$

provided only that δ is sufficiently small. Since in the limit $\delta \to 0$ the basis given for $\Lambda(\delta)$ becomes the basis given for $\Lambda(\frac{1}{2}, 0, 3/2)$ it follows that for all sufficiently small δ the only points of $\Lambda(\delta)$ that can lie in the interior of K are

$$Y_1' = (\tfrac{1}{2} - \delta, \tfrac{1}{2}, -1), Y_2' (\tfrac{1}{2} + \delta, -1, \tfrac{1}{2}), Y_3' = (\tfrac{1}{2} - \delta, 0, \delta),$$

$$Y_1' + Y_2' = (1, -\tfrac{1}{2}, -\tfrac{1}{2}), Y_1' + Y_3' = (1 - 2\delta, \tfrac{1}{2}, \delta - 1), Y_2' + Y_3' = (1, -1, \tfrac{1}{2} + \delta),$$

$$Y_1' - Y_3' = (0, \tfrac{1}{2}, -1 - \delta), Y_2' - Y_3' = (2\delta, -1, \tfrac{1}{2} - \delta), Y_1' + Y_2' - Y_3' = (\tfrac{1}{2} + \delta, -\tfrac{1}{2}, -\tfrac{1}{2} - \delta)$$

together with their reflections in 0. But it is clear that the only ones in the interior of K are $\pm (Y_1' + Y_3')$. Moreover

$$\lim_{\delta \to R} (Y_1' + Y_3') = (1, \tfrac{1}{2}, -1),$$

hence $\Lambda(3/2, 0, \frac{1}{2})$ is free at the point $(1, \frac{1}{2}, -1)$. Therefore K is C-irreducible. This completes the proof of Theorem 2.

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