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On semi-simple radical classes

B.J. Gardner and Patrick N. Stewart

It has been incorrectly asserted that each non-trivial semi-simple radical class of associative rings is a variety defined by an equation of the form $x^n = x$. In this paper we give, for each non-trivial semi-simple radical class of associative rings, a set of equations which does define that class as a variety.

We shall discuss conditions on a class C of associative rings which are equivalent to C being a semi-simple radical class. See [1] for a discussion of the more general case in which C is a class of algebras which are not necessarily associative rings. Our theorem corrects an assertion which appears to have been first made by Snider [6] and which has been widely accepted by other authors.

If a is an element of a ring R, $[a]$ denotes the subring of R which is generated by a, and the class of rings R such that $[a] = [a]^2$ for each $a \in R$ is denoted by B_1.

Let P be a finite non-empty set of prime numbers and, for each $p \in P$, $N(p)$ a finite non-empty set of positive integers. The equations

(1) $$(\Pi\{p : p \in P\})x = 0$$

and

(2) $$\hat{p}x\Pi\left\{x^{p^n-1}-1 : n \in N(p)\right\} = 0 \text{ for each } p \in P ,$$

where $\hat{p} = \Pi\{q \in P : q \neq p\}$, define a variety which we shall denote by $V(P, N)$.

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LEMMA. *Each of the varieties* $V(P, N)$ *is contained in* B_1 *, and if* F *is a field in* $V(P, N)$ *, then* F *is a subfield of the field of order* q^m *for some* $q \in P$ *and* $m \in N(q)$.

Proof. Let a be an element of a ring $R \in V(P, N)$. Since the greatest common divisor of the numbers $\hat{p}$, $p \in P$, is 1 , the equations in (2) force $a \in [a]^2$. Thus, $[a] = [a]^2$ and so $V(P, N) \subseteq B_1$.

Assume that F is a field in $V(P, N)$. From equation (1) we see that F has characteristic q for some $q \in P$. Now, from (2),

$$\Pi\left\{a^{q^n-1}-1 : n \in N(q)\right\} = 0$$

for each non-zero $a \in F$, so F is finite. Let $u \in F$ be such that $[u] = F$. Then $u^{q^m-1} = 1$ for some $m \in N(q)$ and so F is a subfield of the field of order q^m . //

THEOREM. *Let* $C \neq \{0\}$ *be a class of associative rings which is not the class of all associative rings. The following are equivalent:*

(i) C *is a semi-simple radical class;*

(ii) $C = V(P, N)$ *for some* P *and* N *as above;*

(iii) C *is a variety contained in* B_1 .

Proof. We shall use two results from [7]: Every finitely generated ring in B_1 is isomorphic to a finite direct product of finite fields (Theorem 3.4); C is a semi-simple radical class if and only if there is a non-empty finite set F of finite fields, closed under taking subfields and such that a ring R belongs to C if and only if every finitely generated subring of R is isomorphic to a finite direct product of fields in F (Theorem 4.3).

(i) ⇒ *(ii)*. Let C be a semi-simple radical class, F the set of fields whose existence is guaranteed by [7, Theorem 4.3], $P = \{p : \exists$ a field in F of characteristic $p\}$ and, for each $p \in P$, $N(p) = \{n : \dot{\exists}$ a field of order p^n in $F\}$. We will show that $C = V(P, N)$.

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It is clear that $C \subseteq V(P, N)$ because if $0 \neq a \in R \in C$, then $[a]$ is isomorphic to a finite direct product of fields in $F \subseteq V(P, N)$.

Conversely, suppose $R \in V(P, N)$ and S is a (non-zero) finitely generated subring of R. By the lemma above, R belongs to B_1, so S is isomorphic to a direct product of finite fields [7, Theorem 3.4]. Each of these fields must be in $V(P, N)$ and so, using the lemma again, each is a subfield of a field in F. Since F is closed under taking subfields, S is isomorphic to a finite direct product of fields in F. Thus, R is in C by [7, Theorem 4.3].

(ii) $\Rightarrow$ *(iii)*. This is the first assertion in the lemma.

(iii) $\Rightarrow$ *(i)*. Let C be a variety, $C \subseteq B_1$. Let F be the class of finite fields in C. Since C contains a non-zero ring, it follows from [7, Theorem 3.4] that $F \neq \emptyset$. Also, since C is a variety, F is closed under taking subfields and, using [7, Theorem 3.4] again, we see that a ring R is in C if and only if every finitely generated subring of R is isomorphic to a finite direct product of fields in F. In view of [7, Theorem 4.3] it is sufficient to prove that F is finite (we identify isomorphic fields). Suppose $F_1, F_2, \ldots, F_n, \ldots$ are fields in F. For each n, choose $u_n \in F_n$ such that $[u_n] = F_n$. Since $F_n \in C \subseteq B_1$, $[u]$, the subring of ΠF_n which is generated by $u = (u_1, u_2, \ldots, u_n, \ldots)$, is isomorphic to a finite direct product of finite fields [7, Theorem 3.4]. Thus, there are only a finite number of possibilities for the characteristic of F_n. Moreover, there exists an integer k such that $u^k = u$. It follows that there are only a finite number of possibilities for the dimension of F_n over its prime subfield. Hence there are only a finite number of possibilities for the fields $F_1, F_2, \ldots, F_n, \ldots$ and so F is finite. This completes the proof of the theorem. //

For each integer $n \geq 2$, let V_n denote the variety defined by $x^n = x$. It follows from the implication *(iii)* $\Rightarrow$ *(i)* that V_n is a semi-simple radical class. It has been claimed that every semi-simple radical class is one of the varieties V_n (for references see [1]), but this is not correct: each V_n contains the field of order 2 , but $V(P, N)$ does not contain the field of order 2 unless $2 \in P$.

Various other conditions are equivalent to those given in the theorem.

(iv) *C is a homomorphically closed semi-simple class* (see [9, Corollary 32.2] for (i) $\Leftrightarrow$ (iv));

(v) *C is an idempotent* (that is, extension closed) *variety* (see [9, Theorem 34.1] for (iv) $\Leftrightarrow$ (v));

(vi) *C is a variety with attainable identities* (see [1, Theorem 1.5] for (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (vi));

(vii) *C is a variety generated by a finite set of finite fields* (see [3] for (v) $\Leftrightarrow$ (vii));

(viii) *C is a variety consisting entirely of arithmetic rings* (see [2] for (i) $\Leftrightarrow$ (v) $\Leftrightarrow$ (viii) and [4] for (vii) $\Leftrightarrow$ (viii)).

Finally, we note that other equational definitions of these varieties are considered in [5] and [8].

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Department of Mathematics,
University of Tasmania,
Hobart,
Tasmania;

Department of Mathematics,
Dalhousie University,
Halifax,
Nova Scotia,
Canada.

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