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THE CONVERGENCE OF EULER PRODUCTS OVER p-ADIC NUMBER FIELDS

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Abstract We define a topological space over the p-adic numbers, in which Euler products and Dirichlet series converge. We then show how the classical Riemann zeta function has a (p-adic) Euler product structure at the negative integers. Finally, as a corollary of these results, we derive a new formula for the non-Archimedean Euler–Mascheroni constant.

Keywords: number theory; zeta functions; Iwasawa theory; p-adic L-functions; Euler products

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Secondary 11M41; 11R60

1. Introduction

The Riemann zeta function must be one of the most studied objects in mathematics. It is an analytic function on the punctured plane $\mathbb{C} - \{1\}$ with a simple pole at 1. The initial description of zeta is as a Dirichlet series

$$\zeta_\infty(\sigma) = \sum_{n=1}^{\infty} n^{-\sigma}, \quad \text{where } \operatorname{Re}(\sigma) > 1.$$

Furthermore, in the same right half-plane it admits an Euler product expansion

$$\zeta_\infty(\sigma) = \prod_{\text{primes } l} \frac{1}{1 - l^{-\sigma}}.$$

Famously, the difference between the Riemann zeta function at $\sigma = 1$ and the area under the real curve $f(x) = x^{-1}$ is measured by the Euler–Mascheroni constant γ. There are many formulae describing how to compute it numerically.

Let p be a prime number. The p-adic analogue of $\zeta_\infty(\sigma)$ was constructed by Kubota and Leopoldt in the 1950s, exploiting earlier congruences due to Kummer. In this paper, we are interested in addressing the following ugly anomaly.

Question 1.1. Why does the p-adic zeta function of Kubota and Leopoldt appear to be nothing like the classical Riemann zeta function $\zeta_\infty(-)$ mentioned above?

Before we look for an answer, let us first describe the Kubota–Leopoldt L-function. Fix an odd prime number p, and let $\chi : \mathbb{Z} \to \bar{\mathbb{Q}}$ denote any Dirichlet character. Via embeddings $\iota_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, one may view χ as taking both complex and p-adic values. We shall write ω for the Teichmüller character, which maps $x \in \mathbb{Z}_p^\times$ to the unique $(p-1)$st root of unity congruent to x modulo p.

By its very definition, $\boldsymbol{L}_p(s,\chi)$ is a continuous function on the disc $s \in \mathbb{C}_p$, $|s|_p < p^{(p-2)/(p-1)}$ satisfying the interpolation property

$$\boldsymbol{L}_p(1-n,\chi) = \iota_p \circ (\iota_\infty)^{-1}((1-\chi\omega^{-n}(p)p^{n-1}) \times L_\infty(1-n,\chi\omega^{-n}))$$

at all integers $n \geqslant 1$.

Whenever χ is an odd character, i.e. $\chi(-1) = -1$, this interpolation forces $\boldsymbol{L}_p(s,\chi)$ to be identically zero. It is rather annoying to keep writing $\iota_p \circ (\iota_\infty)^{-1}$ and we will drop it altogether in what follows.

The aims of the paper

For each prime $p \neq 2$ and character χ, we aim to prove that

$(\mathrm{A})_p$ the p-adic zeta function $\boldsymbol{L}_p(s,\chi)$ is equal to a Dirichlet series over $\mathbb{C}_p$,

$(\mathrm{B})_p$ the p-adic zeta function possesses an Euler product expansion over $\mathbb{C}_p$,

$(\mathrm{C})_p$ there exists an explicit formula for the p-adic Euler–Mascheroni constant.

The naive approach one might adopt is the following. First normalize the p-adic logarithm so that $\log_p(p) = 0$, and let us write

$$\exp_p(X) = \sum_{m \geqslant 0} \frac{X^m}{m!}$$

for the exponential map.

For all $x \in \mathcal{O}_{\mathbb{C}_p}^\times$, define the function $\langle x \rangle^s := \exp_p(s \log_p(x))$ which converges everywhere on the disc $|s|_p < p^{(p-2)/(p-1)}$. In particular, if $x \in \mathbb{Z}_p^\times$, then $\langle x \rangle^1 = x\omega^{-1}(x)$. We now consider the partial sums/products

$$\sum_{\substack{m=1,\\ p \nmid m}}^{N} \chi(m)\langle m \rangle^{-s} \quad \text{and} \quad \prod_{\substack{\text{primes } l,\\ l \neq p,\, l \leqslant N}} \frac{1}{1-\chi(l)\langle l \rangle^{-s}}.$$

Do either of these converge p-adically as $N \to \infty$?

Clearly, the norm $\|\chi(m)\langle m \rangle^{-s}\| = 1$, and thus the terms in each successive partial summation are *not* tending towards zero. If $\mathfrak{f}_\chi$ denotes the conductor of χ, then

$$\begin{aligned} -\mathrm{order}_{s=0}\biggl(\prod_{l \leqslant N} \frac{1}{1-\chi(l)\langle l \rangle^{-s}}\biggr) &= \#\{\text{primes } l \leqslant N \text{ such that } \chi(l) = 1\} \\ &\geqslant \#\{\text{primes } l \leqslant N \text{ such that } l \equiv 1 \pmod{\mathfrak{f}_\chi}\} \\ &\approx \frac{N}{\phi(\mathfrak{f}_\chi)\log N}, \end{aligned}$$

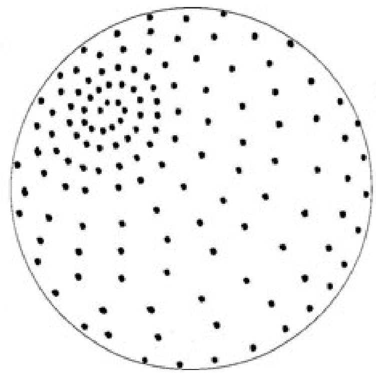

Figure 1. The ∞-polydisc over $\boldsymbol{X}$ which has unit radius showing the contour traced out by $\xi_m \mapsto \omega^{\beta}(m)\langle m\rangle^{-s}$ as s varies ($\underline{X} = (X_0, X_2, X_3, X_5, \dots)$).

which slowly tends to ∞ with N. As a consequence, the infinite product $\prod_{\text{primes } l}(\cdots)$ picks up pole after pole at $s = 0$, and cannot possibly tend to a rigid meromorphic function.

2. The main results

Essentially the method failed because $\langle m\rangle^{-s}$ did not get any smaller as $m \to \infty$. The p-adic topology can see neither the sum nor the product structure in zeta. Making these things converge over the Tate field would appear to be impossible! Since the usual topology has let us down, we replace it with a non-standard one.

In this section and primarily the next, we shall consider a topological space of 'shadow ∇-functions' which exists as a dense subset inside the Iwasawa functions. It is generated over the p-adics by elements ξ_m for each $m \in \mathbb{N}$, viewed as avatars of the complex functions $m^{-\sigma}$ with $\mathrm{Re}(\sigma) > 1$. In particular, ξ_m is small in the shadow topology whenever the integer m is large. In this context, it makes good sense to speak about convergence of Dirichlet series and Euler products.

For each choice of β modulo $p-1$ there is a boundary map ${}_{\beta}\nabla_p$ discontinuously injecting our space of shadow ∇-functions into the larger ring of Iwasawa functions; on finite sums, it sends

$$ {}_{\beta}\nabla_p : \sum_{m=1}^{N} c_m \xi_m \mapsto \sum_{\substack{m=1,\\ p\nmid m}}^{N} c_m \omega^{\beta}(m)\langle m\rangle^{-s} \in \mathbb{C}_p\langle\langle s\rangle\rangle. $$

We refer the reader to § 3 for full details of the construction of the shadow space.

As usual, one writes ϕ for Euler's totient function and μ for the Möbius function. In particular, we define $\hat{\Gamma}_{\chi} := \mathrm{Hom}((\mathbb{Z}/2\mathfrak{f}_{\chi}\mathbb{Z})^{\times}, \mathbb{G}_{\mathrm{m}})$ to be the group of characters with conductors dividing $2\mathfrak{f}_{\chi}$.

Hypothesis 2.1. *Throughout we assume that* $\gcd(p, 2\mathfrak{f}_{\chi} \cdot \phi(\mathfrak{f}_{\chi})) = 1$.

At every divisor k of $2\mathfrak{f}_\chi$ and $\Psi \in \hat{\Gamma}_\chi$, denote by $E_k^{(\Psi)}(\chi)$ the algebraic numbers

$$\sum_{d|\gcd(k,2\mathfrak{f}_\chi/\mathfrak{f}_\Psi)} \frac{\mu(k/d)\Psi(k/d)}{\phi(2\mathfrak{f}_\chi/d)} \times \sum_{\substack{c=1,\\ \gcd(c,2\mathfrak{f}_\chi)=1}}^{2\mathfrak{f}_\chi/d} \Psi^{-1}(c)\Bigg(\sum_{j=\lfloor(cd-1)/2\rfloor+1}^{cd-1} \chi(j) - \sum_{j=1}^{\lfloor(cd-1)/2\rfloor} \chi(j)\Bigg).$$

These scalars occur as structure constants, for the decomposition of the χ-twisted p-adic L-function in terms of shadow Euler products.

Definition 2.2. The χ-twisted zeta-element is defined by the formula

$$\begin{aligned}\Theta^{\mathbf{Hu}} \otimes \chi := \Bigg(L_\infty(0,\chi) + \sum_{k|2\mathfrak{f}_\chi} E_k^{(\mathbf{1})}(\chi)\xi_k\Bigg) \times \prod_{\text{primes } l} \frac{1}{1-\xi_l} \\ + \sum_{\Psi\in\hat{\Gamma}_\chi, \Psi\neq\mathbf{1}} \Bigg(\sum_{k|2\mathfrak{f}_\chi} E_k^{(\Psi)}(\chi)\xi_k\Bigg) \times \prod_{\text{primes } l} \frac{1}{1-\Psi(l)\xi_l}.\end{aligned}$$

Theorem 2.3. *The element $\Theta^{\mathbf{Hu}} \otimes \chi$ exists in the space of shadow ∇-functions, and under the boundary map*

$${}_\beta\nabla_p(\Theta^{\mathbf{Hu}} \otimes \chi) = (2\omega^\beta(2)\langle 2\rangle^{-s} - 1) \times \boldsymbol{L}_p(s, \chi\omega^{1+\beta})$$

for each branch β modulo $p-1$.

This is weaker than $(\mathrm{B})_p$ as it indicates that the p-adic L-function is a sum of at most $\phi(2\mathfrak{f}_\chi)$ Euler products, rather than a single one (much like a Hurwitz zeta function). The proof of Theorem 2.3 for general Dirichlet characters χ is based upon the following estimate coming from fractional calculus.

Theorem 2.4. *Along every branch β mod $p-1$, we have the equality of functions*

$$\boldsymbol{L}_p(s, \chi\omega^{1+\beta}) = (2\omega^\beta(2)\langle 2\rangle^{-s} - 1)^{-1} \times \lim_{t\to\infty} \Bigg(\sum_{\substack{m=1,\\ p\nmid m}}^{p^{t\phi(2\mathfrak{f}_\chi)}} a_m(\chi)\omega^\beta(m)\langle m\rangle^{-s}\Bigg),$$

where each coefficient $a_m(\chi) := L_\infty(0,\chi) + \sum_{j=1}^{m-1}\chi(j) - 2\sum_{j=1}^{\lfloor(m-1)/2\rfloor}\chi(j)$.

This result is important in its own right, and shows that every p-adic L-function attached to an abelian field can be expanded in a Dirichlet series. Thus, property $(\mathrm{A})_p$ is always satisfied by these functions. The terms $a_m(\chi)$ are periodic of modulus $2\mathfrak{f}_\chi$ and intriguingly depend not on the choice of prime p, but only on the character χ.

Remark 2.5. As the referee pointed out, when $\chi = \mathbf{1}$ and $\beta = -1$ the above formula is quite close to one found by Iwasawa [**6**] using Stickelberger elements, where the twisting to make the elements integral is at $c = 2$. While a relatively easy argument shows both formulae to be congruent modulo p^{t-1}, we were unable to show required equivalence modulo p^t (a short proof of this assertion would be nice).

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Certainly Iwasawa's methods are a lot more elementary than the calculations we undertake here, our intention being very much a 'from first principles' approach. The main advantage of using fractional differentiation and reciprocity is that one obtains Dirichlet series expansions whose coefficients are stable from the outset, without requiring any subsequent fiddly modifications.

As a useful illustration of Theorem 2.3, consider the situation when $\chi = \mathbf{1}$ is the trivial character. The element $\Theta^{\mathbf{Hu}} \otimes \mathbf{1}$ is composed of just a single Euler product. On the other hand, at non-positive integers $s = -k$ the p-adic function $\boldsymbol{L}_p(s, \omega^{1+\beta})$ interpolates the critical values $L_\infty(-k, \omega^{\beta-k})$ of its complex counterpart.

Corollary 2.6. *Writing ∂_p^k for the specialization ${}_k\nabla_p|_{s=-k}$, we have the equality*

$$\zeta_\infty(-k) = \frac{1}{(2^{1+k}-1)(1-p^k)} \times \partial_p^k\left((\xi_2 - \tfrac{1}{2}) \times \prod_{\textit{primes } l} \frac{1}{1-\xi_l}\right)$$

for all integers $k \geqslant 0$.

In essence, the shape of the Riemann zeta function at negative integers is that of an Euler product over the p-adics, for every prime number $p \neq 2$. Somewhat abusively, it is tempting to write down the formula

$$(2^{1+k}-1)(1-p^k)\zeta_\infty(-k) = \text{`}(2^{+k} - \tfrac{1}{2}) \times \prod_{l \neq p} \frac{1}{1-l^{+k}}\text{'}$$

and interpret the right-hand side as converging in the space of shadow ∇-functions. With a pleasant degree of symmetry, it is precisely in the right half-plane $\mathrm{Re}(\sigma) > 1$ where $\zeta_\infty(\sigma)$ admits an expansion as a complex Euler product.

Finally, the following formula expresses the p-adic version of Euler's constant to arbitrary accuracy, and in so doing provides a favourable response to point $(\mathrm{C})_p$. By definition, γ_p is taken to be the constant term occurring in the trivial branch of the Kubota–Leopoldt L-function, i.e.

$$\boldsymbol{L}_p(s,\chi)|_{\chi=1} = \frac{1-1/p}{s-1} + \gamma_p + \text{higher-order terms in } (s-1)$$

due to the simple pole at $s = 1$.

Theorem 2.7. *Let $\epsilon_2 = \mathrm{ord}_p(\langle 2\rangle - 1)$. Then γ_p is congruent to the p-adic number*

$$\frac{(\sum_{m=1,\, p\nmid m}^{p^{2(n+\epsilon_2)}}((-1)^{m+1}/2m)\langle m\rangle^{p^{n+\epsilon_2}}) - (1-1/p)\log_p(2)(1+p^{n+\epsilon_2}\frac{1}{2}\log_p(2))}{(1-\langle 2\rangle^{p^{n+\epsilon_2}})}$$

modulo $p^n\mathbb{Z}_p$, for all integers $n \geqslant 1$.

This formula is highly reminiscent of an Archimedean version which was proved by Brent and McMillan in 1980, and relates γ to the difference between a partial harmonic series and the logarithm function.

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For example, at the first three odd primes one finds

$$\gamma_3 = 2\times 3^1 + 2\times 3^2 + 1\times 3^3 + 2\times 3^4 + 1\times 3^5 + 2\times 3^6 + O(3^7),$$
$$\gamma_5 = 1\times 5^1 + 0\times 5^2 + 3\times 5^3 + O(5^4),$$
$$\gamma_7 = 5\times 7^0 + 2\times 7^1 + 4\times 7^2 + O(7^3), \quad \text{and so on...}$$

Question 2.8. Is γ_p an irrational number?

This is still an open question in the Archimedean situation, although it is widely conjectured that $\gamma \notin \mathbb{Q}$.

The plan of the paper is as follows. We shall start by introducing the space of shadow ∇-functions in § 3, and write down some basic topological properties of it. In the next section, we prove a key equality linking $\Theta^{\mathbf{Hu}}$ with a Dirichlet expansion in the shadow space. Then in §§ 5–7 we use fractional differentiation and reciprocity to compute some brand new approximations for the χ-twisted p-adic L-function. We next explain how the ideas outlined in this paper reduce the calculation of the quadratic twist $\boldsymbol{L}_p(s,(D/\cdot))$ to the determination of a finite set of coefficients. Lastly, the missing proof of Theorem 2.7 is contained in § 9.

3. Visibility of Euler products

We begin by constructing the space of shadow ∇-functions out of a projective limit. Let $\mathcal{O}$ denote the ring of integers of a local field K of residue characteristic $p>2$. For any integer $N\geqslant 1$, we shall write Σ_N for the finite set $\{2,3,5,\dots,l_N\}$ of the first N prime numbers in order (so l_N will denote the Nth least prime).

Consider the complete Noetherian local ring

$$\boldsymbol{B}_N := \mathcal{O}[\![X_0, X_2, X_3, \dots, X_{l_N}]\!]$$

as an Iwasawa $\mathcal{O}$-algebra in $N+1$ variables, indexed by the elements of $\Sigma_N \cup \{0\}$. The sequence $\{\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3, \boldsymbol{B}_4, \dots\}$ may then be viewed as a projective system via transition maps $\varphi : \boldsymbol{B}_{N+1} \to \boldsymbol{B}_N$, induced from the deletions

$$\varphi : X_l \mapsto \begin{cases} X_l & \text{if the index } l \neq l_{N+1}, \\ 0 & \text{if the index } l = l_{N+1}. \end{cases}$$

Note that the projective limit $\boldsymbol{B}_\infty := \varprojlim_\varphi \boldsymbol{B}_N$ inherits its topology from the $\boldsymbol{B}_N$s, and consists of power series convergent on the interior of an infinite-dimensional polydisc of radius 1, parametrized by coordinates $\underline{X} = (X_0, X_2, X_3, X_5, \dots)$ (see Figure 1).

Definition 3.1. For each integer $m \geqslant 1$, we shall write ξ_m for the image of the monomial $\prod_{\text{primes } l} (X_0^l X_l)^{\mathrm{ord}_l(m)}$ inside the ring $\boldsymbol{B}_\infty$.

We define the space of $\boldsymbol{\xi}$-tempered functions $\boldsymbol{B}_\infty^\dagger$ to be the $\mathcal{O}$-subalgebra of $\boldsymbol{B}_\infty$ generated by convergent series in the ξ_ms, i.e.

$$\boldsymbol{B}_\infty^\dagger := \left\{ \theta \in \boldsymbol{B}_\infty \text{ such that } \theta \text{ is of the form } \sum_{m=1}^{\infty} c_m \xi_m \text{ with } c_m \in \mathcal{O} \right\}.$$

Listed below are some basic properties.

Lemma 3.2.

(a) $\xi_{mn} = \xi_m \xi_n$ *for all* $m, n \in \mathbb{N}$.

(b) $\lim_{m\to\infty} \xi_m = 0$.

(c) $1/(1-\xi_m) = 1 + \xi_m + \xi_{m^2} + \xi_{m^3} + \cdots$ *lies in* $\boldsymbol{B}^{\dagger}_{\infty}$.

Proof. Only assertion (b) needs justification. Assume that the integer $N \gg m$. On the closed $(N+1)$-dimensional disc $\mathbb{D}_{r,N}$ of fixed radius $\rho_r = p^{-1/r}$,

$$\|\xi_m\|_{\rho_r} = \sup_{\underline{x}\in\mathbb{D}_{r,N}} \left| \prod_l (x_0^l x_l)^{\mathrm{ord}_l(m)} \right|_p = p^{-(1/r)\sum_l (1+l)\,\mathrm{ord}_l(m)}.$$

Since $\sum_l (1+l)\,\mathrm{ord}_l(m)$ tends to $+\infty$ as the number m increases, clearly ξ_m tends to 0 pointwise on any interior region of the infinite-dimensional polydisc. □

One huge advantage of working with the arithmetic of these $\boldsymbol{\xi}$-tempered functions is that the Euler product

$$\prod_{\text{primes } l} \left(\frac{1}{1-\chi(l)\xi_l} \right) = \sum_{m=1}^{\infty} \chi(m)\xi_m \quad \text{exists inside } \boldsymbol{B}^{\dagger}_{\infty}$$

for any purely multiplicative function $\chi : \mathbb{N} \to \mathcal{O}$. However, the infinite product of the terms $(1-\chi(l)\langle l\rangle^{-s})^{-1}$ over $\operatorname{Spec}\mathbb{Z}$ can never converge p-adically.

We next establish connections between our $\boldsymbol{\xi}$-tempered series and the more familiar rigid analytic functions occurring in nature. Fix a branch β modulo $p-1$. Consider the maps ${}_{\beta}\nabla_p^{\boldsymbol{f}} : \boldsymbol{B}^{\dagger}_{\infty} \to \mathcal{O}\langle\!\langle s\rangle\!\rangle^{\mathbb{N}}$, defined for integers $\boldsymbol{f} \geqslant 1$ by

$${}_{\beta}\nabla_p^{\boldsymbol{f}} \left(\sum_{m=1}^{\infty} c_m \xi_m \right)_n := \sum_{\substack{m=1,\\ p\nmid m}}^{p^{n\phi(\boldsymbol{f})}} c_m \omega^{\beta}(m)\langle m\rangle^{-s}.$$

We say that an element $\theta \in \boldsymbol{B}^{\dagger}_{\infty}$ is *visible at level* $\boldsymbol{f}$ if it gives rise to a Cauchy sequence in $\mathcal{O}\langle\!\langle s\rangle\!\rangle$ under ${}_{\beta}\nabla_p^{\boldsymbol{f}}$, along every branch β. Clearly, θ will also be visible at any integer multiple $\boldsymbol{cf}$ of $\boldsymbol{f}$, since $\phi(\boldsymbol{f})$ always divides into $\phi(\boldsymbol{cf})$; it makes good sense to call the minimal such level *the conductor of* θ.

If θ_1 is visible at level $\boldsymbol{f}_1$ and θ_2 is visible at level $\boldsymbol{f}_2$, then their sum $\theta_1+\theta_2$ will always be visible at level $\mathrm{lcm}(\boldsymbol{f}_1, \boldsymbol{f}_2)$. Perversely, the element $\theta_1 \times \theta_2 \in \boldsymbol{B}^{\dagger}_{\infty}$ may not be visible at any level whatsoever, so the product of two visible elements can often become invisible!

Definition 3.3. Let $\mathcal{O}\langle\!\langle s\rangle\!\rangle^{\mathrm{conv}}$ denote the space of Cauchy sequences in $\mathcal{O}\langle\!\langle s\rangle\!\rangle^{\mathbb{N}}$, and write $\mathcal{O}\langle\!\langle s\rangle\!\rangle^{\mathrm{null}}$ for the closed subspace of sequences converging to zero.

We define the space of shadow ∇-functions over K as the direct limit

$$\boldsymbol{A}^{\rm shw} := \varinjlim_{\boldsymbol{f}} \left(\frac{\bigcap_{\beta \bmod p-1} ({}_\beta\nabla_p^{\boldsymbol{f}})^{-1}(\mathcal{O}\langle\langle s\rangle\rangle^{\rm conv} \cap \mathrm{Im}(\nabla_p))}{\bigcap_{\beta \bmod p-1} ({}_\beta\nabla_p^{\boldsymbol{f}})^{-1}(\mathcal{O}\langle\langle s\rangle\rangle^{\rm null} \cap \mathrm{Im}(\nabla_p))} \right) \otimes_{\mathcal{O}} K,$$

where the partial ordering on $\boldsymbol{f} \in \mathbb{N}$ is with respect to divisibility.

In more down-to-earth terms, elements of $\boldsymbol{A}^{\rm shw}$ may be represented by convergent power series $\theta = \sum_{m=1}^{\infty} c_m \xi_m$ with $c_m \in K$, visible beyond some integer level $\boldsymbol{f}_\theta$. Moreover, along each branch β modulo $p-1$,

$${}_\beta\nabla_p : \boldsymbol{A}^{\rm shw} \to \boldsymbol{A}^{\rm rig} \quad \text{by } {}_\beta\nabla_p(\theta) = \lim_{n\to\infty} ({}_\beta\nabla_p^{\boldsymbol{f}_\theta}(\theta)_n),$$

where $\boldsymbol{A}^{\rm rig} := \mathcal{K}\langle\langle s\rangle\rangle$ denotes the affinoid K-algebra of the closed unit disc.

Proposition 3.4. *The map $\oplus_\beta \nabla_p : \boldsymbol{A}^{shw} \to \bigoplus_{\beta \bmod p-1} \boldsymbol{A}^{rig}$ is a non-continuous injection of topological spaces whose image is dense under the rigid topology.*

Proof. Let $\mathcal{T} = \oplus_\beta \nabla_p$; clearly $\mathcal{T}$ is an additive map of infinite-dimensional vector spaces over the field K, since the sum of any two visible elements remains visible (ditto for multiplication by scalars).

The injectivity of $\mathcal{T}$ follows from the fact that any two $\boldsymbol{f}$-visible elements which have the same image under $\mathcal{T}$ differ by an element of $\boldsymbol{B}_\infty^\dagger$ which is zero at level $\boldsymbol{f}$. However, in Definition 3.3 we carefully quotiented out the null sequences under $\mathcal{T}$; hence, their difference in the shadow space must be trivial.

To prove that the image of $\mathcal{T}$ is dense in the Iwasawa functions, it is a well-known fact that the K-linear span of the countable set $\{\langle m\rangle^{-s}$ with $m \in \mathbb{N}$ and $p \nmid m\}$ is p-adically dense. Consequently, any element $\gamma \in \bigoplus_{\beta \bmod p-1} \boldsymbol{A}^{\rm rig}$ may always be approximated by a finite K-linear sum like

$$\gamma_{p^N} = \left(\sum_{\substack{m=1,\\ p\nmid m}}^{p^N} c_{\beta,m,N}(\gamma)\langle m\rangle^{-s} \right)_\beta .$$

Here the $c_{\beta,m,N}(\gamma)$s are the moments of the bounded measure representing $\gamma|_\beta$.

Choose any integer x_N which is a primitive root modulo $p-1$, and such that $\langle x_N\rangle \in 1 + p^N\mathbb{Z}_p$. Consider the element $\mathfrak{h}_{x_N}^{(j)} \in \boldsymbol{B}_\infty^\dagger$ satisfying

$$\mathfrak{h}_{x_N}^{(j)} = \frac{\omega^j(x_N)}{p-1} \times \prod_{\substack{i=0,\\ i\not\equiv j \ (\mathrm{mod}\, p-1)}}^{p-2} (\xi_{x_N} - \omega^i(x_N)).$$

It can be easily checked that

$${}_\beta\nabla_p(\mathfrak{h}_{x_N}^{(j)}) \equiv \begin{cases} 1 \ (\mathrm{mod}\, p^N) & \text{if } \beta \equiv j \ (\mathrm{mod}\, p-1), \\ 0 \ (\mathrm{mod}\, p^N) & \text{if } \beta \not\equiv j \ (\mathrm{mod}\, p-1). \end{cases}$$

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If we define $\theta_{p^N}(\gamma) \in K \otimes_{\mathcal{O}} \boldsymbol{B}_\infty^\dagger$ by

$$\theta_{p^N}(\gamma) := \sum_{j=0}^{p-2} \mathfrak{h}_{x_N}^{(j)} \times \sum_{\substack{m=1,\\ p\nmid m}}^{p^N} \omega^{-j}(m) c_{\beta,m,N}(\gamma) \xi_m,$$

then $\theta_{p^N}(\gamma)$ will be visible at level 1 (and therefore at all levels) as it is a finite combination of ξ_ms. It represents an element of $\boldsymbol{A}^{\mathrm{shw}}$, and one readily verifies that $\mathcal{T}(\theta_{p^N}(\gamma))$ coincides exactly with the approximation γ_{p^N} modulo p^N.

Lastly, to establish the discontinuity of $\mathcal{T}$ we simply observe that the monomials $\xi_m \to 0$ in $\boldsymbol{A}^{\mathrm{shw}}$ as $m \to \infty$. However, $\mathcal{T}(\xi_m) \not\to 0 = \mathcal{T}(0)$ inside $\boldsymbol{A}^{\mathrm{rig}}$ because the Iwasawa functions $\mathcal{T}(\xi_m) = (\omega^\beta(m)\langle m\rangle^{-s})_\beta$ fail to stabilize as m increases. Thus, $\mathcal{T}$ cannot possibly be a continuous mapping. The underlying cause is that at every integer level $\boldsymbol{f} \geqslant 1$ there is a factorization

$${}_\beta\nabla_p^{\boldsymbol{f}} : \boldsymbol{B}_\infty^\dagger \hookrightarrow \varprojlim_{\varphi} \boldsymbol{B}_N \xrightarrow{\text{proj}} \prod_{n\in\mathbb{N}} \left(\varprojlim_{\tilde{\varphi}} \tilde{\boldsymbol{B}}_{N,\boldsymbol{f},n}\right) \xrightarrow{X_m \mapsto \omega^\beta(m)\langle m\rangle^{-s}} \mathcal{O}\langle\langle s\rangle\rangle^{\mathbb{N}},$$

where for each $n \in \mathbb{N}$ the intermediate rings are quotients

$$\tilde{\boldsymbol{B}}_{N,\boldsymbol{f},n} := \frac{\mathcal{O}[\![X_0, X_2, \dots, X_{l_N}]\!]}{\langle X_2^{e_2} X_3^{e_3} \dots X_{l_N}^{e_{l_N}} \mid \sum_{l\in\Sigma_N} e_l > p^{n\phi(\boldsymbol{f})}\rangle}.$$

Since $\tilde{\boldsymbol{B}}_{N,\boldsymbol{f},n}$ contains zero-divisors, the topology on $\varprojlim_{\tilde{\varphi}} \tilde{\boldsymbol{B}}_{N,\boldsymbol{f},n}$ is non-separated. However, $\mathcal{O}\langle\langle s\rangle\rangle^{\mathbb{N}}$ is a complete Hausdorff space, which forces the topology on the space of shadow ∇-functions to be incompatible with the rigid analytic one. □

4. Outline of the proof

Let us see how the main results of this paper can be deduced from the following.

Key equality. Inside the ring $\boldsymbol{B}_\infty^\dagger$, we have $\Theta^{\mathbf{Hu}} \otimes \chi = \sum_{m=1}^\infty a_m(\chi)\xi_m$.

Firstly, assuming the statement of Theorem 2.4 actually holds, we remark that $\sum_{m=1}^\infty a_m(\chi)\xi_m$ represents a $2\mathfrak{f}_\chi$-visible element of $\boldsymbol{A}^{\mathrm{shw}}$ (in the language of § 3).

Thus, along all branches $\beta \mod p-1$,

$$\begin{aligned}
{}_\beta\nabla_p(\Theta^{\mathbf{Hu}} \otimes \chi) &\overset{\text{by key eq.}}{=} {}_\beta\nabla_p\left(\sum_{m=1}^\infty a_m(\chi)\xi_m\right) \\
&= \lim_{t\to\infty} {}_\beta\nabla_p^{2\mathfrak{f}_\chi}\left(\sum_{m=1}^\infty a_m(\chi)\xi_m\right)_t \\
&= \lim_{t\to\infty}\left(\sum_{\substack{m=1,\\ p\nmid m}}^{p^{t\phi(2\mathfrak{f}_\chi)}} a_m(\chi)\omega^\beta(m)\langle m\rangle^{-s}\right) \\
&\overset{\text{by Thm 1.3}}{=} (2\omega^\beta(2)\langle 2\rangle^{-s} - 1) \times \boldsymbol{L}_p(s, \chi\omega^{1+\beta})
\end{aligned}$$

and Theorem 2.4 follows.

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It therefore remains to give the proofs of the key equality and Theorem 2.4. The latter proof is lengthier and occupies the entirety of §§ 5–7. In this section, we shall establish the former.

Proof of key equality. For $m \in \mathbb{N}$ consider

$$\widetilde{a_m(\chi)} := \sum_{j=\lfloor (m-1)/2 \rfloor + 1}^{m-1} \chi(j) - \sum_{j=1}^{\lfloor (m-1)/2 \rfloor} \chi(j).$$

It is easily checked that the $\widetilde{a_m(\chi)}$s are periodic with modulus $2\mathfrak{f}_\chi$. Thus, one may write

$$\sum_{m=1}^{\infty} \widetilde{a_m(\chi)} \xi_m = \sum_{c=1}^{2\mathfrak{f}_\chi} \widetilde{a_c(\chi)} \sum_{\substack{m=1, \\ m \equiv c \,(\mathrm{mod}\, 2\mathfrak{f}_\chi)}}^{\infty} \xi_m = \sum_{d \mid 2\mathfrak{f}_\chi} \sum_{\substack{c=1, \\ (c, 2\mathfrak{f}_\chi)=1}}^{2\mathfrak{f}_\chi/d} \widetilde{a_{cd}(\chi)} \xi_d \sum_{\substack{m=1, \\ m \equiv c \,(\mathrm{mod}(2\mathfrak{f}_\chi/d))}}^{\infty} \xi_m.$$

Focusing on the rightmost summation, we have

$$\begin{aligned} &\sum_{\substack{m=1, \\ m \equiv c \,(\mathrm{mod}(2\mathfrak{f}_\chi/d))}}^{\infty} \xi_m \\ &\qquad = \frac{1}{\#\mathcal{G}_d} \sum_{\Psi \in \mathcal{G}_d} \Psi^{-1}(c) \times \sum_{\substack{m=1, \\ (m, (2\mathfrak{f}_\chi/d))=1}}^{\infty} \Psi(m) \xi_m \\ &\qquad = \phi\left(\frac{2\mathfrak{f}_\chi}{d}\right)^{-1} \sum_{\substack{\Psi \in \mathcal{G}_1, \\ \mathfrak{f}_\Psi \mid 2\mathfrak{f}_\chi/d}} \Psi^{-1}(c) \prod_{\text{primes } l \mid 2\mathfrak{f}_\chi/d} (1 - \Psi(l)\xi_l) \times \sum_{m=1}^{\infty} \Psi(m)\xi_m, \end{aligned}$$

where $\mathcal{G}_d := \mathrm{Hom}((\mathbb{Z}/2\mathfrak{f}_\chi d^{-1}\mathbb{Z})^\times, \bar{\mathcal{O}}^\times)$.

Substituting this formula back into our original expression, we discover that the $\boldsymbol{\xi}$-tempered series $\sum_{m=1}^{\infty} \widetilde{a_m(\chi)} \xi_m$ equals

$$\sum_{\Psi \in \mathcal{G}_1} \sum_{d \mid 2\mathfrak{f}_\chi/\mathfrak{f}_\Psi} \phi\left(\frac{2\mathfrak{f}_\chi}{d}\right)^{-1} \sum_{\substack{c=1, \\ (c, 2\mathfrak{f}_\chi)=1}}^{2\mathfrak{f}_\chi/d} \Psi^{-1}(c) \widetilde{a_{cd}(\chi)} \times \xi_d \sum_{k \mid 2\mathfrak{f}_\chi/d} \mu(k)\Psi(k)\xi_k \times \sum_{m=1}^{\infty} \Psi(m)\xi_m$$

upon surreptitiously replacing $\prod_{\text{primes } l \mid 2\mathfrak{f}_\chi/d}$ above with $\sum_{k \mid 2\mathfrak{f}_\chi/d} \mu(k)$.

Remarks 4.1.

(a) Under the arithmetic of $\boldsymbol{B}_\infty^\dagger$, we naturally identify $\sum_{m=1}^{\infty} \Psi(m)\xi_m$ with the infinite product $\prod_{\text{primes } l} (1 - \Psi(l)\xi_l)^{-1}$.

(b) The dual group $\mathcal{G}_1 = \mathrm{Hom}((\mathbb{Z}/2\mathfrak{f}_\chi\mathbb{Z})^\times, \bar{\mathcal{O}}^\times)$ is isomorphic to $\hat{\Gamma}_\chi$ under ι_p^{-1}.

Using (a) and (b) in tandem, the function $\sum_{m=1}^{\infty}\widetilde{a_m(\chi)}\xi_m$ can be rewritten as

$$\sum_{\Psi\in\hat{\Gamma}_\chi}\prod_{\text{primes } l}\frac{1}{1-\Psi(l)\xi_l}\times\sum_{d|2\mathfrak{f}_\chi/\mathfrak{f}_\Psi}\sum_{k|2\mathfrak{f}_\chi/d}\frac{\mu(k)\Psi(k)}{\phi(2\mathfrak{f}_\chi/d)}\xi_{kd}\times\sum_{\substack{c=1,\\(c,2\mathfrak{f}_\chi)=1}}^{2\mathfrak{f}_\chi/d}\Psi^{-1}(c)\widetilde{a_{cd}(\chi)};$$

furthermore, we can swap around the middle two terms involving d and k since

$$\sum_{d|2\mathfrak{f}_\chi/\mathfrak{f}_\Psi}\sum_{k|2\mathfrak{f}_\chi/d}\mu(k)\Psi(k)\xi_{kd}=\sum_{k|2\mathfrak{f}_\chi}\xi_k\times\sum_{d|\gcd(k,2\mathfrak{f}_\chi/\mathfrak{f}_\Psi)}\mu\left(\frac{k}{d}\right)\Psi\left(\frac{k}{d}\right).$$

It follows directly that

$$\sum_{m=1}^{\infty}\widetilde{a_m(\chi)}\xi_m=\sum_{\Psi\in\hat{\Gamma}_\chi}\prod_{\text{primes } l}\frac{1}{1-\Psi(l)\xi_l}\times\sum_{k|2\mathfrak{f}_\chi}\xi_k\times E_k^{(\Psi)}(\chi),$$

where

$$E_k^{(\Psi)}(\chi)=\sum_{d|\gcd(k,2\mathfrak{f}_\chi/\mathfrak{f}_\Psi)}\frac{\mu(k/d)\Psi(k/d)}{\phi(2\mathfrak{f}_\chi/d)}\times\sum_{\substack{c=1,\\(c,2\mathfrak{f}_\chi)=1}}^{2\mathfrak{f}_\chi/d}\Psi^{-1}(c)\widetilde{a_{cd}(\chi)}$$

are the structural constants appearing in $\Theta^{\mathbf{Hu}}\otimes\chi$ (cf. Definition 2.2).

The final step is merely to point out that $(a_m(\chi))^{\sim}=a_m(\chi)-L_\infty(0,\chi)$, whence

$$\sum_{m=1}^{\infty}a_m(\chi)\xi_m=\sum_{m=1}^{\infty}L_\infty(0,\chi)\xi_m+\sum_{m=1}^{\infty}\widetilde{a_m(\chi)}\xi_m=\Theta^{\mathbf{Hu}}\otimes\chi.$$

The demonstration is finished. □

5. Calculating χ-twisted Dirichlet expansions

Let us now supply the missing proof of Theorem 2.4. We introduce a power series

$$\mathcal{L}_\chi(X):=\sum_{a=1}^{\mathfrak{f}_\chi}\chi(a)\cdot\left(\frac{(1+X)^a}{(1+X)^{\mathfrak{f}_\chi}-1}-2\frac{(1+X)^{2a}}{(1+X)^{2\mathfrak{f}_\chi}-1}\right)$$

which converges everywhere on the open unit disc, centred at the point $X=0$. The validity of the theorem then follows readily from the following two results.

Lemma 5.1. *The Iwasawa function* $(2\omega^\beta(2)\langle 2\rangle^{-s}-1)\cdot\boldsymbol{L}_p(s,\chi\omega^{1+\beta})$ *is always congruent to the summation*

$$\sum_{\substack{m=1,\\p\nmid m}}^{p^n}\left(p^{-n}\sum_{\zeta\in\mu_{p^n}}\mathcal{L}_\chi(\zeta^{-1}-1)\cdot\zeta^m\right)\omega^\beta(m)\langle m\rangle^{-s}\quad\textit{modulo } p^n$$

for all $s\in\mathbb{Z}_p$ *and* $n\in\mathbb{N}$.

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Proposition 5.2. *If $p^n \equiv 1 \pmod{2\mathfrak{f}_\chi}$, then $p^{-n}\sum_{\zeta\in\mu_{p^n}} \mathcal{L}_\chi(\zeta^{-1}-1)\cdot\zeta^m$ equals the coefficient $a_m(\chi)$ for every integer m lying in the range $1 \leqslant m \leqslant p^n - 1$.*

By Fermat's little theorem, whenever n is divisible by $\phi(2\mathfrak{f}_\chi)$ the above condition $p^n \equiv 1 \pmod{2\mathfrak{f}_\chi}$ will automatically be satisfied. Indeed, if $n = t\phi(2\mathfrak{f}_\chi)$, then

$$(2\omega^\beta(2)\langle 2\rangle^{-s}-1)\boldsymbol{L}_p(s,\chi\omega^{1+\beta}) \equiv \sum_{\substack{m=1,\\ p\nmid m}}^{p^{t\phi(2\mathfrak{f}_\chi)}} a_m(\chi)\omega^\beta(m)\langle m\rangle^{-s} \bmod p^{t\phi(2\mathfrak{f}_\chi)}, \tag{5.1}$$

which is strong enough to imply Theorem 2.4.

Proposition 5.2 relies on Iwasawa's cyclotomic reciprocity law [**5**], the implementation of which may be found in the next couple of sections. To prove Lemma 5.1 we employ a bit of fractional calculus.

Proof of Lemma 5.1. We start by logarithmically differentiating our generating function. As in § 4, we assume that $\mathcal{O}$ is a finite integral extension of $\mathbb{Z}_p$ containing the values of the character χ.

The poles of $\mathcal{L}_\chi$ are of the form $\alpha - 1$, where α ranges over the $(2\mathfrak{f}_\chi)$th-roots of unity. Under Hypothesis 2.1, the prime $p \nmid 2\mathfrak{f}_\chi$; therefore, $|\alpha-1|_p = 1$ when $\alpha \neq 1$. Its poles must then be scattered uniformly around the boundary of the unit disc. In particular, $\mathcal{L}_\chi$ lies in $\mathcal{O}[\![X]\!]$, and, changing variables, we obtain

$$\mathcal{L}_\chi(\exp(Z)-1) = \frac{1}{Z}\sum_{a=1}^{\mathfrak{f}_\chi}\left(\frac{\chi(a)\cdot Z\exp(aZ)}{\exp(\mathfrak{f}_\chi Z)-1} - \frac{\chi(a)\cdot 2Z\exp(2Za)}{\exp(\mathfrak{f}_\chi 2Z)-1}\right).$$

Recall that the χ-twisted Bernoulli numbers $B_{n,\chi}$ can be defined as the coefficients in a certain Taylor series, namely

$$\sum_{a=1}^{\mathfrak{f}_\chi}\frac{\chi(a)\cdot Z\exp(aZ)}{\exp(\mathfrak{f}_\chi Z)-1} = \sum_{n=0}^{\infty} B_{n,\chi}\frac{Z^n}{n!}.$$

Thus,

$$\left.\frac{\mathrm{d}^k\mathcal{L}_\chi(X)}{\mathrm{d}\log(1+X)^k}\right|_{X=0} = \left.\frac{\mathrm{d}^k\mathcal{L}_\chi(\exp(Z)-1)}{\mathrm{d}Z^k}\right|_{Z=0} = (1-2^{k+1})\frac{B_{k+1,\chi}}{k+1}$$

for all integers $k \geqslant 0$.

Unfortunately this equation is not quite in the correct form to be interpolated. Instead, consider χ as a single element of $\hat{\mathcal{W}} := \mathrm{Hom}(\mathcal{W}, \bar{\mathcal{O}}^\times)$, where $\mathcal{W} = (\mathbb{Z}/\mathfrak{f}_\chi\mathbb{Z})^\times$. From this viewpoint the set of power series $\{\mathcal{L}_\Psi\}_{\Psi\in\hat{\mathcal{W}}}$ corresponds to an $\bar{\mathcal{O}}$-valued measure $\Phi_{\mathcal{L}}$ on $\mathcal{W}\times\mathbb{Z}_p$, in other words

$$\left.\frac{\mathrm{d}^k\mathcal{L}_\Psi(X)}{\mathrm{d}\log(1+X)^k}\right|_{X=0} = \int_{\mathcal{W}\times\mathbb{Z}_p}\Psi(x)x^k\cdot\mathrm{d}\Phi_{\mathcal{L}}(x) \quad \text{for all } k \geqslant 0.$$

Let us now apply Coleman's idempotent $\psi \in \mathrm{End}(\mathcal{O}[\![X]\!])$, constructed in [**2**] via the formula

$$\psi(g(X)) = g(X) - \frac{1}{p}\sum_{\zeta\in\mu_p} g(\zeta(1+X)-1) \quad \text{for all } g(X)\in\mathcal{O}[\![X]\!].$$

In terms of the underlying distribution theory, it restricts a bounded measure on the topological space $\mathcal{W}\times\mathbb{Z}_p$ (corresponding to the series $g(X)$) to a bounded measure supported on the subgroup $\mathcal{W}\times\mathbb{Z}_p^\times$ of invertible elements.

Again, the hypothesis we make (Hypothesis 2.1) forces the action of p on $\mathcal{W}$ to be invertible, since p is assumed to be coprime to $\#\mathcal{W} = \phi(\mathfrak{f}_\chi)$. Writing the Lie group $\mathcal{W}\times\mathbb{Z}_p^\times$ as a subtraction $\mathcal{W}\times\mathbb{Z}_p - \mathcal{W}\times p\mathbb{Z}_p = \mathcal{W}\times\mathbb{Z}_p - p(\mathcal{W}\times\mathbb{Z}_p)$, it follows that

$$\begin{aligned}\left.\frac{\mathrm{d}^k\psi\circ\mathcal{L}_\chi(X)}{\mathrm{d}\log(1+X)^k}\right|_{X=0} &= \int_{\mathcal{W}\times\mathbb{Z}_p^\times}\chi(x)x^k\cdot\mathrm{d}\Phi_{\mathcal{L}}(x)\\ &= \int_{\mathcal{W}\times\mathbb{Z}_p}\chi(x)x^k\cdot\mathrm{d}\Phi_{\mathcal{L}}(x) - \int_{\mathcal{W}\times\mathbb{Z}_p}\chi(px)(px)^k\cdot\mathrm{d}\Phi_{\mathcal{L}}(x)\\ &= (1-\chi(p)p^k)\left.\frac{\mathrm{d}^k\mathcal{L}_\chi(X)}{\mathrm{d}\log(1+X)^k}\right|_{X=0}.\end{aligned}$$

We have just computed the latter quantity, which equals

$$(1-2^{k+1})(1-\chi(p)p^k)\frac{B_{k+1,\chi}}{k+1}.$$

Furthermore, $B_{k+1,\chi}/(k+1)$ is *minus* the special value of the χ-twisted zeta function at the critical point $-k$ (unless $\chi=\mathbf{1}$ and $k=0$). We therefore deduce that

$$\left.\frac{\mathrm{d}^k\psi\circ\mathcal{L}_\chi(X)}{\mathrm{d}\log(1+X)^k}\right|_{X=0} = -(1-2^{k+1})\times(1-\chi(p)p^k)L_\infty(-k,\chi) \quad \text{at all } k\in\mathbb{Z},\ k\geqslant 0.$$

This is now in the right form to be interpolated. Fix a class $\beta\in\mathbb{Z}/(p-1)\mathbb{Z}$, and let us consider exclusively integers $k\geqslant 0$ which are congruent to $\beta\,(\mathrm{mod}\,p-1)$. We focus first on the right-hand side of the above equation. The p-adic L-function $\boldsymbol{L}_p(-s,\chi\omega^{1+\beta})$ coincides with $(1-\chi(p)p^k)L_\infty(-k,\chi)$ at $s=k\equiv\beta\,(\mathrm{mod}\,p-1)$. Similarly, the Iwasawa function $(2\omega^\beta(2)\langle 2\rangle^s-1)$ will interpolate $-(1-2^{k+1})$ at these non-negative integer values. What about the left-hand side?

In [**3**] we defined a fractional differential operator ${}_\beta\boldsymbol{D}^s : \mathcal{O}[\![X]\!]^{\psi=1}\to\mathcal{O}[\![X]\!]^{\psi=1}$ varying continuously in the parameter $s\in\mathbb{Z}_p$, and satisfying

$${}_\beta\boldsymbol{D}^k = \frac{\mathrm{d}^k}{\mathrm{d}\log(1+X)^k} \quad \text{at every } k\geqslant 0 \text{ with } k\equiv\beta\,(\mathrm{mod}\,p-1).$$

By [**3**, Lemma 3.1] we have the numerical approximation

$${}_\beta\boldsymbol{D}^s(\psi\circ g(X)) \equiv \sum_{\substack{m=1,\\ p\nmid m}}^{p^n}\left(p^{-n}\sum_{\zeta\in\mu_{p^n}} g(\zeta^{-1}-1)\cdot\zeta^m\right)\omega^\beta(m)\langle m\rangle^s\cdot(1+X)^m$$

modulo $(p,X)^n\mathcal{O}[\![X]\!]\langle\!\langle s\rangle\!\rangle$, for all elements $g(X)\in\mathcal{O}[\![X]\!]$.

Putting $g(X) = \mathcal{L}_\chi(X)$ and evaluating at $X = 0$, it follows directly from our approximation that the p-adic function $(2\omega^\beta(2)\langle 2\rangle^s - 1)\cdot \boldsymbol{L}_p(-s, \chi\omega^{1+\beta})$ is congruent to

$$\sum_{\substack{m=1,\\ p\nmid m}}^{p^n}\biggl(p^{-n}\sum_{\zeta\in\mu_{p^n}}\mathcal{L}_\chi(\zeta^{-1}-1)\cdot\zeta^m\biggr)\omega^\beta(m)\langle m\rangle^s \quad \text{modulo } p^n\mathcal{O}\langle\!\langle s\rangle\!\rangle.$$

Finally, the substitution $s \mapsto -s$ yields the formula stated in Lemma 5.1. □

6. The base case

All that remains is to give the demonstration Proposition 5.2. Let us recall that we need to equate the sum

$$\sum_{\zeta\in\mu_{p^n}}\mathcal{L}_\chi(\zeta^{-1}-1)\cdot\zeta^m$$

with the coefficient $p^n a_m(\chi)$. We shall use induction; the initialization step is treated in this section, while the inductive step is given in § 7.

We start with a basic yet useful result. If $m \in \mathbb{Z}$, let $\lambda_{p^n}(m)$ denote the unique integer satisfying

$$1 \leqslant \lambda_{p^n}(m) \leqslant p^n \quad \text{and} \quad \lambda_{p^n}(m) \equiv m \,(\operatorname{mod} p^n).$$

Lemma 6.1. *For all $m \in \mathbb{N}$, we have the equality*

$$\sum_{\substack{\zeta\in\mu_{p^n},\\ \zeta\neq 1}}\frac{\zeta^m}{\zeta-1} = \frac{p^n+1}{2} - \lambda_{p^n}(m).$$

Proof. We can carry out this calculation equally well inside the complex numbers, courtesy of the embedding $\iota_\infty \circ (\iota_p)^{-1}$. Firstly, if $m \equiv 1 \,(\operatorname{mod} p^n)$, then

$$\begin{aligned}\sum_{\zeta\neq 1}\frac{\zeta^1}{\zeta-1} &= \sum_{\zeta\neq 1}\biggl(1+\frac{1}{\zeta-1}\biggr)\\ &= p^n - 1 + \sum_{\zeta\neq 1}\frac{1}{\zeta-1}\\ &= p^n - 1 + \frac{1}{2}\sum_{\zeta\neq 1}\frac{\bar\zeta-1}{1-\operatorname{Re}(\zeta)}.\end{aligned}$$

Now

$$\sum_{\zeta\neq 1}\frac{\bar\zeta-1}{1-\operatorname{Re}(\zeta)} = \sum_{\zeta\neq 1}\biggl(-1+\frac{\operatorname{Im}(\zeta)}{1-\operatorname{Re}(\zeta)}\biggr),$$

and the term $\operatorname{Im}(\zeta)/(1-\operatorname{Re}(\zeta))$ cancels out the term $\operatorname{Im}(\bar\zeta)/(1-\operatorname{Re}(\bar\zeta))$ involving the conjugate root $\bar\zeta$. Consequently,

$$\sum_{\zeta\neq 1}\frac{\zeta^1}{\zeta-1} = p^n - 1 + \frac{1}{2}\sum_{\zeta\neq 1} -1 = \frac{p^n-1}{2},$$

so the result is true for $m \equiv 1$.

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We now assume that $m \not\equiv 1 \pmod{p^n}$ and proceed inductively:

$$\sum_{\zeta \neq 1} \frac{\zeta^m}{\zeta - 1} = \sum_{\zeta \neq 1} \frac{\zeta^{m-1}(\zeta - 1 + 1)}{\zeta - 1} = \sum_{\zeta \neq 1} \zeta^{m-1} + \sum_{\zeta \neq 1} \frac{\zeta^{m-1}}{\zeta - 1}.$$

The first sum is -1 since $m - 1 \not\equiv 0$, and the second sum equals $\frac{1}{2}(p^n + 1) - \lambda_{p^n}(m-1)$ by our inductive hypothesis. The lemma then follows because

$$\sum_{\zeta \neq 1} \frac{\zeta^m}{\zeta - 1} = -1 + \left(\frac{p^n + 1}{2} - \lambda_{p^n}(m-1) \right) = \frac{p^n + 1}{2} - \lambda_{p^n}(m).$$

□

Let us next focus our attention on computing the value of $\mathcal{L}_\chi(X)$ at zero. Expanding this function as a Taylor series in the variable X yields

$$\begin{aligned} \mathcal{L}_\chi(X) &= \sum_{a=1}^{\mathfrak{f}_\chi} \chi(a) \cdot \left(\frac{(1+X)^a((1+X)^{\mathfrak{f}_\chi} + 1) - 2(1+X)^{2a}}{(1+X)^{2\mathfrak{f}_\chi} - 1} \right) \\ &= \sum_{a=1}^{\mathfrak{f}_\chi} \chi(a) \cdot \left(\frac{(\mathfrak{f}_\chi - 2a)X + \left(\binom{a+\mathfrak{f}_\chi}{2} + \binom{a}{2} - 2\binom{2a}{2}\right)X^2 + \cdots}{2\mathfrak{f}_\chi X + \binom{2\mathfrak{f}_\chi}{2} X^2 + \cdots} \right); \end{aligned}$$

hence,

$$\mathcal{L}_\chi(0) = \sum_{a=1}^{\mathfrak{f}_\chi} \chi(a) \cdot \left(\frac{\mathfrak{f}_\chi - 2a}{2\mathfrak{f}_\chi} \right).$$

Definition 6.2. For all $m, n \in \mathbb{N}$, let $\Omega_{n,\chi}^{(m)} := \sum_{\zeta \in \mu_{p^n}} \mathcal{L}_\chi(\zeta^{-1} - 1) \cdot \zeta^m$.

The calculation of this quantity will be our primary objective. Taking the definition above as our starting point, we derive the expression

$$\begin{aligned} \Omega_{n,\chi}^{(m)} &= \mathcal{L}_\chi(0) + \sum_{\substack{\zeta \in \mu_{p^n}, \\ \zeta \neq 1}} \mathcal{L}_\chi(\zeta^{-1} - 1) \cdot \zeta^m \\ &= \sum_{a=1}^{\mathfrak{f}_\chi} \chi(a) \cdot \left(\frac{\mathfrak{f}_\chi - 2a}{2\mathfrak{f}_\chi} \right) + \sum_{\substack{\zeta \in \mu_{p^n}, \\ \zeta \neq 1}} \zeta^m \sum_{a=1}^{\mathfrak{f}_\chi} \chi(a) \left(\frac{\zeta^{-a-\mathfrak{f}_\chi} + \zeta^{-a} - 2\zeta^{-2a}}{\zeta^{-2\mathfrak{f}_\chi} - 1} \right) \\ &= \sum_{a=1}^{\mathfrak{f}_\chi} \chi(a) \cdot \left(\frac{1}{2} - \frac{a}{\mathfrak{f}_\chi} + \sum_{\substack{\zeta \in \mu_{p^n}, \\ \zeta \neq 1}} \frac{\zeta^{m-a-\mathfrak{f}_\chi} + \zeta^{m-a} - 2\zeta^{m-2a}}{\zeta^{-2\mathfrak{f}_\chi} - 1} \right). \end{aligned}$$

Remark 6.3. Under the conditions of Proposition 5.2, we assumed $p^n \equiv 1 \pmod{2\mathfrak{f}_\chi}$. One nice consequence is that $\boldsymbol{e}_n := (p^n - 1)/2\mathfrak{f}_\chi$ will always be a positive integer; in fact, each $\boldsymbol{e}_n$ is a multiplicative inverse of $-2\mathfrak{f}_\chi$ modulo p^n.

Since the map $\zeta \mapsto \zeta^{\boldsymbol{e}_n}$ extends linearly to an automorphism of $\mathcal{O}[\mu_{p^n}]$, we clearly have

$$\sum_{\substack{\zeta\in\mu_{p^n},\\ \zeta\neq 1}} \frac{\zeta^{m-a-\mathfrak{f}_\chi}+\zeta^{m-a}-2\zeta^{m-2a}}{\zeta^{-2\mathfrak{f}_\chi}-1} = \sum_{\substack{\zeta\in\mu_{p^n},\\ \zeta\neq 1}} \frac{\zeta^{\boldsymbol{e}_n(m-a-\mathfrak{f}_\chi)}+\zeta^{\boldsymbol{e}_n(m-a)}-2\zeta^{\boldsymbol{e}_n(m-2a)}}{\zeta^{\boldsymbol{e}_n(-2\mathfrak{f}_\chi)}-1}$$

and the denominator equals $\zeta-1$ by the above remark.

Substituting back into our expression for $\Omega_{n,\chi}^{(m)}$, we obtain the closed formula

$$\Omega_{n,\chi}^{(m)} = \sum_{a=1}^{\mathfrak{f}_\chi}\chi(a)\cdot\left(\frac{1}{2}-\frac{a}{\mathfrak{f}_\chi}+\sum_{\substack{\zeta\in\mu_{p^n},\\ \zeta\neq 1}}\frac{\zeta^{\boldsymbol{e}_n(m-a-\mathfrak{f}_\chi)}}{\zeta-1}+\frac{\zeta^{\boldsymbol{e}_n(m-a)}}{\zeta-1}-2\frac{\zeta^{\boldsymbol{e}_n(m-2a)}}{\zeta-1}\right) \qquad (6.1)$$

which we now have the necessary tools to work out.

Proof of Proposition 5.2 when $m=1$.

Applying Lemma 6.1 three times to (6.1), we see immediately that

$$\Omega_{n,\chi}^{(1)} = \sum_{a=1}^{\mathfrak{f}_\chi}\chi(a)\cdot\left(\frac{1}{2}-\frac{a}{\mathfrak{f}_\chi}+\frac{p^n+1}{2}-\lambda_{p^n}(\boldsymbol{e}_n(1-a-\mathfrak{f}_\chi))+\frac{p^n+1}{2}-\lambda_{p^n}(\boldsymbol{e}_n(1-a)) \right.$$
$$\left. -2\left(\frac{p^n+1}{2}-\lambda_{p^n}(\boldsymbol{e}_n(1-2a))\right)\right).$$

Upon remembering that $\boldsymbol{e}_n$ represented $(p^n-1)/2\mathfrak{f}_\chi$, a direct evaluation of the term

$$-\lambda_{p^n}(\boldsymbol{e}_n(1-a-\mathfrak{f}_\chi))-\lambda_{p^n}(\boldsymbol{e}_n(1-a))+2\lambda_{p^n}(\boldsymbol{e}_n(1-2a))$$

yields the rational number

$$-\left(p^n+\frac{p^n-1}{2\mathfrak{f}_\chi}(1-a-\mathfrak{f}_\chi)\right)-\left(p^n+\frac{p^n-1}{2\mathfrak{f}_\chi}(1-a)\right)+2\left(p^n+\frac{p^n-1}{2\mathfrak{f}_\chi}(1-2a)\right).$$

The above quantity can be further simplified to give

$$p^n\left(\frac{1}{2}-\frac{a}{\mathfrak{f}_\chi}\right)+\frac{a}{\mathfrak{f}_\chi}-\frac{1}{2};$$

hence, $\Omega_{n,\chi}^{(1)}$ reduces to the algebraic element

$$\sum_{a=1}^{\mathfrak{f}_\chi}\chi(a)\cdot\left(\frac{1}{2}-\frac{a}{\mathfrak{f}_\chi}+\left(p^n\left(\frac{1}{2}-\frac{a}{\mathfrak{f}_\chi}\right)+\frac{a}{\mathfrak{f}_\chi}-\frac{1}{2}\right)\right)=p^n\times\sum_{a=1}^{\mathfrak{f}_\chi}\chi(a)\cdot\left(\frac{1}{2}-\frac{a}{\mathfrak{f}_\chi}\right).$$

If χ is the trivial character, then $\Omega_{n,\chi}^{(1)}=p^n\times-\frac{1}{2}$, i.e. p^n times the value of the Riemann zeta function at zero. Similarly, if $\chi\neq\mathbf{1}$, then

$$\begin{aligned}\Omega_{n,\chi}^{(1)} &= \frac{p^n}{2}\times\sum_{a=1}^{\mathfrak{f}_\chi}\chi(a)-p^n\times\sum_{a=1}^{\mathfrak{f}_\chi}\chi(a)\cdot\frac{a}{\mathfrak{f}_\chi}\\ &= \frac{p^n}{2}\times 0+p^n\times L_\infty(0,\chi).\end{aligned}$$

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Thus, in both cases

$$\Omega^{(1)}_{n,\chi} = \sum_{\zeta\in\mu_{p^n}} \mathcal{L}_\chi(\zeta^{-1}-1)\cdot\zeta^1 = p^n \cdot L_\infty(0,\chi),$$

as asserted in the statement of Proposition 5.2. □

7. The reciprocity step

In order to finish the demonstration of Proposition 5.2, we are required to show that $\Omega^{(m)}_{n,\chi} - \Omega^{(1)}_{n,\chi} = p^n(a_m(\chi) - a_1(\chi))$ for a general $m \in \{1, 2, \ldots, p^n - 1\}$. The delicate part is to switch between the arithmetic of the tower $\{\mathbb{Q}(\mu_{p^n})\}_{n\in\mathbb{N}}$ and that of the field $\mathbb{Q}(\mu_{2\mathfrak{f}_\chi})$, a typical instance of reciprocity in action [**5**].

Assume that $m > 1$. Subtracting equation (6.1) for the index m from the same equation at index $m+1$, we discover that $\Omega^{(m+1)}_{n,\chi} - \Omega^{(m)}_{n,\chi}$ has the value

$$\sum_{a=1}^{\mathfrak{f}_\chi} \chi(a)\cdot\left(\sum_{\substack{\zeta\in\mu_{p^n},\\ \zeta\neq 1}} \frac{\zeta^{\boldsymbol{e}_n}-1}{\zeta-1}(\zeta^{\boldsymbol{e}_n(m-a-\mathfrak{f}_\chi)} + \zeta^{\boldsymbol{e}_n(m-a)} - 2\zeta^{\boldsymbol{e}_n(m-2a)})\right),$$

which looks particularly unpleasant.

Let us concentrate on the outer brackets. Since the terms in the geometric progression $1, \zeta, \zeta^2, \ldots, \zeta^{\boldsymbol{e}_n-1}$ add up to $(\zeta^{\boldsymbol{e}_n}-1)/(\zeta-1)$, clearly

$$\begin{aligned}\sum_{\substack{\zeta\in\mu_{p^n},\\ \zeta\neq 1}} \frac{\zeta^{\boldsymbol{e}_n}-1}{\zeta-1}(\cdots) &= \sum_{\substack{\zeta\in\mu_{p^n},\\ \zeta\neq 1}} \sum_{j=0}^{\boldsymbol{e}_n-1} \zeta^j \times (\cdots) \\ &= \sum_{j=0}^{\boldsymbol{e}_n-1} \sum_{\substack{\zeta\in\mu_{p^n},\\ \zeta\neq 1}} (\zeta^{j+\boldsymbol{e}_n(m-a-\mathfrak{f}_\chi)} + \zeta^{j+\boldsymbol{e}_n(m-a)} - 2\zeta^{j+\boldsymbol{e}_n(m-2a)}).\end{aligned}$$

In fact, we can even replace $\sum_{\zeta\in\mu_{p^n},\zeta\neq 1}$ above with $\sum_{\zeta\in\mu_{p^n}}$ without changing anything. One deduces that

$$\Omega^{(m+1)}_{n,\chi} - \Omega^{(m)}_{n,\chi} = \sum_{a=1}^{\mathfrak{f}_\chi} \chi(a) \times \beta(a, m, n, \mathfrak{f}_\chi),$$

where

$$\beta(a, m, n, \mathfrak{f}_\chi) = \sum_{j=0}^{\boldsymbol{e}_n-1} \sum_{\zeta\in\mu_{p^n}} (\zeta^{j+\boldsymbol{e}_n(m-a-\mathfrak{f}_\chi)} + \zeta^{j+\boldsymbol{e}_n(m-a)} - 2\zeta^{j+\boldsymbol{e}_n(m-2a)}).$$

Definition 7.1. If $x, y, d \in \mathbb{Z}$ with $d > 0$, then define

$$\Delta_{x\equiv y\,(\mathrm{mod}\,d)} := \begin{cases} 1 & \text{if } x \equiv y \text{ modulo } d, \\ 0 & \text{if } x \not\equiv y \text{ modulo } d. \end{cases}$$

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For example,

$$\sum_{\zeta\in\mu_{p^n}} \zeta^{j+x} = p^n \times \Delta_{j+x\equiv 0\,(\mathrm{mod}\,p^n)},$$

which implies that

$$\beta(a,m,n,\mathfrak{f}_\chi) = \sum_{j=0}^{\boldsymbol{e}_n-1} p^n\big(\Delta_{j+\boldsymbol{e}_n(m-a-\mathfrak{f}_\chi)\equiv 0\,(\mathrm{mod}\,p^n)} + \Delta_{j+\boldsymbol{e}_n(m-a)\equiv 0\,(\mathrm{mod}\,p^n)} - 2\Delta_{j+\boldsymbol{e}_n(m-2a)\equiv 0\,(\mathrm{mod}\,p^n)}\big).$$

Lemma 7.2. *With the same assumptions and notation as before:*

(a) $\displaystyle\sum_{j=0}^{\boldsymbol{e}_n-1} \Delta_{j+\boldsymbol{e}_n(m-a-\mathfrak{f}_\chi)\equiv 0\,(\mathrm{mod}\,p^n)} = \Delta_{m\equiv a+\mathfrak{f}_\chi\,(\mathrm{mod}\,2\mathfrak{f}_\chi)}$;

(b) $\displaystyle\sum_{j=0}^{\boldsymbol{e}_n-1} \Delta_{j+\boldsymbol{e}_n(m-a)\equiv 0\,(\mathrm{mod}\,p^n)} = \Delta_{m\equiv a\,(\mathrm{mod}\,2\mathfrak{f}_\chi)}$;

(c) $\displaystyle\sum_{j=0}^{\boldsymbol{e}_n-1} \Delta_{j+\boldsymbol{e}_n(m-2a)\equiv 0\,(\mathrm{mod}\,p^n)} = \Delta_{m\equiv 0\,(\mathrm{mod}\,2)} \times \Delta_{m/2\equiv a\,(\mathrm{mod}\,\mathfrak{f}_\chi)}$.

Deferring the proof for a moment, we explain quickly how the inductive argument for Proposition 5.2 follows from them. Applying (a)–(c), we discover that the coefficient $\beta(a,m,n,\mathfrak{f}_\chi)$ equals

$$p^n\big(\Delta_{m\equiv a+\mathfrak{f}_\chi\,(\mathrm{mod}\,2\mathfrak{f}_\chi)} + \Delta_{m\equiv a\,(\mathrm{mod}\,2\mathfrak{f}_\chi)} - 2\Delta_{m\equiv 0\,(\mathrm{mod}\,2)} \times \Delta_{m/2\equiv a\,(\mathrm{mod}\,\mathfrak{f}_\chi)}\big),$$

or, even more succinctly,

$$\beta(a,m,n,\mathfrak{f}_\chi) = \begin{cases} p^n\Delta_{m\equiv a\,(\mathrm{mod}\,\mathfrak{f}_\chi)} - 2p^n\Delta_{m/2\equiv a\,(\mathrm{mod}\,\mathfrak{f}_\chi)} & \text{if } m \text{ is even,} \\ p^n\Delta_{m\equiv a\,(\mathrm{mod}\,\mathfrak{f}_\chi)} & \text{if } m \text{ is odd.} \end{cases}$$

Plugging this back into our formula for the difference of the $\Omega^{(-)}_{n,\chi}$s, we obtain

$$\begin{aligned} \Omega^{(m+1)}_{n,\chi} - \Omega^{(m)}_{n,\chi} &= \sum_{a=1}^{\mathfrak{f}_\chi} \chi(a) \times \beta(a,m,n,\mathfrak{f}_\chi) \\ &= \begin{cases} p^n\chi(m) - 2p^n\chi\left(\dfrac{m}{2}\right) & \text{if } m \text{ is even,} \\ p^n\chi(m) & \text{if } m \text{ is odd.} \end{cases} \end{aligned}$$

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Consequently, for all $m > 1$, it follows that

$$\begin{aligned}
\Omega_{n,\chi}^{(m)} - \Omega_{n,\chi}^{(1)} &= (\Omega_{n,\chi}^{(m)} - \Omega_{n,\chi}^{(m-1)}) + (\Omega_{n,\chi}^{(m-1)} - \Omega_{n,\chi}^{(m-2)}) + \cdots + (\Omega_{n,\chi}^{(2)} - \Omega_{n,\chi}^{(1)}) \\
&= p^n \sum_{j=1}^{m-1} \chi(j) - 2p^n \sum_{\substack{j=1,\\ j \text{ even}}}^{m-1} \chi\left(\frac{j}{2}\right) \\
&= p^n \left(\sum_{j=1}^{m-1} \chi(j) - 2 \sum_{j=1}^{\lfloor (m-1)/2 \rfloor} \chi(j) \right) \\
&= p^n (a_m(\chi) - a_1(\chi)),
\end{aligned}$$

which completes the induction step.

Remark 7.3. Before we give the proof of Lemma 7.2, it is worth asking why the argument breaks down when $p = 2$. The power series $\mathcal{L}_\chi(X)$ has a pole at $X = -2$ which lies in the interior of the 2-adic unit disc; hence, $\mathcal{L}_\chi(X)$ cannot possibly be an element of $\bar{\mathbb{Z}}_2[\![X]\!]$. Despite trying several other generating series, we have yet to find a Dirichlet expansion which stabilizes independent of 2^n. Is this intractable?

Proof of Lemma 7.2.

Let us prove the first assertion. We need to determine whether the congruence

$$j + \boldsymbol{e}_n(m - a - \mathfrak{f}_\chi) \equiv 0 \pmod{p^n} \quad \text{with } 0 \leqslant j \leqslant \boldsymbol{e}_n - 1$$

has any solutions in (m, j) for fixed n, and then count up the number of solutions. Since $\boldsymbol{e}_n = (p^n - 1)/2\mathfrak{f}_\chi$ is congruent to $(-2\mathfrak{f}_\chi)^{-1}$ modulo p^n, we are asking under what conditions the equation $j - (m - a - \mathfrak{f}_\chi)/2\mathfrak{f}_\chi \equiv 0 \pmod{p^n}$ is solvable, i.e.

$$m \equiv a + \mathfrak{f}_\chi + 2\mathfrak{f}_\chi j \pmod{p^n}, \quad \text{where } 0 \leqslant j \leqslant \frac{p^n - 1 - 2\mathfrak{f}_\chi}{2\mathfrak{f}_\chi}.$$

We make the important observation that both m and $a + \mathfrak{f}_\chi + 2\mathfrak{f}_\chi j$ are integers lying between 1 and $p^n - 1$; not only is the above a congruence modulo p^n, it is actually an equality.

Thus, to have an integer solution in the pair (m, j), it is a necessary condition that $m \equiv a + \mathfrak{f}_\chi \pmod{2\mathfrak{f}_\chi}$. Conversely, if this condition is satisfied, then

$$(m, j) = \left(m, \frac{m - a - \mathfrak{f}_\chi}{2\mathfrak{f}_\chi} \right) \text{ will be the \textit{unique} solution}$$

for j in the interval $[0, (p^n - 1 - 2\mathfrak{f}_\chi)/2\mathfrak{f}_\chi]$. It follows that

$$\sum_{j=0}^{\boldsymbol{e}_n - 1} \Delta_{j + \boldsymbol{e}_n(m - a - \mathfrak{f}_\chi) \equiv 0 \pmod{p^n}}$$

must equal $\Delta_{m \equiv a + \mathfrak{f}_\chi \pmod{2\mathfrak{f}_\chi}}$ and takes either the value 1 or 0, depending on whether or not there is a solution. The proof of statement (a) is now complete.

The demonstration of (b) is almost identical, so we shall leave it to the reader. To prove part (c), let us consider the congruence

$$j + \boldsymbol{e}_n(m-2a) \equiv 0 \pmod{p^n}, \quad \text{where } 0 \leqslant j \leqslant \boldsymbol{e}_n - 1.$$

Exploiting the fact that $\boldsymbol{e}_n \equiv (-2\mathfrak{f}_\chi)^{-1}$ modulo p^n, after rearranging the above we obtain the equivalent equation

$$m \equiv 2a + 2\mathfrak{f}_\chi j \pmod{p^n} \quad \text{with } 0 \leqslant j \leqslant \frac{p^n - 1 - 2\mathfrak{f}_\chi}{2\mathfrak{f}_\chi}.$$

Both sides of the equation are integers lying between 1 and $p^n - 1$; hence, this becomes an equality, namely $m = 2a + 2\mathfrak{f}_\chi j$.

Clearly, to have a solution pair (m, j), it is necessary that both $m \equiv 0 \pmod 2$ and $m2 \equiv a \pmod{\mathfrak{f}_\chi}$ are satisfied. On the other hand, if both congruences hold true simultaneously, then

$$(m, j) = \left(m, \frac{m/2 - a}{\mathfrak{f}_\chi}\right) \quad \text{will be the } \textit{only} \text{ solution pair}$$

under the constraint that the non-negative integer j is bounded above by

$$(p^n - 1 - 2\mathfrak{f}_\chi)/2\mathfrak{f}_\chi.$$

Reasoning as in the proof of Lemma 7.2 (a), the summation

$$\sum_{j=0}^{\boldsymbol{e}_n - 1} \Delta_{j + \boldsymbol{e}_n(m-2a) \equiv 0 \pmod{p^n}}$$

must equal the product $\Delta_{m \equiv 0 \pmod 2} \times \Delta_{m/2 \equiv a \pmod{\mathfrak{f}_\chi}}$.

The result now follows. □

8. Quadratic fields with small conductor

Let d be a square-free integer, so the discriminant of $\mathbb{Q}(\sqrt{d})$ is

$$D = \begin{cases} d & \text{if } d \text{ is congruent to 1 modulo 4,} \\ 4d & \text{if } d \text{ is congruent to 2 or 3 modulo 4.} \end{cases}$$

Table 1 gives coefficients $a_m(\chi_d)$ from Theorem 2.4 which occur in the Dirichlet expansion of the p-adic zeta function, twisted by the non-trivial quadratic character $\chi_d : (\mathbb{Z}/d\mathbb{Z})^\times \to \{\pm 1\}$ of $\mathbb{Q}(\sqrt{d})$.

Let us outline how to use the table in conjunction with previous formulae. Recall once more Hypothesis 2.1: $\gcd(p, 2|D| \cdot \phi(|D|)) = 1$.

For all integers $t \geqslant 1$, the Iwasawa function $(2\omega^\beta(2)\langle 2\rangle^{-s} - 1)\boldsymbol{L}_p(s, \chi_d \cdot \omega^{1+\beta})$ is approximated by

$$\sum_{\substack{m=1,\\ p\nmid m}}^{p^{t\phi(2|D|)}} a_m(\chi_d)\omega^\beta(m)\langle m\rangle^{-s} \quad \text{modulo } p^{t\phi(2|D|)}\mathbb{Z}_p\langle\langle s\rangle\rangle;$$

this is equation (5.1).

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Table 1. *The values of $a_m(\chi)$ for quadratic fields with $|D| \leqslant 20$*

$\mathbb{Q}(\sqrt{d})$	$\lvert D\rvert$	$a_1(\chi_d), \dots, a_{2\lvert D\rvert}(\chi_d)$
$\mathbb{Q}$	1	$-\frac{1}{2}, \frac{1}{2}$
$\mathbb{Q}(\sqrt{-3})$	3	$\frac{1}{3}, \frac{4}{3}, -\frac{5}{3}, -\frac{5}{3}, \frac{4}{3}, \frac{1}{3}$
$\mathbb{Q}(i)$	4	$\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{1}{2}$
$\mathbb{Q}(\sqrt{5})$	5	$0, 1, -2, -3, 0, 0, 3, 2, -1, 0$
$\mathbb{Q}(\sqrt{-7})$	7	$1, 2, 1, 0, -1, -2, -1, -1, -2, -1, 0, 1, 2, 1$
$\mathbb{Q}(\sqrt{2})$	8	$0, 1, -1, -2, -2, -3, -1, 0, 0, 1, 3, 2, 2, 1, -1, 0$
$\mathbb{Q}(\sqrt{-2})$	8	$1, 2, 0, 1, 1, 0, -2, -3, -3, -2, 0, 1, 1, 0, 2, 1$
$\mathbb{Q}(\sqrt{-11})$	11	$1, 2, -1, 0, 3, 4, 1, 0, -3, -2, -5, -5, -2, -3, 0, 1, 4, 3, 0, -1, 2, 1$
$\mathbb{Q}(\sqrt{3})$	12	$0, 1, -1, -1, -1, -2, -2, -3, -3, -3, -1, 0, 0, 1, 3, 3, 3, 2, 2, 1, 1, 1, -1, 0$
$\mathbb{Q}(\sqrt{13})$	13	$0, 1, -2, -1, 2, 1, -2, -3, -6, -5, -2, -3,$ $0, 0, 3, 2, 5, 6, 3, 2, -1, -2, 1, 2, -1, 0$
$\mathbb{Q}(\sqrt{-15})$	15	$2, 3, 2, 2, 1, 1, 1, 0, -1, -1, -1, -2, -2, -3, -2, -2,$ $-3, -2, -2, -1, -1, -1, 0, 1, 1, 1, 2, 2, 3, 2$
$\mathbb{Q}(\sqrt{17})$	17	$0, 1, 0, -1, -2, -3, -2, -3, -4, -3, -2, -3, -2, -1, 0,$ $1, 0, 0, -1, 0, 1, 2, 3, 2, 3, 4, 3, 2, 3, 2, 1, 0, -1, 0$
$\mathbb{Q}(\sqrt{-19})$	19	$1, 2, -1, -2, 1, 2, 5, 6, 3, 4, 1, 2, -1, -2, -5, -6, -3, -2, -5, -5,$ $-2, -3, -6, -5, -2, -1, 2, 1, 4, 3, 6, 5, 2, 1, -2, -1, 2, 1$
$\mathbb{Q}(\sqrt{-5})$	20	$2, 3, 1, 2, 2, 2, 0, 1, 1, 2, 2, 1, 1, 0, -2, -2, -2, -3, -5, -6, -6,$ $-5, -3, -2, -2, -2, 0, 1, 1, 2, 2, 1, 1, 0, 2, 2, 2, 1, 3, 2$

8.1. A worked example

Suppose that we choose $d = -1$ so χ_{-1} will be the imaginary character of conductor 4. Fix our prime number $p \geqslant 3$. Here the $a_m(\chi_d)$s have period $2\mathfrak{f}_{\chi_d} = 2|D| = 8$. If $2^{1+\beta} \not\equiv 1 \pmod{p}$, then the factor $(2\omega^\beta(2)\langle 2\rangle^{-s} - 1)$ must become invertible. For all such even branches β, the function $\boldsymbol{L}_p(s, \chi_{-1} \cdot \omega^{1+\beta})$ is congruent to

$$(2\omega^\beta(2)\langle 2\rangle^{-s} - 1)^{-1} \sum_{c=1}^{8} a_c(\chi_{-1}) \sum_{\substack{m=1,\, p\nmid m \\ m\equiv c \,(\mathrm{mod}\, 8)}}^{p^{4t}} \omega^\beta(m)\langle m\rangle^{-s} \pmod{p^{4t}},$$

where $a_1(\chi_{-1}), \dots, a_8(\chi_{-1})$ can then be read off from the third line of the table. Clearly, if β is odd, the p-adic L-functions $\boldsymbol{L}_p(s, \chi_{-1} \cdot \omega^{1+\beta})$ are identically zero.

Remarks 8.1.

(i) Washington [**8**, Theorem 1] shows, for all integers k, that

$$\sum_{\substack{m=1,\\ p\nmid m}}^{Np} m^{-k} = -\sum_{r=1}^{\infty} \binom{-k}{r} \boldsymbol{L}_p(k + r, \omega^{1-r-k})(Np)^r,$$

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interpreting

$$\binom{-k}{1-k} \boldsymbol{L}_p(1, \omega^0)$$

as

$$\lim_{s \to -k} \binom{s}{1-k} \boldsymbol{L}_p(k+s+1, \omega^0) = \frac{-1}{1-k}(1-p^{-1}).$$

This was originally proved by Boyd [**1**] in the case when $k=1$ using harmonic series. Results like these express a sum of powers m^{-k} in terms of a rapidly convergent series whose *coefficients are p-adic L-values.*

(ii) In stark contrast, Theorem 2.4 tells us that if $\chi = \mathbf{1}$,

$$\boldsymbol{L}_p(k, \omega^{1-k}) = (2^{1-k}-1)^{-1} \times \lim_{t \to \infty} \sum_{\substack{m=1,\\ p \nmid m}}^{p^t} \frac{(-1)^m}{2} m^{-k}.$$

Therefore, p-adic L-values may, in turn, be expressed through a convergent power series whose *coefficients are combinations of* $m^{-k}s$, the opposite situation to (i).

It does not seem possible to obtain our results from [**1**, **8**] (or vice versa).

(iii) Were the two approaches to be fused together in some way, this may imply algebraicity of $\boldsymbol{L}_p(4, \omega^{-3}), \boldsymbol{L}_p(6, \omega^{-5}), \boldsymbol{L}_p(8, \omega^{-7}), \dots$ over the field $\mathbb{Q}(\boldsymbol{L}_p(2, \omega^{-1}))$. It is well known that $\zeta_\infty(4), \zeta_\infty(6), \zeta_\infty(8), \dots$ are rational over $\mathbb{Q}(\zeta_\infty(2)) = \mathbb{Q}(\pi^2)$ in the classical case, a very nice consequence of the complex functional equation.

9. The non-Archimedean Euler constant

The Euler–Mascheroni constant [**4**] is defined to be the positive real number

$$\gamma := \lim_{N \to \infty} \left(\sum_{m=1}^{N} m^{-1} - \ln(N) \right) = \lim_{\sigma \to 1^+} \left(\zeta_\infty(\sigma) - \frac{\operatorname{Res}_{\sigma=1} \zeta_\infty(\sigma)}{\sigma - 1} \right),$$

where $\zeta_\infty(\sigma)$ denotes the Riemann zeta function (note that the residue at $\sigma = 1$ is one).

In other words, γ represents the constant term in its Laurent expansion about $\sigma = 1$. Numerically, the value of Euler's constant has been computed at

$$\gamma = 0.577\,215\,664\,901\,532\,860\,61\dots \quad \text{to 20 decimal places.}$$

What is its non-Archimedean analogue?

Let us recall that the trivial branch of the Kubota–Leopoldt L-function

$$\zeta_{p\text{-adic}}(s) := \boldsymbol{L}_p(s, \chi)|_{\chi=1}$$

is a meromorphic function, with a simple pole at the point $s = 1$ of residue $1 - 1/p$. Following [**7**], one defines

$$\gamma_p := \lim_{s \to 1,\, s \in \mathbb{Z}_p} \left(\zeta_{p\text{-adic}}(s) - \frac{1 - 1/p}{s - 1} \right)$$

as the p-adic analogue of the quantity γ.

We are therefore required to show that γ_p is congruent to the p-adic number

$$\frac{(\sum_{m=1,\,p\nmid m}^{p^{2(n+\epsilon_2)}}((-1)^{m+1}/2m)\langle m\rangle^{p^{n+\epsilon_2}}) - (1-(1/p))\log_p(2)(1+p^{n+\epsilon_2}\frac{1}{2}\log_p(2))}{(1-\langle 2\rangle^{p^{n+\epsilon_2}})}$$

modulo $p^n\mathbb{Z}_p$, where the constant $\epsilon_2 = \mathrm{ord}_p(\langle 2\rangle - 1)$.

Proof of Theorem 2.7. The argument is a fairly elementary consequence of Theorem 2.4.

We begin by replacing the denominator in the definition of γ_p with an Euler factor, so we may rewrite

$$\gamma_p = \lim_{s\to 1}(1-\langle 2\rangle^{1-s})^{-1}\left((1-\langle 2\rangle^{1-s})\zeta_{p\text{-adic}}(s) + \left(1-\frac{1}{p}\right)\frac{\langle 2\rangle^{1-s}-1}{s-1}\right).$$

The rightmost term has the Taylor series development

$$\frac{\langle 2\rangle^{1-s}-1}{s-1} = -\log_p(2) + \left(\frac{s-1}{2}\right)\log_p^2(2) + O((s-1)^2),$$

which means that the limit becomes

$$\gamma_p = \lim_{s\to 1}\left(\frac{(1-\langle 2\rangle^{1-s})\zeta_{p\text{-adic}}(s) - (1-(1/p))\log_p(2)(1-\frac{1}{2}(s-1)\log_p(2))}{(1-\langle 2\rangle^{1-s})}\right).$$

We shall now assume that s is close to 1, so $s = 1 - \mathfrak{u}p^M$ for some $\mathfrak{u}\in\mathbb{Z}_p^\times$ and $M \geqslant 1$. Reading off the first line in Table 1, we obtain the expansion

$$(1-\langle 2\rangle^{1-s})\zeta_{p\text{-adic}}(s) \approx \sum_{\substack{m=1,\\ p\nmid m}}^{p^{2M}}\frac{(-1)^{m+1}}{2}\omega^{-1}(m)\langle m\rangle^{-s} + O(p^{2M}\mathbb{Z}_p\langle\!\langle s\rangle\!\rangle)$$

and it follows that

$$\gamma_p \approx \frac{\begin{array}{l}\sum_{m=1,\,p\nmid m}^{p^{2M}}((-1)^{m+1}/2m)\langle m\rangle^{\mathfrak{u}p^M}\\ \qquad\qquad -(1-(1/p))\log_p(2)(1+(\frac{1}{2}\mathfrak{u}p^M)\log_p(2)) + O(p^{2M})\end{array}}{(1-\langle 2\rangle^{\mathfrak{u}p^M})}.$$

We must be careful to adjust for the size of the denominator, namely

$$|1-\langle 2\rangle^{\mathfrak{u}p^M}|_p = \left|-\mathfrak{u}p^M\log_p(2) - \frac{(\mathfrak{u}p^M)^2}{2}\log_p^2(2) + \cdots\right|_p = p^{-M-\mathrm{ord}_p(\log_p(2))}.$$

In fact, $\mathrm{ord}_p(\log_p(2)) = \mathrm{ord}_p(\langle 2\rangle - 1)$, which is precisely ϵ_2.

We have some freedom in fixing the unit $\mathfrak{u}$ and make our favourite choice $\mathfrak{u} = 1$. Hence, γ_p is approximated by

$$\frac{\sum_{m=1,\,p\nmid m}^{p^{2M}}((-1)^{m+1}/2m)\langle m\rangle^{p^M} - (1-(1/p))\log_p(2)(1+p^M\frac{1}{2}(\log_p(2)))}{(1-\langle 2\rangle^{p^M})} + O(p^{M-\epsilon_2}).$$

Finally, setting $M = n + \epsilon_2$, the stated result follows. $\square$

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