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REARRANGEMENT INEQUALITIES

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1. Introduction. In recent years a number of inequalities have appeared which involve rearrangements of vectors in $\mathbf{R}^n$ and of measurable functions on a finite measure space. These inequalities are not only interesting in themselves, but also are important in investigations involving rearrangement invariant Banach function spaces and interpolation theorems for these spaces [**2**; **8**; **9**].

The most famous inequality of this type for vectors is due to Hardy-Littlewood and Polya [**4**, Theorem 368]:

$$\sum_{i=1}^{m} a_i^* b_i' \leqq \sum_{i=1}^{m} a_i b_i \leqq \sum_{i=1}^{m} a_i^* b_i^* \tag{1.1}$$

with equality on the left (right) if and only if $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$ are oppositely (similarly) ordered. Here the $a_i^*(a_i')$ are the numbers a_i in decreasing (increasing) order.

An example involving more than two vectors is the following one of H. D. Ruderman [**12**]:

$$\prod_{j=1}^{m} \sum_{k=1}^{n} a_{k,j} \geqq \prod_{j=1}^{m} \sum_{k=1}^{n} a_{k,j}^* \tag{1.2}$$

where $a_{k,j} > 0$ for all k, j, and for each k the $a_{k,j}^*$ are the numbers $a_{k,1}, \ldots, a_{k,m}$ in decreasing order. A condition for equality was not given.

Other inequalities of these types are possible, and general theorems have been given by G. G. Lorentz [**7**] and D. London [**6**].

Workers with inequalities generally recognize that many inequalities which are proved for real numbers by real variable methods also hold in more general systems. In Section 3 we let $\varphi : T_1 \times T_2 \to G$ where T_1, T_2 are ordered sets, and G is a partially ordered abelian group, and we give a necessary and sufficient condition on φ so that

$$\sum_{j=1}^{n} \varphi(a_j^*, b_j') \leqq \sum_{j=1}^{n} \varphi(a_j, b_j) \leqq \sum_{j=1}^{n} \varphi(a_j^*, b_j^*)$$

for all chains $\mathbf{a} \in T_1^n$, $\mathbf{b} \in T_2^n$. Also we give a necessary and sufficient condition on φ so that equality holds on the right (left) if and only if $\mathbf{a}$ and $\mathbf{b}$ are similarly (oppositely) ordered. We give a sufficient condition so that $\varphi(\mathbf{a}^*, \mathbf{b}') \ll \varphi(\mathbf{a}, \mathbf{b}) \ll \varphi(\mathbf{a}^*, \mathbf{b}^*)$, where $\ll$ denotes a preorder relation of

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Hardy, Littlewood, and Polya. Similar results to these are given when φ is a function of n variables.

W. A. J. Luxemburg [**9**] has proved analogs of discrete rearrangement inequalities for measurable functions on a finite measure space. In Section 5, all our discrete results are generalized for real valued essentially bounded measurable functions on a finite measure space. For specific choices of φ the inequalities are shown to hold for even larger classes of functions. The concept of "similarly ordered" is generalized for measurable functions to give a necessary and sufficient condition for equality.

Finally in Sections 4 and 6 we give numerous examples to show how to obtain many known rearrangement inequalities. Our analysis gives conditions for equality, in many cases for the first time.

2. Definitions and notation. Let T be a partially ordered set. If $\mathbf{a} = (a_1, \ldots, a_m) \in T^m$, then $\mathbf{a}$ will be called a *chain* if $\{a_1, \ldots, a_m\}$ is linearly ordered. If $\mathbf{a}$ is a chain, then $\mathbf{a}^* = (a_1^*, \ldots, a_m^*)$ $(\mathbf{a}' = (a_1', \ldots, a_m'))$ denotes the vector obtained by rearranging the components of $\mathbf{a}$ in decreasing (increasing) order. If $\mathbf{a}$ and $\mathbf{b}$ are chains in a partially ordered abelian group G (written additively), then $\mathbf{b} \ll \mathbf{a}$ means $\sum_{i=1}^{k} b_i^* \leqq \sum_{i=1}^{k} a_i^*$ for all $1 \leqq k \leqq m$; and $\mathbf{b} \prec \mathbf{a}$ means $\mathbf{b} \ll \mathbf{a}$ and $\sum_{i=1}^{m} b_i^* = \sum_{i=1}^{m} a_i^*$. It will be notationally simpler and should cause no confusion to denote every partial order under consideration by $\leqq$. A partial order is understood to be anti-symmetric, and $x < y$ is used to mean $x \leqq y$ and $x \neq y$.

Let T_1 and T_2 be partially ordered sets. Chains $\mathbf{a} \in T_1^m$ and $\mathbf{b} \in T_2^m$ are said to be *similarly* (*oppositely*) *ordered* if for every $1 \leqq i, j \leqq m$, $a_i < a_j$ implies $b_i \leqq b_j$ $(b_j \leqq b_i)$.

Let $T_1, \ldots, T_n$ be partially ordered sets, and let

$$\mathbf{a}_k = (a_{k,1}, \ldots, a_{k,m}) \in (T_k)^m.$$

It is sometimes necessary to substitute values for some of the variables x_i in $(x_1, \ldots, x_n)$ and then consider the result as a function of the remaining x_i. Let I, J, and K be disjoint subsets of $N = \{1, \ldots, n\}$. To denote the result of substituting $a_{i,j}$ for x_i when $i \in I$, $a_{i,k}$ for x_i when $i \in J$, and $a_{i,l}$ for x_i when $i \in K$, we use the notation $(a_{I,j}, a_{J,k}, a_{K,l})$. In addition, $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ denotes the sequence of vectors given by $j \mapsto (a_{i,j}, \ldots, a_{n,j})$, and similarly for $(\mathbf{a}_I^*, \mathbf{a}_J')$ when $\{I, J\}$ is a partition of $\{1, \ldots, n\}$.

Let $\varphi : T_1 \times \ldots \times T_n \to G$. When I and J are partitions of $N = \{1, \ldots, n\}$, we define conditions (A) and (A^*) on φ as follows.

(A) $\quad$ $[(A^*)]$ If $x_i, y_i \in T_i$ with $x_i < y_i$, and $k \neq i$,

then $\varphi(y_i) - \varphi(x_i)$ is [strictly] increasing in u_k when k and i are in the same set I or J, and [strictly] decreasing in u_k when k and i are in different sets I and J, for all $1 \leqq i, k \leqq n$.

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If $G = \mathbf{R}$, if $T_k = [r_k, s_k]$ with $r_k < s_k$, if the first partials of φ are separately continuous on $T_1 \times \ldots \times T_n$, and if the second partials of φ exist on $T =]r_1, s_1[\times \ldots \times]r_n, s_n[$, then [**1**, Theorems 5–7 and 5–10] implies that condition (A) is equivalent to:

$$(A)' \quad \partial^2\varphi/\partial u_i \partial u_j \geqq 0, \text{ when } i \text{ and } j \text{ are in the same set } I \text{ or } J$$
$$\leqq 0, \text{ when } i \text{ and } j \text{ are in different sets } I \text{ and } J$$

on T for all $1 \leqq i \neq j \leqq n$.

A sufficient differentiability condition for (A^*) is $(A^*)'$:

φ satisfies $(A)'$ and in addition, $\{u_i \in]r_i, s_i[: \partial^2\varphi/\partial u_i \partial u_j = 0\}$ contains no open interval for all $r_k < u_k < s_k$, and $1 \leqq k \neq i \leqq n$.

Let (X, Λ, μ) be a finite measure space with $\alpha = \mu(X) < \infty$, and let $M = M(X, \mu)$ denote the set of all extended real valued measurable functions on X. If $f \in M$, then the *decreasing rearrangement* δ_f of f is defined by

$$\delta_f(t) = \inf\{s \in \mathbf{R} : \mu(\{x : f(x) > s\}) \leqq t\} \text{ for } 0 \leqq t \leqq \alpha.$$

Also $\iota_f(t) = \delta_f((\alpha - t)-)$ denotes the *increasing rearrangement* of f, 1_E denotes the *characteristic function* of $E \in \Lambda$; $f|E$ denotes the *restriction of* f *to* E; and we let $I_f = [\text{ess. inf} f, \text{ess. sup} f]$.

If $f, g \in M$ then $f \sim g$ means $\delta_f = \delta_g$. This is equivalent to having $\mu(\{f > t\}) = \mu(\{g > t\})$ for all $t \in \mathbf{R}$. Let $(t_1, \ldots, t_n) > (u_1, \ldots, u_n)$ mean $t_i > u_i$, $1 \leqq i \leqq n$. For measurable $\mathbf{f}, \mathbf{g} : X \to \mathbf{R}^n$ we define $\mathbf{f} \sim \mathbf{g}$ to mean $\mu(\{\mathbf{f} > \mathbf{t}\}) = \mu(\{\mathbf{g} > \mathbf{t}\})$ for all $\mathbf{t} \in \mathbf{R}^n$.

We will say that $f, g \in M$ are *similarly ordered* if ess. sup $f|A <$ ess. inf $f|B$ implies ess. sup $g|A \leqq$ ess. inf $g|B$ whenever $A, B \in \Lambda$ are disjoint and each has positive measure. Analogously, $f, g \in M$ are called *oppositely ordered* if f and $-g$ are similarly ordered.

3. The discrete case. This section is devoted to the proof of the following theorem.

(3.1) THEOREM. *Let* $\varphi : T_1 \times \ldots \times T_n \to G$, *where each* T_k $(k = 1, \ldots, n)$ *is linearly ordered, and* G *is a partially ordered abelian group. Let* $\{I, J\}$ *be a partition of* $N = \{1, \ldots, n\}$.

(i) φ *satisfies condition* (A) *if and only if*

$$(1) \qquad \sum_{j=1}^{m} \varphi(a_{i,j}, \ldots, a_{n,j}) \leqq \sum_{j=1}^{m} \varphi(a_{I,j}^*, a_{J,j}')$$

for all $\mathbf{a}_k = (a_{k,1}, \ldots, a_{k,m}) \in (T_k)^m$, $k = 1, \ldots, n$.

(ii) φ *satisfies condition* (A^*) *if and only if the following are equivalent for all* $\mathbf{a}_k \in (T_k)^m$, $k = 1, \ldots, n$.

(a) *Equality occurs in* (1).

(b) $\mathbf{a}_p$ *and* $\mathbf{a}_q$ *are similarly ordered whenever* p *and* q *are in the same set*

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I or J, and oppositely ordered when p and q are in different sets I and J, for all $1 \leqq p, q \leqq n$.

(c) $\varphi(\mathbf{a}_1, \ldots, \mathbf{a}_n) \sim \varphi(\mathbf{a}_I^*, \mathbf{a}_J')$.

(iii) *Suppose the range of* φ *is linearly ordered. If* φ *satisfies condition* (A) *and is increasing (respectively decreasing) in* u_k *for* $k \in I$ *and decreasing (respectively increasing) in* u_k *for* $k \in J$, *then*

$$\varphi(\mathbf{a}_1, \ldots, \mathbf{a}_n) \ll \varphi(\mathbf{a}_I^*, \mathbf{a}_J') \tag{2}$$

for all chains $\mathbf{a}_k \in T_k^m$ $(k = 1, \ldots, n)$.

Proof. To prove necessity of (A) for (1), let $1 \leqq k, i \leqq n$, let $x_i, y_i \in T_i$ with $x_i < y_i$, let $\mathbf{a}_i = (x_i, y_i, \ldots, y_i)$, let $u_k, v_k \in T_k$ with $u_k < v_k$, and for $j \neq i, k$ let $u_j \in T_j$ and $\mathbf{a}_j = (u_j, \ldots, u_j)$. Case 1: k, i are in the same set I or J. Let $\mathbf{a}_k = (v_k, u_k, \ldots, u_k)$. After cancelling terms in (1) we obtain

$$\varphi(x_i, v_k) + \varphi(y_i, u_k) \leqq \varphi(y_i, v_k) + \varphi(x_i, u_k),$$

so

$$\varphi(y_i, u_k) - \varphi(x_i, u_k) \leqq \varphi(y_i, v_k) - \varphi(x_i, v_k),$$

and hence (A) is true in this case. Case 2: k, i are in different sets I and J. Let $\mathbf{a}_k = (u_k, v_k, \ldots, v_k)$. The proof is similar to Case 1. This completes the proof of necessity.

Before continuing we introduce some notation. For $\mathbf{a}_k \in T_k^m$ write $\mathbf{b}_N = S_{i,j}\mathbf{a}_N$ if $1 \leqq i < j \leqq m$ are such that for $P = \{k \in I : a_{k,i} < a_{k,j}\}$, and $Q = \{k \in J : a_{k,i} > a_{k,j}\}$ we have: $\mathbf{b}_k$ for $k \in P \cup Q$ is the sequence obtained from $\mathbf{a}_k$ by interchanging $a_{k,i}$ and $a_{k,j}$, while $\mathbf{b}_k = \mathbf{a}_k$ for other k.

Assume $\mathbf{b}_N = S_{i,j}\mathbf{a}_N$ with P and Q as above, and let $\psi = \varphi(a_{P,i}, a_{Q,i}) - \varphi(a_{P,j}, a_{Q,j})$. Also, for $0 \leqq k \leqq n$ let

$$P_k = P \cap \{0, \ldots, k\} \quad \text{and} \quad Q_k = Q \cap \{0, \ldots, k\}.$$

Then

$$\begin{aligned}\psi = &\sum_{k=0}^{n-1} [\varphi(a_{P,i}, a_{Q-Q_k,i}, a_{Q_k,j}) - \varphi(a_{P,i}, a_{Q-Q_{k+1},i}, a_{Q_{k+1},j})] \\ &+ \sum_{k=0}^{n-1} [\varphi(a_{P-P_k,i}, a_{P_k,j}, a_{Q,j}) - \varphi(a_{P-P_{k+1},i}, a_{P_{k+1},j}, a_{Q,j})]\end{aligned}$$

is a sum of differences like that in (A), so

$$\psi(a_{I-P,i}, a_{J-Q,i}) \leqq \psi(a_{I-P,j}, a_{J-Q,j}). \tag{3}$$

On writing it out, this is the same as

$$\varphi(a_{N,i}) + \varphi(a_{N,j}) \leqq \varphi(b_{N,i}) + \varphi(b_{N,j}), \tag{4}$$

so

$$\sum_{r=1}^{m} \varphi(a_{N,r}) \leqq \sum_{r=1}^{m} \varphi(b_{N,r}). \tag{5}$$

If (A^*) holds, inequality (3) and hence (5) will be strict unless $P \cup Q \neq \emptyset$ or $a_{k,i} = a_{k,j}$ for all $k \in (I - P) \cup (J - Q)$.

There are $\mathbf{b}(1), \ldots, \mathbf{b}(q)$ such that $\mathbf{b}(1) = \mathbf{a}_N, \mathbf{b}(q) = (\mathbf{a}_I^*, \mathbf{a}_J')$ and for each $1 \leqq k \leqq n - 1$ there are i and j such that $\mathbf{b}(k+1) = S_{i,j}\mathbf{b}(k)$. Hence $\sum_{j=1}^m \varphi(b(1)_j) \leqq \ldots \leqq \sum_{j=1}^m \varphi(b(q)_j)$, which proves (1).

In (ii) it is clear that (b) $\Rightarrow$ (c) $\Rightarrow$ (a) always. We begin by assuming (A^*) holds and show that (a) $\Rightarrow$ (b). Suppose (b) does not hold. Then an examination of cases shows there are $1 \leqq i < j \leqq m$ such that for P and Q as above we have $P \cup Q \neq \emptyset$, and there is a $k \in (I - P) \cup (J - Q)$ such that $a_{k,i} \neq a_{k,j}$. Hence letting $\mathbf{b}_N = S_{i,j}\mathbf{a}_N$ we have $\sum_{r=1}^m \varphi(a_{N,r}) < \sum_1^m \varphi(b_{N,r}) \leqq \sum_1^m \varphi(b_{I,r}^*, b_{J,r}') = \sum_1^m \varphi(a_{I,r}^*, a_{J,r}')$, since $\mathbf{b}_k^* = \mathbf{a}_k^*$, $k = 1, \ldots, n$. Conversely if (a) $\Rightarrow$ (b), then the arguments used in proving necessity of (A) for (1) show that (A^*) holds.

We turn now to the proof of (iii). Since $\varphi(\mathbf{a}_I^*, \mathbf{a}_J') \sim \varphi(\mathbf{a}_I', \mathbf{a}_J^*)$, it suffices to prove (2) assuming φ is increasing in the I-variables and decreasing in the J-variables. In this case let $\mathbf{b}_N = S_{i,j}\mathbf{a}_N$. Then

$$(6) \qquad \varphi(b_{N,j}) \leqq \varphi(a_{N,i}),\ \varphi(a_{N,j}) \leqq \varphi(b_{N,i}).$$

We call $\varphi(a_{N,i})$ and $\varphi(a_{N,j})$ the "old terms", and $\varphi(b_{N,i})$ and $\varphi(b_{N,j})$ the "new terms". These are the only terms where $\varphi(\mathbf{a}_N)$ and $\varphi(\mathbf{b}_N)$ differ.

Let $1 \leqq k \leqq m$, define sequences

$$\boldsymbol{\alpha} = (\varphi(a_N)_r^* : 1 \leqq r \leqq k), \quad \boldsymbol{\beta} = (\varphi(b_N)_r^* : 1 \leqq r \leqq k),$$

let $\sum \boldsymbol{\alpha} = \sum_{r=1}^k \varphi(a_N)_r^*$ and define $\sum \boldsymbol{\beta}$ similarly. We show that $\sum \boldsymbol{\alpha} \leqq \sum \boldsymbol{\beta}$.

If exactly one of the old terms occurs in $\boldsymbol{\alpha}$, then (6) implies that the only new term in $\boldsymbol{\beta}$ is $\varphi(b_{N,i})$. For if $\varphi(b_{N,j})$ is in $\boldsymbol{\beta}$, then (6) implies that $\boldsymbol{\beta}$ contains both new terms, so there are $m - k$ terms of $\varphi(\mathbf{a}_N)$ which are $\leqq \varphi(b_{N,j})$, in which case (6) implies that both old terms occur in $\boldsymbol{\alpha}$. Hence $\boldsymbol{\beta}$ is obtained from $\boldsymbol{\alpha}$ by replacing an old term by the larger term $\varphi(b_{N,i})$. Thus $\sum \boldsymbol{\alpha} \leqq \sum \boldsymbol{\beta}$.

If both old terms occur in $\boldsymbol{\alpha}$, then (4) implies their sum is $\leqq$ the sum of the new terms, which is $\leqq$ the sum of $\varphi(b_{N,i})$ and any term $\geqq \varphi(b_{N,j})$, in case $\varphi(b_{N,j})$ is not in $\boldsymbol{\beta}$. Hence $\sum \boldsymbol{\alpha} \leqq \sum \boldsymbol{\beta}$.

If none of the old terms occur in $\boldsymbol{\alpha}$, then either $\boldsymbol{\alpha} = \boldsymbol{\beta}$, or $\boldsymbol{\beta}$ is obtained from $\boldsymbol{\alpha}$ by replacing one term of $\boldsymbol{\alpha}$ by the larger term $\varphi(b_{N,i})$. Thus $\sum \boldsymbol{\alpha} \leqq \sum \boldsymbol{\beta}$. The proof of (iii) is finished as in (i). This completes the proof of the theorem.

When φ is a function of two variables, conditions (A) and (A^*) simplify, and the arguments proving (3.1) have a symmetry which shows how small the sums can get.

(3.2) COROLLARY. *Let* $\varphi : T_1 \times T_2 \to G$.

(i) *The inequality*

$$(1) \qquad \sum_{j=1}^m \varphi(a_j^*, b_j') \leqq \sum_{j=1}^m \varphi(a_j, b_j) \leqq \sum_{j=1}^m \varphi(a_j^*, b_j^*)$$

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holds for all $\mathbf{a} \in (T_1)^m$ *and* $\mathbf{b} \in (T_2)^m$ *if and only if* $\Delta_{c,d}\varphi(y) = \varphi(d, y) - \varphi(c, y)$ *is increasing in* $y \in T_2$ *whenever* $d > c$, $d, c \in T_1$.

(ii) $\Delta_{c,d}\varphi$ *is strictly increasing whenever* $d > c$ *if and only if the following are equivalent:* (a) *Equality occurs in* (1) *on the left* (*right*); (b) $\mathbf{a}$ *and* $\mathbf{b}$ *are oppositely* (*similarly*) *ordered;* (c) $\varphi(\mathbf{a}^*, \mathbf{b}') \sim \varphi(\mathbf{a}, \mathbf{b})$ $(\varphi(\mathbf{a}^*, \mathbf{b}^*) \sim \varphi(\mathbf{a}, \mathbf{b}))$.

(iii) *If the range of* φ *is totally ordered, and in addition to* (i), φ *is increasing* (*or decreasing*) *in both variables, then* $\varphi(\mathbf{a}^*, \mathbf{b}') \ll \varphi(\mathbf{a}, \mathbf{b}) \ll \varphi(\mathbf{a}^*, \mathbf{b}^*)$.

(3.3) *Remarks.* (i) In (3.2.i) above, replacing φ by $-\varphi$ gives the condition when the inequalities (1) reverse. The corresponding condition in (iii) is that φ be increasing in one variable and decreasing in the other, in which case, $\varphi(\mathbf{a}^*, \mathbf{b}^*) \ll \varphi(\mathbf{a}, \mathbf{b}) \ll \varphi(\mathbf{a}^*, \mathbf{b}')$.

(ii) The inequalities in (3.1), (3.2) and (3.3.i) may be written equivalently by interchanging primes and asterisks, since, for example, $\varphi(\mathbf{a}^*, \mathbf{b}') \sim \varphi(\mathbf{a}', \mathbf{b}^*)$.

4. Examples for the discrete case. In this section we illustrate the previous theorems for particular choices of φ. In all cases, $G = \mathbf{R}$.

(4.1) $T_1 = T_2 = \mathbf{R}$ and $\varphi(x, y) = x + y : \mathbf{a}^* + \mathbf{b}' \prec \mathbf{a} + \mathbf{b} \prec \mathbf{a}^* + \mathbf{b}^*$.

(4.2) $T_1 = T_2 = \mathbf{R}$ and $\varphi(x, y) = x - y : \mathbf{a}^* - \mathbf{b}^* \prec \mathbf{a} - \mathbf{b} \prec \mathbf{a}^* - \mathbf{b}'$.

(4.3) $$\varphi(x, y) = xy : \text{For } T_1 = T_2 = \mathbf{R}$$

we obtain (1.1) with the indicated condition for equality.

For $T_1 = T_2 = [0, \infty[$ or $T_1 = T_2 =]-\infty, 0]$ we obtain $\mathbf{a}^*\mathbf{b}' \ll \mathbf{ab} \ll \mathbf{a}^*\mathbf{b}^*$ whenever

$$\mathbf{a}, \mathbf{b} \in [0, \infty[^m \quad \text{or} \quad \mathbf{a}, \mathbf{b} \in]-\infty, 0]^m.$$

When $T_k = [0, \infty[$ $(k = 1, \ldots, n)$, $I = \{1, \ldots, n\}$ and $J = \emptyset$ then $\varphi(u_1, \ldots, u_n) = u_1 \ldots u_n$ satisfies (A) and we obtain a companion to (1.4), also proved by Ruderman:

$$\sum_{j=1}^{m} \prod_{i=1}^{n} a_{i,j} \leqq \sum_{j=1}^{m} \prod_{i=1}^{n} a_{i,j}^*.$$

If all $a_{i,j} > 0$, then the inequality is strict unless all of the sequences $\mathbf{a}_k = (a_{k,1}, \ldots, a_{k,m})$ are similarly ordered.

(4.4) $$\varphi(x, y) = \log(1 + xy)$$

with $T_1 \times T_2 \subset \{(x, y) : xy > -1\}$ gives:

$$\prod_{i=1}^{m} (1 + a_i^*b_i') \leqq \prod_{i=1}^{m} (1 + a_ib_i) \leqq \prod_{i=1}^{m} (1 + a_i^*b_i^*)$$

whenever $a_i^*b_i' > -1$ for $i = 1$ and $i = m$. The inequality is strict except as indicated in (3.2.ii). The choice $T_1 = T_2 = [0, \infty[$ or $]-\infty, 0]$ gives:

$$\log(1 + \mathbf{a}^*\mathbf{b}') \ll \log(1 + \mathbf{ab}) \ll \log(1 + \mathbf{a}^*\mathbf{b}^*)$$

whenever $\mathbf{a}, \mathbf{b} \in [0, \infty[^m$ or $]-\infty, 0]^m$.

(4.5) $\varphi(x, y) = -\log(x + y)$, $T_1 \times T_2 \subset \{(x, y) : x + y > 0\}$:

$$-\log(\mathbf{a}^* + \mathbf{b}') \ll -\log(\mathbf{a} + \mathbf{b}) \ll -\log(\mathbf{a}^* + \mathbf{b}^*)$$

whenever $a_m{}^* + b_m{}^* > 0$, and in particular we get an inequality of Minc [**10**]:

$$\prod_{i=1}^{m} (a_i{}^* + b_i{}^*) \leqq \prod_{i=1}^{m} (a_i + b_i) \leqq \prod_{i=1}^{m} (a_i{}^* + b_i{}').$$

The inequality is strict except as indicated by (3.2.i). The example $\mathbf{a} = (6, 5, 2, 1)$ $\mathbf{b} = (-3, -4, -2, 1)$ shows this inequality may fail under the condition $a_i + b_i \geqq 0$ for all i (but it will hold for vectors of length $\leqq 3$). This inequality is also easily seen to hold for all $a_i, b_i \geqq 0$.

Analogously, $\varphi(u_1, \ldots, u_n) = -\log(u_1 + \ldots + u_n)$ with

$$T_1 \times \ldots \times T_n \subset \{(u_1, \ldots, u_n) : u_1 + \ldots + u_n > 0\}$$

gives Ruderman's Inequality (1.2) whenever $\sum_{k=1}^{n} a_{k,m}{}^* > 0$. The inequality is strict unless all the $\mathbf{a}_k$ are similarly ordered.

(4.6) Suppose φ satisfies the hypotheses of (3.1.iii) and H is increasing and convex on an interval containing the range of φ. Then $\varphi_1 = H \circ \varphi$ satisfies condition (A). In this way [**11**, p. 165, Theorem 2] and (3.1.i) may be used to prove (3.1.iii). If in addition, φ satisfies (A^*) and H is strictly convex, then φ_1 satisfies (A^*). The proof follows easily from [**11**, p. 164, the third inequality from the bottom].

(4.7) Two theorems of D. London [**6**] may be obtained using (3.2) and (4.6). Replace a_i by $1/a_i$, so that his results are stated without quotients. His conditions on F in both theorems are the same as saying that F is convex and increasing on $[0, \infty[$. Hence let $H = F$, let $\varphi(x, y) = \log(1 + xy)$ for Theorem 1, and let $\varphi(x, y) = xy$ for Theorem 2. If F is strictly convex, we obtain his conditions for equality.

(4.8) Ruderman [**12**] has observed that (1.2) generalizes the inequality between the arithmetic and geometric means. Using (3.1) we may obtain the following inequality for certain quasi-arithmetic symmetric means. Let U be an open interval of $\mathbf{R}$, let $f, g : U \to \mathbf{R}$ be strictly monotone and let $f \circ g^{-1}$ be convex on $g[U]$. If f is increasing then

$$g^{-1}([g(r_1) + \ldots + g(r_n)]/n) \leqq f^{-1}([f(r_1) + \ldots + f(r_n)]/n)$$

for all $r_1, \ldots, r_n \in U$, while if f is decreasing, the inequality reverses. If $f \circ g^{-1}$ is strictly convex, the inequality is strict unless $r_1 = \ldots = r_n$. To prove this, in (3.1.i.1) let

$$\mathbf{a}_1 = (r_1, r_2, \ldots, r_{n-1}, r_n), \quad \mathbf{a}_2 = (r_2, r_3, \ldots, r_n, r_1), \ldots, \mathbf{a}_n = (r_n, r_1, \ldots, r_{n-2}, r_{n-1})$$

and note that

$$\varphi(u_1, \ldots, u_n) = f \circ g^{-1}([g(u_1) + \ldots + g(u_n)]/n)$$

satisfies (A) with $I = \{1, \ldots, n\}$. If $f \circ g^{-1}$ is strictly convex, then φ satisfies (A^*), and the inequality is strict unless all the $\mathbf{a}_k$ are similarly ordered, in which case $r_1 = \ldots = r_n$.

(4.9) For $\varphi(x, y) = (x + y)^p$ with real $p > 0$ we have:

(i) $(\mathbf{a}^* + \mathbf{b}')^p \ll (\mathbf{a} + \mathbf{b})^p \ll (\mathbf{a}^* + \mathbf{b}^*)^p$ if $p > 1$,

(ii) $\sum_{j=1}^{m} (a_j^* + b_j^*)^p \leqq \sum_{j=1}^{m} (a_j + b_j)^p \leqq \sum_{j=1}^{m} (a_j^* + b_j')^p$ if $p < 1$,

whenever $a_m^* + b_m^* \geqq 0$. The inequalities are strict except as indicated in (3.2) and (3.3). If p is an integer, then (i) holds for all $\mathbf{a}, \mathbf{b} \in \mathbf{R}^m$. The example $\mathbf{a} = (1, 2, 3)$, $\mathbf{b} = (3, 1, 2)$ shows that relation $\ll$ cannot be used in (ii).

5. The continuous case. In this section we show how to generalize Theorems (3.1) and (3.2) for L^∞ functions on a finite measure space (X, Λ, μ) when φ is jointly continuous. If $f_1, \ldots, f_n \in L^\infty$ and $\varphi : I_{f_1} \times \ldots \times I_{f_n} \to \mathbf{R}$ is bounded, then the function $\varphi(f_1, \ldots, f_n)$ defined by $x \mapsto \varphi(f_1(x), \ldots, f_n(x))$ is in L^∞. If $\{I, J\}$ is a partition of $\{1, \ldots, n\}$ then $(\delta_{\mathbf{f}_I}, \iota_{\mathbf{f}_J})$ denotes $(g_1, \ldots, g_n)$ where $g_i = \delta_{f_i}$ for $i \in I$ and $g_i = \iota_{f_i}$ for $i \in J$.

(5.1) THEOREM. *Let $\varphi : T_1 \times \ldots \times T_n \to \mathbf{R}$ be continuous, where $T_1, \ldots, T_n$ are intervals of $\mathbf{R}$, and let $\{I, J\}$ be a partition of $\{1, \ldots, n\}$.*

(i) *If φ satisfies condition (A) then*

$$(1) \qquad \int \varphi(f_1, \ldots, f_n) d\mu \leqq \int_0^\alpha \varphi(\delta_{\mathbf{f}_I}, \iota_{\mathbf{f}_J})$$

for all $f_i \in L^\infty$ such that $I_{f_i} \subset T_i$, $i = 1, \ldots, n$. If (X, Λ, μ) is non-atomic, then (A) is necessary for (1).

(ii) *If φ satisfies (A^*) then the following are equivalent:*

(a) *Equality holds in* (1).

(b) *f_i and f_j are similarly ordered whenever i and j are in the same set I or J, and oppositely ordered whenever i and j are in different sets I and J for all $1 \leqq i, j \leqq n$.*

(c) $\varphi(f_1, \ldots, f_n) \sim \varphi(\delta_{\mathbf{f}_I}, \iota_{\mathbf{f}_J})$.

(iii) *If φ satisfies (A) and is increasing (respectively decreasing) in u_i for $i \in I$ and decreasing (respectively increasing) for $i \in J$, then for all f_i as in* (i) *we have*

$$\varphi(f_1, \ldots, f_n) \ll \varphi(\delta_{\mathbf{f}_I}, \iota_{\mathbf{f}_J}).$$

(5.2) COROLLARY. *Let $\varphi : T_1 \times T_2 \to \mathbf{R}$ be continuous, where T_1 and T_2 are intervals of $\mathbf{R}$, and let $f, g \in L^\infty$ with $I_f \subset T_1$ and $I_g \subset T_2$.*

(i) *If $\Delta_{c,d}\varphi(y)$ is increasing in $y \in T_2$ whenever $d > c$ and $d, c \in T_1$, then*

$$(1) \qquad \int_0^\alpha \varphi(\delta_f, \iota_g) \leqq \int \varphi(f, g) d\mu \leqq \int_0^\alpha \varphi(\delta_f, \delta_g).$$

(ii) *If* $\Delta_{c,d}\varphi$ *is strictly increasing, then the following are equivalent:* (a) *Equality occurs in* (1) *on the left* (*right*)*;* (b) *f and g are oppositely* (*similarly*) *ordered;* (c) $\varphi(\delta_f, \iota_g) \sim \varphi(f, g)$ $(\varphi(\delta_f, \delta_g) \sim \varphi(f, g))$.

(iii) *If in addition to* (i) φ *is increasing in both variables or decreasing in both variables, then*

$$\varphi(\delta_f, \iota_g) \ll \varphi(f, g) \ll \varphi(\delta_f, \delta_g). \tag{2}$$

(5.3) *Remark.* The conditions that the inequalities in (5.2) reverse are the same as in (3.3). If φ satisfies these conditions, then (5.2) may be applied to $\varphi_1(x, y) = \varphi(x, r + s - y)$, f, and $g_1 = r + s - g$, where $I_g = [r, s]$.

We begin by showing that it suffices to prove (5.1) and (5.2) for non-atomic measure spaces by embedding (X, Λ, μ) in a non-atomic m.s. $(X^\#, \Lambda^\#, \mu^\#)$. (See [**9**] or [**2**] for details of this method.) If $f \in M(X, \mu)$, then the corresponding member of $M(X^\#, \mu^\#)$ is denoted by $f^\#$. Then $\varphi(f_1^\#, \ldots, f_n^\#) = \varphi(f_1, \ldots, f_n)^\# \sim \varphi(f_1, \ldots, f_n)$. In addition it is not hard to see that f and g are similarly (oppositely) ordered if and only if $f^\#$ and $g^\#$ are similarly (oppositely) ordered. Thus if (5.1) and (5.2) are true when (X, Λ, μ) is non-atomic, then they are true for any finite m.s.

Before proceeding with the proof when (X, Λ, μ) is non-atomic, we require some lemmas. The first two are needed when the measure space is not separable, for otherwise it is measure theoretically $[0, \alpha]$.

(5.4) LEMMA. *Let* (X, Λ, μ) *be non-atomic. Suppose* $\{D_k\}_{k=1}^N$ *is a partition of X by measurable sets. If* $\epsilon > 0$, *then there is a partition* $\{E_i\}_{i=1}^n$ *of X by measurable sets such that* $\mu(E_i) = \mu(X)/n$ $(i = 1, \ldots, n)$ *and* $\mu(\bigcup\{E_i : E_i$ *intersects more than one* $D_k\}) < \epsilon$.

Proof. Let $\alpha = \mu(X)$. If $\alpha = 0$, the lemma is trivially true. Otherwise, rename the sets D_k so that $\mu(D_k) = 0$ for $1 \leqq k < p$ and $\mu(D_k) > 0$ for $p \leqq k \leqq N$. There is a $\phi : [0, \alpha] \to \Lambda$ such that $\mu(\phi(t)) = t$, $t \leqq u$ implies $\phi(t) \subset \phi(u)$, $\phi(0) = \bigcup_{1 \leqq k < p} D_k$, and $\phi(\sum_{1 \leqq k \leqq q} \mu(D_k)) = \bigcup_{1 \leqq k \leqq q} D_k$ for $q = p, \ldots, N$ (use [**2**, (5.6)]). For any n such that $\alpha/n \leqq \min\{\mu(D_k) : p \leqq k \leqq N\}$ and for $E_i = \phi(\alpha i/n) - \phi(\alpha(i-1)/n)$ $(i = 1, \ldots, n)$ we have that each E_i intersects at most two sets D_k of positive measure, and at most N $- 1$ of these E_i intersect more than one D_k. To finish the proof, choose n so that also $\alpha(N - 1)/n < \epsilon$.

(5.5) LEMMA. *Suppose* (X, Λ, μ) *is non-atomic. Let* $\{s(k)_i\}_{i=1}^\infty$ $(k = 1, \ldots, n)$ *be n sequences of simple functions. Then there are n sequences* $\{t(k)_i\}_{i=1}^\infty$, $(k = 1, \ldots, n)$ *of simple functions such that*

(i) *For each i,* $t(1)_i, \ldots, t(n)_i$ *have the same sets of constancy, and these sets have equal measure;*

(ii) *For each* $k = 1, \ldots, n$, $s(k)_i - t(k)_i \to 0$ μ*-almost everywhere as* $i \to \infty$*;*

(iii) *For each* $k = 1, \ldots, n$ *and* $i \geqq 1$, $|t(k)_i| \leqq |s(k)_i|$.

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Proof. For clarity of exposition, we prove the lemma in the case $n = 2$. The proof for larger n will be readily apparent. Before considering sequences, let $s(1) = \sum_{i=1}^{n} a_i 1_{A_i}$ and $s(2) = \sum_{j=1}^{p} b_j 1_{B_j}$ where $\{A_i\}$ and $\{B_j\}$ partition X, and let $\{D_k\}_{k=1}^{N} = \{A_i \cap B_j : 1 \leqq i \leqq n, 1 \leqq j \leqq p\}$. Let $\epsilon > 0$. Then there is a measurable partition $\{E_q\}_{q=1}^{r}$ as in Lemma (5.4). For each $q = 1, \ldots, r$, if E_q intersects only $A_i \cap B_j$ then $E_q \subset A_i \cap B_j$, and for $k = 1, 2$ we define $t(k)|E_q = s(k)|(A_i \cap B_j)$; we define $t(k) = 0$ elsewhere. Then $|t(k)| \leqq |s(k)|$ and $\mu(\{s(k) \neq t(k)\}) < \epsilon$. Hence given $\{s(k)_i\}_{i=1}^{\infty}$ there are sequences $\{t(k)_i\}_{i=1}^{\infty}$ satisfying (i) and (iii) such that $\mu(\{s(k)_i \neq t(k)_i\}) < 2^{-i}$. Then

$$\mu(\{s(k)_i - t(k)_i \not\to 0\}) = \mu\left(\bigcup_{q=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{i=N}^{\infty} \{|s(k)_i - t(k)_i| > 1/q\}\right) \leqq \lim_{q\to\infty} \lim_{N\to\infty} \sum_{i=N}^{\infty} 2^{-i} = 0,$$

and the proof is finished.

(5.6) PROPOSITION. *Suppose (X, Λ, μ) is non-atomic, let $f_1, \ldots, f_n \in M(X, \mu)$, let $\{I, J\}$ be a partition of $\{1, \ldots, n\}$, and let $F_1, \ldots, F_n \in [0, \alpha]$ with F_i right continuous and decreasing (increasing) when $i \in I$ ($i \in J$). Then the following three conditions are equivalent.*

(i) $(f_1, \ldots, f_n) \sim (F_1, \ldots, F_n)$.

(ii) *There is a measure preserving map $\sigma : X \to [0, \alpha]$ such that $F_i \circ \sigma = f_i$ μ-almost everywhere, $1 \leqq i \leqq n$.*

(iii) *f_i and f_j are similarly ordered when i and j are in the same set I or J, oppositely ordered when i and j are in different sets I and J, and $F_i = \delta_{f_i}$ for $i \in I$, $F_j = \iota_{f_j}$ for $j \in J$.*

Proof. Let $A \subseteq B[\mu]$ mean $\mu(A \backslash B) = 0$, i.e., $1_A \leqq 1_B$ μ-almost everywhere. Writing $\mathbf{f} = (f_1, \ldots, f_n)$, $\mathbf{F} = (F_1, \ldots, F_n)$, and $\mathbf{t} = (t_1, \ldots, t_n)$, the proof given in [**2**, Theorem (6.2)] shows (i) $\Rightarrow$ (ii). Also, (ii) $\Rightarrow$ (iii) is straightforward.

We prove (iii) $\Rightarrow$ (i) first in the case $J = \emptyset$.

I. If f and g are similarly ordered, then for all $t \in \mathbf{R}$, ess.sup $g|\{f \leqq t\} \leqq$ ess.inf $g|\{f > t\}$. This follows from ess.sup $g|\{f > t + 1/n\} \to$ ess.sup $g|\{f > t\}$ as $n \to \infty$.

II. If f and g are similarly ordered, and $t, u \in \mathbf{R}$, then $\{f > t\} \cap \{g > u\} = \{f > t\}$ or $\{g > u\}$ $[\mu]$. Indeed, let

$$A = \{f \leqq t\} \cap \{g > u\}, \qquad B = \{f > t\} \cap \{g \leqq u\},$$

and suppose both $\mu(A), \mu(B) > 0$. Then ess.inf $g|B \leqq u <$ ess.sup $g|A$, while (I) implies ess.sup $g|A \leqq$ ess.inf $g|B$, a contradiction. Hence $\mu(A) = 0$ or $\mu(B) = 0$.

III. If $\{f > t\} \subset \{g > u\}$ $[\mu]$ then $\{\delta_f > t\} \subset \{\delta_g > u\}$. Indeed, $\{\delta_f > t\} = [0, \mu\{f > t\}[\subset [0, \mu\{g > u\}[= \{\delta_g > u\}$.

IV. It follows from (II) and (III) that for all $\mathbf{t} \in \mathbf{R}^n$, $\mu\{\mathbf{f} > \mathbf{t}\} = \mu(\cap \{f_i > t_i\}) = m(\cap \{\delta_{f_i} > t_i\}) = m\{\mathbf{F} > \mathbf{t}\}$, so $\mathbf{f} \sim \mathbf{F}$.

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To deduce the general case from this one, let $\varphi(t_1, \ldots, t_n) = (u_1, \ldots, u_n)$, where $u_i = t_i$ if $i \in I$, $= -t_i$ if $i \in J$, let $(f_1', \ldots, f_n') = \varphi(f_1, \ldots, f_n)$, and let $F_i' = \delta_{f_i'}$. By the $J = \emptyset$ case, $\mathbf{F}' \sim \mathbf{f}'$, so $\mathbf{F} = \varphi(\mathbf{F}') \sim \varphi(\mathbf{f}') = \mathbf{f}$ (because $\delta_{-f} = -\iota_f$).

We can now prove (5.1) and (5.2). For clarity of exposition we will only present a proof of (5.2). The proof of (5.1) will then be clear. With regard to (5.1.ii) we remark that (5.6) shows that (b) $\Rightarrow$ (c) $\Rightarrow$ (a) always. The proof of (5.2) will illustrate the proof of (a) $\Rightarrow$ (b) when $n = 2$.

Proof of (5.2). Let $v = \sum_{j=1}^m a_j 1_{E_j}$ and $w = \sum_{j=1}^m b_j 1_{E_j}$, where $a_j \in T_1$, $b_j \in T_2$ $(1 \leqq j \leqq m)$ and $\mu(E_j) = \alpha/m$. In case (i), (3.2.i) gives

$$(*) \qquad \int_0^\alpha \varphi(\delta_v, \iota_w) \leqq \int \varphi(v, w) d\mu \leqq \int_0^\alpha \varphi(\delta_v, \delta_w)$$

while in case (ii), (3.2.iii) gives for $t = k\alpha/p$ $(1 \leqq k \leqq m)$

$$(**) \qquad \int_0^t \delta_{\varphi(\delta_v, \iota_w)} \leqq \int_0^t \delta_{\varphi(v,w)} \leqq \int_0^t \delta_{\varphi(\delta_v, \delta_w)}.$$

Now in (**) each of the integrands is constant on each of the intervals $[(j-1)\alpha/n, j\alpha/n[$, so the integrals are linear functions of t on these intervals, and hence (**) holds for all $0 \leqq t \leqq \alpha$. Using now (5.5) there are sequences v_i and w_i of simple functions like v and w above such that $v_i \to f$, $w_i \to g$, $|v_i| \leqq |f|$ and $|w_i| \leqq |g|$, so $\delta_{v_i} \to \delta_f$ and $\delta_{w_i} \to \delta_g$ almost everywhere. Since φ is bounded on $I_f \times I_g$, each integrand in (*) or (**) is bounded by a constant depending only on f and g. Taking limits and using the dominated convergence theorem, we have that (*) or (**) holds with v and w replaced by f and g respectively.

We now show the condition for equality on the right in (3.2.i.1). Assume φ satisfies (A^*), suppose f and g are not similarly ordered, and we will show that the inequality on the right is strict. There are disjoint sets A and B of positive measure such that

$$\text{ess.sup}\, f|A < \text{ess.inf}\, f|B \qquad \text{and} \qquad t = \text{ess.sup}\, g|A > \text{ess.inf}\, g|B = r.$$

Let $r < s_1 < s_2 < t$ and let

$$D \subset \{x \in A : g(x) \geqq s_2\} \qquad \text{and} \qquad E \subset \{x \in B : g(x) \leqq s_1\}$$

with $0 < \mu(D) = \mu(E) = \beta$. Then let $\sigma_D : D \to [0, \beta[$ and $\sigma_E : E \to [0, \beta[$ be measure preserving and define

$$f' = \delta_{f|D} \circ \sigma_D \text{ on } D, = \delta_{f|E} \circ \sigma_E \text{ on } E, \text{ and } = f \text{ elsewhere};$$

$$g' = \delta_{g|E} \circ \sigma_D \text{ on } D, = \delta_{g|D} \circ \sigma_E \text{ on } E, \text{ and } = g \text{ elsewhere}.$$

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Then $f' \sim f$, $g' \sim g$, $\delta_{f|D} < \delta_{f|E}$, and $\delta_{g|E} < \delta_{g|D}$. Hence

$$\int_D \varphi(f,g)d\mu + \int_E \varphi(f,g)d\mu \leqq \int_0^\beta [\varphi(\delta_{f|D}, \delta_{g|D}) + \varphi(\delta_{f|E}, \delta_{g|E})]$$
$$< \int_0^\beta [\varphi(\delta_{f|D}, \delta_{g|E}) + \varphi(\delta_{f|E}, \delta_{g|D})]$$
$$= \int_D \varphi(f', g')d\mu + \int_E \varphi(f\ , g')d\mu.$$

Adding

$$\int_{X-(D\cup E)} \varphi(f,g)d\mu = \int_{X-(D\cup E)} \varphi(f', g')d\mu$$

we obtain

$$\int \varphi(f,g)d\mu < \int \varphi(f', g')d\mu \leqq \int_0^\alpha \varphi(\delta_{f'}, \delta_{g'}) = \int_0^\alpha \varphi(\delta_f, \delta_g),$$

and the proof is finished.

(5.7) *Remark*. Depending on the choice of φ and the intervals T_i, Theorems (5.1) and (5.2) may hold for a larger set of functions than L^∞. Indeed, the proof shows that in (5.2) inequalities (1) or (2) will hold whenever limit and integral can be interchanged in (*) or (**). The condition for equality holds if (5.2.1) holds for $f|A$ and $g|A$ for all $A \in \Lambda$ whenever it holds for f and g.

For example, suppose $f_1, \ldots, f_m \in L^p$ implies $\varphi(f_1, \ldots, f_m) \in L^1$. Now it follows from [**9**, p. 93] that $|v| \leqq |f|$ implies $|\delta_v| \leqq |\delta_f|$ and $|\iota_v| \leqq |\iota_f|$, so we may use [**3**] and the dominated convergence theorem to conclude that (5.1.1) and (5.2.1) hold for all L^p functions. Finally, since $f_1, \ldots, f_m \in L^p$ implies $f_1|A, \ldots, f_m|A \in L^p$, the condition for equality also holds for all L^p functions. Other illustrations appear in the following examples.

6. Examples for the continuous case.

(6.1) (i) $\delta_f + \iota_g \prec f + g \prec \delta_f + \delta_g$ for all $f, g \in L^1$.

(ii) $\delta_f - \delta_g \prec f - g \prec \delta_f - \iota_g$ for all $f, g \in L^1$.

The (i) and (ii) are easily seen to be equivalent using [**9**, p. 93]. While $\delta_{f+g} \prec \delta_f + \delta_g$ is well-known (see [**9**, p. 108]), the fact that $\delta_f - \delta_g \prec f - g$ is new. Then a theorem of Luxemburg [**9**, p. 107] implies $|\delta_f - \delta_g| \ll |f - g|$, generalizing [**8**, Proposition 1, p. 34]. It then follows that $||f_\beta - f||_1 \to 0$ implies $||\delta_{f_\beta} - \delta_f||_1 \to 0$, where $\{f_\beta\}$ is a net. Using [**9**, (9.1)], the inequality $\delta_f - \delta_g \prec f - g$ can be written equivalently:

$$\int_E \delta_f + \int_E \delta_g(\alpha - t)dt \leqq \int_0^{m(E)} \delta_{f+g}$$

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for all Lebesgue measurable $E \subset [0, \alpha]$, where m denotes Lebesgue measure. This is an interesting generalization of [**9**, (10.1)].

(6.2) An inequality of Hardy-Littlewood-Polya-Luxemburg:

$$\int_0^\alpha \delta_f \iota_g \leqq \int fg \, d\mu \leqq \int_0^\alpha \delta_f \delta_g$$

holds for all $f, g \in L^\infty$, and, using monotone convergence, it is easily seen to hold for all $0 \leqq f, g \in M$. Then as in [**9**, p. 102], it may be shown to hold whenever $\delta_{|f|}\delta_{|g|} \in L^1[0, \alpha]$. The inequalities are strict except as indicated in (5.2). Similarly, $\delta_f \iota_g \ll fg \ll \delta_f \delta_g$ for all $0 \leqq f, g \in M$ such that $\delta_f \delta_g \in L^1[0, \alpha]$.

$$\text{(6.3) (i)} \quad \int_0^\alpha \log(1 + \delta_f \iota_g) \leqq \int \log(1 + fg) d\mu \leqq \int_0^\alpha \log(1 + \delta_f \delta_g)$$

holds for all $f, g \in L^\infty$ satisfying both

$$\text{(ii)} \qquad \delta_f(0)\iota_g(0) > -1 \text{ and } \delta_f(\alpha-)\iota_g(\alpha-) > -1,$$

because (ii) is equivalent to: $I_f \times I_g \subset \{(x, y) : xy > -1\}$. In addition, using monotone convergence, (i) can be shown to hold if $0 \leqq f, g \in M$ or $0 \geqq f, g \in M$. Then (i) can be shown to hold for all $f, g \in M$ satisfying (ii) using the following observations. First, $\log(1 + fg) = \log(1 + f^+g^+) + \log(1 - f^+g^-) + \log(1 - f^-g^+) + \log(1 + f^-g^-)$. Next, when (ii) holds for the pair f, g it also holds for each of the pairs: f^+, g^+; $f^+, -g^-$; $-f^-, g^+$; $-f^-, -g^-$. Finally, when (ii) holds, then: f unbounded above implies $g \geqq 0$; f unbounded below implies $g \leqq 0$; and the same is true when f and g are interchanged. Clearly if $f, g \in M$ satisfy (ii) so do $f|A$ and $g|A$ for any $A \in \Lambda$. Hence the inequalities are strict as indicated in (5.2).

Similarly, $\log(1 + \delta_g \iota_g) \ll \log(1 + fg) \ll \log(1 + \delta_f \delta_g)$ for all $0 \leqq f, g \in M$ or $0 \geqq f, g \in M$ such that $\log(1 + \delta_f \delta_g) \in L^1[0, \alpha]$.

$$\text{(6.4) (i)} \quad \int_0^\alpha \log(\delta_f + \delta_g) \leqq \int \log(f + g) d\mu \leqq \int_0^\alpha \log(\delta_f + \iota_g)$$

for all $f, g \in L^\infty$ such that

$$\text{(ii)} \qquad \delta_f(\alpha-) + \delta_g(\alpha-) > 0,$$

since (ii) is equivalent to $I_f \times I_g \subset \{(x, y) : x + y > 0\}$. Actually, (i) holds for all $f, g \in M$ satisfying (ii) since f and g are then bounded below, so we may approximate them by increasing sequences of bounded functions satisfying (ii) and use the B. Levi monotone convergence theorem [**5**, p. 172]. The inequalities are strict except as indicated in (5.3). Similarly, if $f, g \in M$ satisfy (ii) and $\log(\delta_f + \iota_g) \in L^1[0, \alpha]$ then $-\log(\delta_f + \iota_g) \ll -\log(f + g) \ll -\log(\delta_f + \delta_g)$.

(6.5) We have the following continuous version of London's Theorems. Suppose $0 \leqq f, g \in M$ or $0 \geqq f, g \in M$.

(i) If H is convex, increasing and continuous on $[0, \infty[$, then

$$\int_0^\alpha H(\delta_f \iota_g) \leqq \int H(fg)d\mu \leqq \int_0^\alpha H(\delta_f \delta_g).$$

(ii) If $H(e^x)$ is convex, increasing and continuous on $[0, \infty[$, then

$$\int_0^\alpha H(1 + \delta_f \iota_g) \leqq \int H(1 + fg)d\mu \leqq \int_0^\alpha H(1 + \delta_f \delta_g).$$

In either case, if H is strictly convex, then we have equality on the left (right) if and only if f and g are oppositely (similarly) ordered if and only if $\delta_f \iota_g \sim fg$ $(\delta_f \delta_g \sim fg)$.

(6.6) For real $p > 0$ we have:

(i) $(\delta_f + \iota_g)^p \ll (f + g)^p \ll (\delta_f + \delta_g)^p$ if $p > 1$,

(ii) $\displaystyle\int_0^\alpha (\delta_f + \delta_g)^p \leqq \int (f + g)^p d\mu \leqq \int_0^\alpha (\delta_f + \iota_g)^p$ if $p < 1$,

whenever (a) $\delta_f(\alpha -) + \delta_h(\alpha -) \geqq 0$ and $f, g \in L^p$; or (b) $0 \leqq f, g \in M$; or (c) p is an integer and $f, g \in L^p$. The (i) gives a lower bound to an inequality of Chong and Rice [**2**, p. 88]. The inequalities are strict except as indicated in (5.2) and (5.3).

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