COMPOSITIO MATHEMATICA

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Compositio Math. **152** (2016), 556–666.

doi:10.1112/S0010437X15007459

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY EST. 1865

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doi:10.1112/S0010437X15007459

Étale motives

Denis-Charles Cisinski and Frédéric Déglise

Abstract

We define a theory of étale motives over a noetherian scheme. This provides a system of categories of complexes of motivic sheaves with integral coefficients which is closed under the six operations of Grothendieck. The rational part of these categories coincides with the triangulated categories of Beilinson motives (and is thus strongly related to algebraic K-theory). We extend the rigidity theorem of Suslin and Voevodsky over a general base scheme. This can be reformulated by saying that torsion étale motives essentially coincide with the usual complexes of torsion étale sheaves (at least if we restrict ourselves to torsion prime to the residue characteristics). As a consequence, we obtain the expected results of absolute purity, of finiteness, and of Grothendieck duality for étale motives with integral coefficients, by putting together their counterparts for Beilinson motives and for torsion étale sheaves. Following Thomason's insights, this also provides a conceptual and convenient construction of the ℓ-adic realization of motives, as the homotopy ℓ-completion functor.

Contents

Received 14 October 2013, accepted in final form 4 December 2014, published online 24 September 2015.
2010 Mathematics Subject Classification 14F20, 14F42, 14F05, 19E15, 14C25 (primary).
Keywords: étale motives, h-topology, Grothendieck six functors, rigidity theorem, l-adic realization, l-adic completion.

Partially supported by the ANR (grant No. ANR-12-BS01-0002).

Introduction

The theory of mixed motives, or mixed motivic complexes as conjecturally described by Beilinson, has evolved a lot in the last twenty years, according to the fundamental work of Voevodsky. One of the main recent evolutions is the extension of the stable homotopy theory of schemes of Morel and Voevodsky to a complete formalism of Grothendieck's six operations, as in the case of étale coefficients (SGA4, SGA5). This was made possible, following an initial idea of Voevodsky [Del01], by the work of Ayoub [Ayo07]. The stable homotopy categories of Morel and Voevodsky define the universal system of triangulated categories satisfying the formalism of Grothendieck's six operations. The triangulated categories of mixed motives should be the universal system of triangulated categories satisfying the formalism of Grothendieck's six operations and which is oriented with additive formal group law (i.e. with a theory of Chern classes behaving as in ordinary intersection theory). While such a theory already exists with rational coefficients (see [CD12] for the construction and comparison of various candidates), the construction of a version with integral coefficients is sill problematic: we can only check all the expected properties (such as proper base change formulas, finiteness theorems and duality theorems, as well as the universal property formulated above) in equal characteristics, and at the price of inverting the exponential characteristic of the ground field in the coefficients; see [CD15]. This difficulty can be explained by the fact that the usual realization functors do not define a conservative family, so that, even conjecturally, there is no hope to describe integral mixed motives in a concrete way (e.g. using the language of representations of groups). On the other hand, the strong relationship of mixed motives with classical Chow groups makes them play a central role in the understanding of

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intersection theory. Another feature which makes them interesting is that they admit a theory of weights à la Deligne with integral coefficients (such a construction is initiated by Bondarko in [Bon13]).

The aim of the present article is to study the theory of mixed motivic complexes over a general base locally for the étale topology, leading to triangulated categories of étale mixed motives. These do form the universal system of triangulated categories satisfying the formalism of Grothendieck's six operations which is oriented with additive formal group law and which satisfies étale descent. Étale mixed motives are interesting in themselves because, at least conjecturally, they fit in a tannakian picture with integral coefficients: there should exist (perverse) motivic t-structures on triangulated categories of (constructible) étale mixed motives, which, in the case of a field, should define tannakian categories defined over $\mathbf{Z}$ (note that Voevodsky has shown that there is no motivic t-structure on the triangulated category of mixed motives $\mathrm{DM}_{\mathrm{gm}}(k)$ with integral coefficients; see [VSF00, Proposition 4.8]). This is related to the fact, pointed out by Rosenschon and Srinivas [RS14], that integral versions of the Hodge conjecture and of the Tate conjecture are reasonable if we consider étale versions of Chow groups. Similarly, the triangulated category of étale mixed motives over a scheme of finite type over $\mathbf{C}$ is expected to be equivalent to the bounded derived category of the abelian category of Nori's motives. On the other hand, there is no theory of weights for étale motives with integral coefficients.[1] But with rational coefficients, these two notions of motives must coincide, so that, conjecturally, $\mathbf{Q}$-linear mixed motives should have all the advantages of these two theories (e.g. relation with classical Chow groups, weights, and motivic t-structures).

More explicitly, and in a less speculative way, over a field, Voevodsky's triangulated category of mixed motives $\mathrm{DM}(k)$ comes with its étale counterpart $\mathrm{DM}_{\text{ét}}(k)$ (see [VSF00]). These two categories coincide with $\mathbf{Q}$-coefficients, which means, for instance, that $\mathrm{DM}_{\text{ét}}(k, \mathbf{Q})$ can be used to understand algebraic K-theory up to torsion. On the other hand, as far as torsion coefficients are involved, the category $\mathrm{DM}_{\text{ét}}(k)$ is much closer to the topological world. Indeed, the rigidity theorem of Suslin and Voevodsky [SV96] means that for any positive integer n, prime to the characteristic of k, $\mathrm{DM}_{\text{ét}}(k, \mathbf{Z}/n\mathbf{Z})$ is equivalent to the derived category of $\mathbf{Z}/n\mathbf{Z}$-linear Galois modules. This is why one should expect that, over general base schemes, the use of the étale topology will make the situation better. The underlying principle which we will use repeatedly is that to prove properties of étale motives with integral coefficients, one should reduce to the case of rational coefficients, and then to the case of torsion coefficients (the latter being well understood since it belongs to the well-established realm of étale cohomology). Still it remains to find the good framework in which to define a category of étale motives with integral coefficients.[2]

There are several directions to do so. Interestingly enough, the first construction of triangulated categories of (effective) mixed motives over an arbitrary (noetherian) base goes back to 1992, in the PhD thesis of Voevodsky (see [Voe96]). It is defined in terms of $\mathbf{A}^1$-homotopy theory of complexes of sheaves with respect to the h-topology (i.e. considering étale descent together with proper descent), and, as pointed out at that time by Voevodsky himself, is a serious candidate for a theory of étale motives. Still following Voevodsky's path, there is a second possible construction using the $\mathbf{A}^1$-homotopy theory of complexes of étale sheaves with transfers (based on the theory of relative cycles by Suslin and Voevodsky; see [VSF00, ch. 2], [CD12, §§ 8 and 9]). Finally, following Morel's insights, a third possibility consists in considering the $\mathbf{A}^1$-homotopy theory of complexes of étale sheaves. The latter construction is studied by

[1] See Remark 7.2.26 for explicit obstructions.

[2] Recall this problem was originally suggested by Lichtenbaum in [Lic84].

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Ayoub [Ayo14]. In this article, we will focus on the first two constructions, and will then, using Ayoub's results, compare these with the third.

We now turn to the contributions of this article. We first consider the version of étale motives, over a general noetherian scheme X with coefficients in a ring R, defined as the category $\mathrm{DM}_{\text{ét}}(X,R)$, obtained by the $\mathbf{A}^1$-localization and $\mathbf{P}^1$-stabilization of the category of complexes of étale sheaves of R-modules with transfers over the smooth-étale site of X. If R is of positive characteristic n, we are able to establish all the expected properties.

- *Localization (Theorem 4.3.1).* Given a closed immersion $i : Z \to X$ with open complement $j : U \to X$, for any étale motive M over X, there is a canonical distinguished triangle of the form
$$j_!j^*(M) \to M \to i_*i^*(M) \to j_!j^*(M)[1].$$
- *Absolute purity (Theorem 4.6.1).* Given a closed immersion of codimension c between regular schemes $i : Z \to X$, for any étale motive M over X, there is a natural isomorphism
$$i^!(M) \simeq i^*(M)(-c)[-2c].$$
- *Rigidity (Theorem 4.5.2).* When n is invertible on X, $\mathrm{DM}_{\text{ét}}(X,R)$ is canonically equivalent to $\mathrm{D}(X_{\text{ét}},R)$, the (unbounded) derived category of étale sheaves of R-modules on the small étale site of X.

In fact, with torsion coefficients we even get the strong form of the cancellation theorem for étale motives, namely that the Tate twist in the effective category $\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(X,R)$ (obtained before applying the $\mathbf{P}^1$-stabilization process) is already invertible. The last property of the one listed above is called the *rigidity property* as it generalizes the original rigidity theorem of Suslin and Voevodsky (in the form of [VSF00, ch. 5, 3.3.3]). Moreover, in the context of sheaves with transfers we can give to this theorem a more concrete form, closer to the original result of Suslin and Voevodsky [VSF00, ch. 3, Theorem 5.25].

Theorem (see Corollary 4.5.4). *Assume that R is of positive characteristic n, and consider a noetherian scheme X with residue characteristics prime to n. For an étale sheaf with transfers of R-modules F over X, the following conditions are equivalent.*

(i) *The functor F is $\mathbf{A}^1$-local: for any smooth X-scheme Y, $H^*_{\text{ét}}(Y;F) \to H^*_{\text{ét}}(\mathbf{A}^1_Y;F)$ is an isomorphism.*

(ii) *The functor F comes from the small étale site of X: for any smooth morphism $p : Y \to X$, the transition maps $p^*(F|_{X_{\text{ét}}}) \to F|_{Y_{\text{ét}}}$ are isomorphisms.*

We also derive some pretty consequences of our work for the classical étale theory: first, we extend the main theorems, proper and smooth base changes, to the unbounded derived category (see § 1) and we also extend the theory of traces to the case of more general finite morphisms (see § 6.1 for more details).

However, to treat the integral case, we fall on the problem that, with rational coefficients, the étale and Nisnevich topologies give the same answer, and thus suffer the same defect. In particular, we only know that $\mathrm{DM}_{\text{ét}}(X,\mathbf{Q})$ is well behaved when X is quasi-excellent and geometrically unibranch (according to [CD12, Theorem 16.1.4]).

This leads us to the second possibility mentioned above, the setting of the h-topology introduced by Voevodsky at the very beginning of his theory of motives. The category $\underline{\mathrm{DM}}_{\mathrm{h}}(X,R)$, is the category obtained from the derived category of h-sheaves of R-modules after $\mathbf{A}^1$-localization and $\mathbf{P}^1$-stabilization. We then consider the category $\mathrm{DM}_{\mathrm{h}}(X,R)$, defined as the smallest thick subcategory of $\underline{\mathrm{DM}}_{\mathrm{h}}(X,R)$ closed under small sums and containing Tate twists of motives of smooth X-schemes. When R is a $\mathbf{Q}$-vector space, it is known to coincide with all the various notions of $\mathbf{Q}$-linear mixed motives which have the expected properties (mainly, the expected relation with the graded piece of algebraic K-theory with respect to the γ-filtration, good behavior with respect to the six operations of Grothendieck, and absolute purity): this is the subject of [CD12], in which we prove that five possible constructions of $\mathbf{Q}$-linear mixed motives are equivalent, the category $\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Q})$ being one of them.[3] In fact, the effective version (before $\mathbf{P}^1$-stabilization) of $\underline{\mathrm{DM}}_{\mathrm{h}}(X,R)$ was the very first construction of a triangulated category of motives considered by Voevodsky; see [Voe96]. In this article, we will see that for any ring R of positive characteristic, $\mathrm{DM}_{\mathrm{h}}(X,R)$ coincides with $\mathrm{DM}_{\text{ét}}(X,R)$ (Theorem 5.5.3), and thus, in the case where the characteristic of R is invertible in $\mathscr{O}_X$, and according to the rigidity property mentioned above, with the derived category $\mathrm{D}(X_{\text{ét}},R)$. Note that, while all these equivalences of categories appear in the work of Suslin and Voevodsky in the case where X is the spectrum of a field, the proofs we give here do not rely on these particular cases.[4] The consequence of these comparison theorems, respectively with rational coefficients and with torsion coefficients, and together with a little game with the Artin–Schreier short exact sequence, is that h-motives, with any coefficients, are well behaved with respect to the six operations, and in fact gives a common framework to the étale torsion coefficients of SGA4 and rational mixed motives. Moreover, the absolute purity theorem holds with integral coefficients for h-motives (Theorem 5.6.2).

In order to get duality properties on h-motives, a finiteness condition is needed on the objects of $\mathrm{DM}_{\mathrm{h}}(X,R)$. The category $\mathrm{DM}_{\mathrm{h},c}(X,R)$ of constructible h-motives is defined as the smallest thick subcategory containing Tate twists of motives of separated smooth X-schemes (see, more precisely, Definition 5.1.3). This notion was first introduced by Voevodsky as the finite type (effective) h-motives in [Voe96] and recast by Ayoub in the axiomatic treatment of [Ayo07]. This notion of constructibility (which we already considered in our work on motives with rational coefficients [CD12]) is good enough for most of our purposes (one can prove its compatibility with the six operations), but suffers little drawbacks: it is not local with respect to the étale topology, and, in the case of torsion coefficients, does not always coincide through the equivalence $\mathrm{D}(X_{\text{ét}},R)\simeq\mathrm{DM}_{\mathrm{h}}(X,R)$ with the notions of constructibility which are traditionally used in the context of (torsion) étale sheaves. This is why we also study the triangulated categories $\mathrm{DM}_{\mathrm{h,lc}}(X,R)$ of locally constructible motives (i.e. of h-motives which are locally constructible in the above sense, with respect to the étale topology); see Definition 6.3.1. Let us summarize the main properties of (locally) constructible h-motives over noetherian schemes of finite dimension that we prove here.

- With rational coefficients, both notions of constructible h-motives and of locally constructible h-motives coincide and are also equivalent to the purely categorical notion of compact object (Theorem 5.2.2); this remains true with integral coefficients if the base scheme is of finite type over a strictly henselian noetherian scheme, or, more generally, if the étale cohomological dimension of its residue fields is uniformly bounded (Theorem 5.2.4).

[3] However, in [CD12], we prove that the categories $\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Q})$ are well behaved (i.e. are suitably related to (homotopy) K-theory, and are closed under the six operations) only when X is quasi-excellent, noetherian, and of finite dimension. In these notes, we extend this result to the case of noetherian schemes of finite dimension; see Theorem 5.2.2.

[4] It is noteworthy that the proofs of Suslin and Voevodsky involve resolution of singularities (at least under the form of de Jong alterations), while we do not need anything like this to prove these comparison theorems in full generality.

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• Both constructible and locally constructible h-motives with integral coefficients are stable with respect to the six operations for quasi-excellent schemes (Corollaries 6.2.14 and 6.3.15).

• Both constructible and locally constructible h-motives are compatible with projective limits in the base schemes with integral coefficients, the property that we called *continuity* in [CD12, 4.3] (see Theorems 6.3.9 and 6.3.12, respectively).

• Under a mild assumption on the base scheme, there exists a dualizing object for constructible h-motives with integral coefficients satisfying the expected properties of Grothendieck–Verdier duality (Theorem 6.2.17), and this duality extends to locally constructible h-motives (Corollary 6.3.15).

• For any surjective morphism of finite type $f : X \to S$ between noetherian schemes of finite dimension, and for any object M of $\mathrm{DM}_{\mathrm{h}}(S, \mathbf{Z})$, if $f^*(M)$ is locally constructible, then so is M; and if S is quasi-excellent, the same is true if we replace $f^*(M)$ by $f^!(M)$ (Proposition 6.3.18).

• For any noetherian ring of coefficients R, whose characteristic is invertible in $\mathscr{O}_X$, we have a canonical identification of locally constructible h-motives over X with bounded complexes of sheaves of R-modules on the small étale site of X which are of finite Tor-dimension and have constructible cohomology sheaves in the sense of SGA4: $\mathrm{D}^b_{\mathrm{ctf}}(X_{\text{ét}}, R) \simeq \mathrm{DM}_{\mathrm{h,lc}}(X, R)$ (Theorem 6.3.11). This correspondence is compatible with the six operations.

For further explanations concerning (locally) constructible motives, the reader may have a look at Remarks 5.4.10 and 5.5.11 (about the abundance of non-compact constructible h-motives), and at Remark 6.3.2 (for a digression on the meaning of locally constructible h-motives). An alternative characterization of locally constructible h-motives with integral coefficients is given by Theorem 6.3.26.

Among the applications of this formalism, we study the *étale motivic cohomology* (also known as the Lichtenbaum motivic cohomology) of X, understood here as the usual extension groups computed in $\mathrm{DM}_{\mathrm{h}}(X, \mathbf{Z})$:

$$\mathrm{H}^{r,n}_{\text{ét}}(X) = \mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(X)}(\mathbf{Z}_X, \mathbf{Z}_X(n)[r]).$$

First, we recall that, when X is a scheme of finite type over a field k, up to inverting the exponential characteristic of k, it coincides with the étale hypercohomology of the Bloch cycle complex (Theorem 7.1.2). Secondly, when X is a regular noetherian scheme of finite dimension, we construct the cycle class map with values in étale motivic cohomology

$$CH^n(X) \to \mathrm{H}^{2n,n}_{\text{ét}}(X),$$

and show it is an isomorphism after inverting all primes, or if $n = 1$ after inverting the set N of the exponential characteristics of the residue fields of X; we also show it is a monomorphism if $n = 2$ after inverting N (Theorem 7.1.11). This is achieved via a study of the coniveau spectral sequence of étale motivic cohomology, which uses the absolute purity theorem for h-motives with integral coefficients, as well as the validity of the Bloch–Kato conjecture. The regularity assumption on X can be avoided if we replace étale motivic cohomology by étale motivic Borel–Moore homology.

The main interest of the formalism described above is to provide an integral part to the torsion étale theory of [SGA4]. We exploit this fact, for any prime number ℓ, by considering the ℓ-adic completion of $\mathrm{DM}_{\mathrm{h}}(X, \mathbf{Z})$ from a homotopical (or derived) perspective. The immediate advantage of this construction is that the resulting category, denoted by $\mathrm{DM}_{\mathrm{h}}(X, \hat{\mathbf{Z}}_\ell)$ in Definition 7.2.1, readily has all the advantage of its integral model: six operations, and absolute purity.

We exhibit two natural notions of finiteness for these ℓ-adic h-motives: constructibility and geometricity (Definition 7.2.13). Both notions are stable by the six operations (Remark 7.2.15

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and Theorem 7.2.16). Using our comparison theorem in the case of torsion coefficients, we show that, when restricted to schemes of residue characteristics prime to ℓ, the category of constructible ℓ-adic h-motives not only extends the classical definitions of Deligne [BBD82] (whenever that makes sense, see Proposition 7.2.19) but in fact coincides in full generality with the constructible ℓ-adic systems defined by Ekedahl in [Eke90] (see Proposition 7.2.21), in a compatible way with the six operations (for Ekedahl's ℓ-adic systems, all this remains true for non-necessarily constructible objects). This has various nice consequences such as showing that Ekedahl constructible ℓ-adic systems are stable by the six operations over any quasi-excellent schemes, and giving a t-structure on ℓ-adic constructible h-motives.

Finally, the crux is reached as ℓ-adic systems are in fact h-motives: for any noetherian $\mathrm{Spec}(\mathbf{Z}[\ell^{-1}])$-scheme of finite dimension, they form a full triangulated subcategory of $\mathrm{DM}_{\mathrm{h}}(X, \mathbf{Z})$, and the inclusion functor has a symmetric monoidal left adjoint,

$$\hat{\rho}_\ell^* : \mathrm{DM}_{\mathrm{h}}(X, \mathbf{Z}) \to \mathrm{DM}_{\mathrm{h}}(X, \hat{\mathbf{Z}}_\ell) \simeq \mathrm{D}(X, \mathbf{Z}_\ell),$$

see (7.2.4.a). This is the ℓ-adic realization functor: it commutes with all of the six operations (including for non-necessarily constructible objects), and sends (locally) constructible h-motives to constructible ℓ-adic systems.

On the homotopy level, ℓ-adic realization is the same thing as homotopy ℓ-completion (see Proposition 7.2.8); with a little abuse of notations, we have

$$\hat{\rho}_\ell^*(M) = \mathbf{R} \varprojlim_r M/\ell^r.$$

We cannot resist to give here an analogy with the situation of the derived category of abelian groups: it is easy to prove that the homotopy ℓ-adic completion is conservative rationally, once restricted to perfect complexes of abelian groups. This gives a new light on the conservativity conjecture of Beilinson which can be stated as the hope that, for any noetherian scheme of finite dimension, the functor

$$\hat{\rho}_\ell^* \otimes \mathbf{Q} : \mathrm{DM}_{\mathrm{h},c}(X, \mathbf{Q}) \to \mathrm{D}^b_c(X, \mathbf{Q}_\ell)$$

is conservative [Bei87, § 5.10, end of A]. This conjecture would imply another one, which is also natural if we think of motives as a generalization of abelian groups: for any noetherian scheme of finite dimension, the family of integral ℓ-adic realization functors below, indexed by all prime ℓ,

$$\mathrm{DM}_{\mathrm{h,lc}}(X, \mathbf{Z}) \xrightarrow{\text{restriction}} \mathrm{DM}_{\mathrm{h,lc}}(X \times \mathrm{Spec}\, \mathbf{Z}[\ell^{-1}], \mathbf{Z}) \xrightarrow{\hat{\rho}_\ell^*} \mathrm{D}^b_c(X \times \mathrm{Spec}\, \mathbf{Z}[\ell^{-1}], \mathbf{Z}_\ell),$$

should form a conservative family. Equivalently, this would mean that, for an object M of $\mathrm{DM}_{\mathrm{h}}(X, \mathbf{Z})$, if, for *any* prime ℓ, the ℓ-adic completion $\mathbf{R} \varprojlim_r M/\ell^r$ vanishes, then $M \simeq 0$. In other words we expect that $\mathbf{Q}$-linear h-motives cannot be (locally) constructible when seen in $\mathrm{DM}_{\mathrm{h}}(X, \mathbf{Z})$.

To be complete, we give a comparison statement (Corollary 5.5.7) between the approach of this article and the one of [Ayo14]: for any noetherian scheme of finite dimension X and any ring R such that either X is of characteristic zero or that 2 is invertible in R, the canonical functor

$$\mathrm{D}_{\mathbf{A}^1,\text{ét}}(X, R) \to \mathrm{DM}_{\mathrm{h}}(X, R)$$

is an equivalence of triangulated categories (which is compatible with the six operations), where the left-hand side is the homotopy category of the $\mathbf{P}^1$-localization of the $\mathbf{A}^1$-localization of the model category of complexes of sheaves of R-modules on the smooth-étale site of X. The reason

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why we think 2-torsion is problematic in the whole article [Ayo14] (except if we restrict ourselves to schemes of characteristic zero) is explained in Remark 5.5.8, in which we also explain why the recent work of Morel should allow us to solve this puzzle. We also emphasize that Ayoub always works with a ring of coefficients R such that any prime number p is invertible either in R or in the structural sheaf of the base scheme, so that he never considers étale motives with integral coefficients in mixed characteristic. Note that Ayoub also considers the comparison of $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(X, R)$ with its counterparts with transfers $\mathrm{DM}_{\text{ét}}(X, R)$, but, even in the case when R is of positive characteristic, he only does it for X normal (and, if X is not of characteristic zero, there is also the problem with 2-torsion as above). Finally, for theorems about the stability of constructible objects under the six operations and duality theorems in $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(X, R)$, Ayoub always makes the assumption that the étale cohomological dimension of the residue fields of X with R-linear coefficients is uniformly bounded (this means that he always works in a context where constructible objects precisely are the compact objects). In particular, and a little bit ironically, for schemes of finite type over $\mathbf{Q}$, one still needs to avoid 2-torsion to apply the full strength of Ayoub's article.

As for the organization of this article, we will use the language we are the most familiar with: the one of [CD12]. A little recollection is given in the Appendix A, in which one can find some complements about the notion of absolute purity and about the effect of the Artin–Schreier exact sequence in étale $\mathbf{A}^1$-homotopy theory, as well as a few remarks on idempotent completion and localization of coefficients in abstract triangulated categories.

The first section of this paper consists in formulating classical results of étale cohomology (such as the proper base change theorem, the smooth base change theorem, or cohomological descent) in terms of unbounded complexes for arbitrary noetherian schemes. Except for the proper base change formula, this extension to unbounded complexes uses non-trivial results of Gabber on the étale cohomological dimension; however, if one is only interested in excellent schemes of characteristic zero or in schemes of finite type over an excellent schemes of dimension less than or equal to 1, one can rely on more classical results from SGA4 (see Remark 1.1.6). Part of the results of this section are abstract because we will need such a level of generality later on, to deal with the problem of cohomological descent with unbounded complexes without any assumptions on the cohomological dimension.

These classical results are then used in §§ 2–4 to study the triangulated categories $\mathrm{DM}_{\text{ét}}(X, R)$: in § 2, we recall the theory of étale sheaves with transfers over general bases with coefficients in an arbitrary ring R. Its effective version was first introduced over fields of finite cohomological dimension by Voevodsky. We establish all the good properties of these sheaves using the framework of [CD12, part 3], without assuming finite cohomological dimension of the base scheme: namely, it forms an abelian premotivic category (see Appendix A.1 for recall on that later notion), and moreover satisfies a weak form of the localization property (Proposition 2.3.5). This leads in particular to the effective (respectively stable) $\mathbf{A}^1$-derived category of sheaves with transfers $\mathrm{DM}^{\text{eff}}_{\text{ét}}(-, R)$ (respectively $\mathrm{DM}_{\text{ét}}(-, R)$); see § 2.2.4.

In § 3, we begin to investigate the link between étale sheaves of R-modules on the small site and sheaves with transfers. The main result is that, for any ring R and over any base, these sheaves uniquely admits transfers (Proposition 3.1.4). When R is of positive characteristic n, and n is invertible on X, we deduce an embedding of the derived category of such sheaves to $\mathrm{DM}^{\text{eff}}_{\text{ét}}(-, R)$ (Proposition 3.1.7).

Using all these preparatory results, the crux is reached in § 4 with the first version of the rigidity theorem: the equivalence between the categories $\mathrm{DM}^{\text{eff}}_{\text{ét}}(X, R)$ and $\mathrm{D}(X_{\text{ét}}, R)$ for a ring R of positive characteristic invertible on X (Theorem 4.5.2). Beside classical properties of étale

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cohomology, the main point here is that, with this constraint on the coefficient ring R, we prove in § 4.3 the localization property (recall Definition A.1.12) for $\mathrm{DM}_{\acute{e}t}(X, R)$. In the theory of sheaves with transfers, and more generally in the study of algebraic cycles, this property is a crucial point, as shown for example by the difficulty of proving that Bloch higher Chow groups have a localization long exact sequence, which is still open in the unequal characteristic case. So far, with integral coefficients, this property is unknown for the Nisnevich topology, and for non-geometrically unibranch schemes for the étale topology.[5]

Section 5, is devoted to the study of the triangulated categories of h-motives $\mathrm{DM}_{\mathrm{h}}(X, R)$. It is organized as follows. Section 5.1 is devoted to the basic definitions of h-motives. The comparison of h-motives and Beilinson motives was proved in [CD12] for quasi-excellent schemes, and § 5.2 is devoted to the proof that we can remove this assumption, and get a comparison theorem for noetherian schemes of finite dimension (Theorem 5.2.2). In § 5.3, we extend the proper descent theorem in torsion étale cohomology to unbounded complexes with the help of the results of the first section, but also of a non-trivial result of Goodwillie and Lichtenbaum on the cohomological dimension for the h-topology. Section 5.4 contains basic results on the effect of changing the coefficient ring R. In § 5.5, we prove a comparison theorem relating h-motives with torsion coefficients with the étale version that we have studied in §§ 2–4. This is also where we compare h-motives with $\mathrm{D}_{\mathbf{A}^1,\acute{e}t}(X, R)$. We explain how to use this, together with the results of § 5.2, to understand the behavior of direct image functors with small sums and arbitrary change of coefficients. In § 5.6, we show that h-motives with an arbitrary ring of coefficients satisfy the complete six functors formalism (at least over noetherian schemes of finite dimension).

Section 6.1 contains preliminary results for the study of constructible h-motives, on the existence of rather general trace maps, which correspond to the structure of presheaf with transfers, for h-motives. In § 6.2, constructible h-motives are studied thoroughly: the main point is the fact f_* respects constructibility (Theorem 6.2.13), which yields the same property for all of the six functors, and the duality Theorem 6.2.17. Most of the proof of this non-trivial property is an adaptation of arguments and results of Gabber. Section 6.3 is devoted to the compatibility of constructible h-motives with projective limits of schemes (*continuity*) as well as to the study of locally constructible h-motives: stability under the six operations, and comparison with $\mathrm{D}^b_{\mathrm{ctf}}(X_{\acute{e}t}, R)$.

Section 7.1 is devoted to étale motivic cohomology (defined as extension groups in DM_{h}) and to its relation with classical (possibly higher) Chow groups (as already mentioned above). Finally, § 7.2 studies (derived) ℓ-adic completion of h-motives, its link with ℓ-adic systems and ℓ-adic realization.

Conventions

Unless stated otherwise, *all schemes are assumed to be noetherian*. In particular, premotivic categories in the text (recall in Appendix A.1) are assumed to be fibered over the category of noetherian schemes. When dealing with rational or integral coefficients, we will need to restrict ourselves to schemes which are in addition finite dimensional. This will always be indicated.

Unless stated otherwise, the word *'smooth' (respectively 'étale') means smooth (respectively étale) and separated of finite type*. We will consider the following classes of morphisms of schemes:

[5] For the étale topology, the case of geometrically unibranch scheme is a consequence of Corollary 5.5.5 and Theorem 5.6.2.

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- Ét for the class of étale morphisms;
- Sm for the class of smooth morphisms;
- $\mathscr{S}^{\mathrm{ft}}$ for the class of morphisms of finite type.

Given a base scheme S, we let $X_{\text{ét}}$ (respectively Sm_S, $\mathscr{S}^{\mathrm{ft}}_S$) be the category of (noetherian) S-schemes whose structural morphism is in Ét (respectively Sm, $\mathscr{S}^{\mathrm{ft}}$).

The dimension of a smooth morphism (respectively codimension of a regular immersion) will be understood as the corresponding Zariski locally constant function d (respectively c) on the source scheme. The twist by d (respectively c) will be the obvious sum of twists obtained by additivity.

Given any adjunction (F, G) of categories, we will denote generically by

$$ad(F, G) : 1 \to GF \quad \text{and} \quad ad'(F, G) : FG \to 1$$

the unit and counit of the given adjunction, respectively.

The letter R will denote a commutative ring which will serve as a ring of coefficients for all our sheaves. In § 4 only, it will be implicitly assumed to be of positive characteristic n.

The letter Λ will denote a localization of $\mathbf{Z}$ which will serve as a ring of coefficients for all our cycles. We assume that R is a Λ-algebra.

We will freely use results on triangulated categories from Neeman's book [Nee01], without warning. We simply recall that, in a given triangulated category T, a family of objects G *generates* T is, for any object M of T, if $\mathrm{Hom}_T(X, M[n]) \simeq 0$ for any X in G and any integer n, then $M \simeq 0$.

1. Unbounded derived categories of étale sheaves

In this section we give a reminder of the properties of étale cohomology, as developed by Grothendieck and Artin in [SGA4]. There is nothing new, except some little complements about unbounded derived categories of étale sheaves. This section is the only one of this paper in which schemes are not supposed to be noetherian.

1.1 Cohomological dimension

1.1.1. Let X be a scheme. We denote by $X_{\text{ét}}$ the topos of sheaves on the small étale site of X. Given a ring R, we write $\mathrm{Sh}(X_{\text{ét}}, R)$ for the category of sheaves of R-modules on $X_{\text{ét}}$. We will denote by $\mathrm{D}(X_{\text{ét}}, R)$ the unbounded derived category of the abelian category $\mathrm{Sh}(X_{\text{ét}}, R)$. Given an étale scheme U over X, we will write $R(U)$ for the sheaf representing evaluation at U (i.e. the étale sheaf associated with the presheaf $R\langle \mathrm{Hom}_X(-, U)\rangle$).

Definition 1.1.2. Let R be a ring of coefficients. A scheme X is of *finite étale cohomological dimension* with R-linear coefficients if there exists an integer n such that $H^i_{\text{ét}}(X, F) = 0$ for any sheaf of R-modules F over $X_{\text{ét}}$ and any integer $i > n$. In the case where $R = \mathbf{Z}$, we will simply say that X is of finite étale cohomological dimension.

Let ℓ be a prime number.

A scheme X is of *finite ℓ-cohomological dimension* if there exists an integer n such that $H^i_{\text{ét}}(X, F) = 0$ for any sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules F over $X_{\text{ét}}$ and any integer $i > n$. We denote by $\mathrm{cd}_\ell(X)$ the smallest integer n with the property above.

A field k is *of finite ℓ-cohomological dimension* if $\mathrm{Spec}(k)$ has this property.

THEOREM 1.1.3 (Gabber). *Let X be a strictly local noetherian scheme of dimension $d > 0$, and ℓ a prime which is distinct of the residue characteristic of X. Then, for any open subscheme $U \subset X$, we have $\mathrm{cd}_\ell(U) \leqslant 2d - 1$.*

For a proof, see [ILO14, Exposé $\mathrm{XVIII_A}$, Theorem 1.1].

LEMMA 1.1.4. *Let X be a noetherian scheme of dimension d. Then, for any sheaf of $\mathbf{Q}$-vector spaces F over $X_{\text{ét}}$, we have $H^i_{\text{ét}}(X, F) = 0$ for $i > d$.*

Proof. Nisnevich cohomology and étale cohomology with coefficients in étale sheaves of $\mathbf{Q}$-vector spaces coincide, and Nisnevich cohomological dimension is bounded by the dimension, which proves this assertion. □

THEOREM 1.1.5 (Gabber). *Let S be a strictly local noetherian scheme and X a S-scheme of finite type. Then X is of finite étale cohomological dimension, and the residue fields of X are uniformly of finite étale cohomological dimension.*

Proof. An easy Mayer–Vietoris argument shows that it is sufficient to prove the theorem in the case where X is affine. For a point $x \in X$ with image $s \in S$, we write $d(x)$ for the degree of transcendence of the residue field $\kappa(x)$ over $\kappa(s)$. Note that, for any prime ℓ which is invertible in $\kappa(x)$, we have $\mathrm{cd}_\ell(\kappa(x)) \leqslant d(x) + \mathrm{cd}_\ell(\kappa(s))$; see [SGA4, Exposé X, Théorème 2.1]. Therefore, by virtue of Gabber's Theorem 1.1.3, we have $\mathrm{cd}_\ell(\kappa(x)) \leqslant d(x) + 2\dim(S) - 1$. Let us define

$$N = \max\{\dim(X), \sup_{x\in X}(2\dim(S) + 1 + d(x) + 2\,\mathrm{codim}(x))\}.$$

We will prove that $H^i_{\text{ét}}(X, F) = 0$ for any sheaf F over $X_{\text{ét}}$ and any $i > N$. As X is quasi-compact and quasi-separated, the functors $H^i_{\text{ét}}(X, -)$ commute with filtered colimits; see [SGA4, Exposé VII, Proposition 3.3]. Therefore, we may assume that F is constructible; see [SGA4, Exposé IX, Corollaire 2.7.2]. We have an exact sequence of the form

$$0 \to T \to F \to C \to 0$$

where T is torsion and C is without torsion (in particular, C is flat over $\mathbf{Z}$). Therefore, we may assume that $F = T$ or $F = C$. We also have a short exact sequence

$$0 \to C \to C \otimes \mathbf{Q} \to C \otimes \mathbf{Q}/\mathbf{Z} \to 0$$

from which we deduce that

$$H^i_{\text{ét}}(X, C \otimes \mathbf{Q}/\mathbf{Z}) \simeq \varinjlim_n H^i_{\text{ét}}(X, C \otimes \mathbf{Z}/n\mathbf{Z})$$

for all i. Lemma 1.1.4 thus shows that it is sufficient to consider the case where F is the form T or $C \otimes \mathbf{Z}/n\mathbf{Z}$. But, as T is torsion and constructible, it is a $\mathbf{Z}/n\mathbf{Z}$-module for some integer $n \geqslant 1$. We are reduced to the case where F is a constructible sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules for some integer $n \geqslant 1$. We can find a finite filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_k = F$$

such that F_{j+1}/F_j is a $\mathbf{Z}/\ell_j\mathbf{Z}$-module for any j, with ℓ_j a prime number: this follows from the fact such a filtration exists in the category of finite abelian groups, using [SGA4, Exposé IX, Proposition 2.14]. Therefore, we may assume that $n = \ell$ is a prime number.

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We will prove that, for any sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules F over $X_{\text{ét}}$, we have $H^a_{\text{ét}}(X, F) = 0$ for $a > N$. Let $Z = \operatorname{Spec}(\mathbf{Z}/\ell\mathbf{Z}) \times X$ and $U = X - Z$. We have a closed immersion $i : Z \to X$ and its open complement $j : U \to X$, which gives rise to a distinguished triangle

$$i_*\mathbf{R}i^!(F) \to F \to \mathbf{R}j_*j^*(F) \to i_*\mathbf{R}i^!(F)[1]$$

and thus to an exact sequence

$$0 \to i_*i^!(F) \to F \to j_*j^*(F) \to i_*\mathbf{R}^1i^!(F) \to 0$$

together with isomorphisms

$$\mathbf{R}^bj_*j^*(F) \simeq i_*\mathbf{R}^{b+1}i^!(F) \quad \text{for } b \geqslant 1.$$

On the other hand, we have, for any étale X-scheme V

$$H^b_{\text{ét}}(U \times_X V, j^*(F)) = 0 \quad \text{for any integer } b > \delta = \sup_{x \in U}(\operatorname{cd}_\ell(k(x)) + 2\operatorname{codim}(x))$$

(see [ILO14, Exposé XVIII$_{\text{A}}$, Lemma 2.2] and [SGA4, Exposé IX, Corollaire 4.3]). Therefore, we have $\mathbf{R}^bj_*j^*(F) = 0$ for $b > \delta$. Hence $\mathbf{R}^bi^!(F) = 0$ for $b > \delta + 1$. By virtue of [SGA4, Exposé X, Théorème 5.1], as Z is affine, we also have $H^i_{\text{ét}}(Z, G) = 0$ for $i > 1$ and for any sheaf of $\mathbf{Z}/\ell\mathbf{Z}$-modules G. The spectral sequence

$$H^a_{\text{ét}}(Z, \mathbf{R}^bi^!(F)) \Rightarrow H^{a+b}_{\text{ét}}(Z, \mathbf{R}i^!(F))$$

thus implies that $H^a_{\text{ét}}(Z, \mathbf{R}i^!(F)) = 0$ for $a > \delta + 2$. In conclusion, the long exact sequence

$$H^a_{\text{ét}}(Z, \mathbf{R}i^!(F)) \to H^a_{\text{ét}}(X, F) \to H^a_{\text{ét}}(U, j^*(F)) \to H^{a+1}_{\text{ét}}(Z, \mathbf{R}i^!(F))$$

gives $H^a_{\text{ét}}(X, F) = 0$ for $a > \delta + 2$. □

Remark 1.1.6. Gabber also proved the affine Lefschetz theorem: if X is an excellent strictly local scheme of dimension d, for any open subscheme $U \subset X$, we have $\operatorname{cd}_\ell(U) \leqslant d$; see [ILO14, Exposé XV, Corollaire 1.2.4]. In the case of excellent schemes of characteristic zero, this had been proved by Artin, using Hironaka's resolution of singularities; see [SGA4, Exposé XIX, Corollaire 6.3]. The case of a scheme of finite type over an excellent scheme of dimension less than or equal to 1 was also known (this follows easily from [SGA4, Exposé X, Proposition 3.2]).

Lemma 1.1.7. *Let $\mathscr{A}$ be a Grothendieck abelian category. We also consider a left exact functor*

$$F : \mathscr{A} \to \mathbf{Z}\text{-Mod},$$

and we denote by

$$\mathbf{R}F : \mathrm{D}(\mathscr{A}) \to \mathrm{D}(\mathbf{Z}\text{-Mod})$$

its total right derived functor. We suppose that the functor

$$\mathscr{A} \to \mathbf{Z}\text{-Mod}, \quad A \mapsto \mathbf{R}^nF(A)$$

commutes with small filtered colimits for any integer $n \geqslant 0$.

Then, the following conditions are equivalent.

(i) *The functor*

$$\mathrm{C}(\mathscr{A}) \to \mathbf{Z}\text{-Mod}, \quad K \mapsto H^0\mathbf{R}F(K)$$

commutes with small filtered colimits.

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(ii) *The functor* $\mathbf{R}F$ *commutes with small sums.*
(iii) *The functor* $\mathbf{R}F$ *commutes with countable sums.*
(iv) *For any degree-wise* F*-acyclic complex* K*, the natural map* $F(K) \to \mathbf{R}F(K)$ *is an isomorphism in* $\mathrm{D}(\mathbf{Z}\text{-Mod})$.

Moreover, the four conditions above are verified whenever the functor F *is of finite cohomological dimension.*

Proof. It is clear that (i)⇒(ii)⇒(iii). It is also easy to see that property (iv) implies property (i). Indeed, our assumption on F implies that the class of F-acyclic objects is closed under filtered colimits, which implies that the class of degree-wise F-acyclic complexes has the same property. On the other hand, property (iv) implies that the functor $\mathbf{R}F$ may be constructed using resolutions by degree-wise F-acyclic complexes, from which property (i) follows immediately.

Let us show that condition (iii) implies condition (iv). Consider a sequence of morphisms of complexes of $\mathscr{A}$:

$$K_0 \to K_1 \to \cdots \to K_n \to K_{n+1} \to \cdots, \quad n \geqslant 0.$$

We then have a map

$$1 - d : \bigoplus_n K_n \to \bigoplus K_n,$$

where d is the morphism induced by the maps $K_n \to K_{n+1}$. The cone of $1 - d$ (the cokernel of $1 - d$, respectively) is the homotopy colimit (the colimit, respectively) of the diagram $\{K_n\}$. Moreover, as filtered colimits are exact in $\mathscr{A}$, the canonical map

$$\mathbf{L}\varinjlim_n K_n \to \varinjlim_n K$$

is an isomorphism in $\mathrm{D}(\mathscr{A})$. As a consequence, it follows from condition (iii) that, if K belongs to $\mathrm{C}(\mathscr{A})$, we have a natural long exact sequence of shape

$$\cdots \to \bigoplus_n H^i\mathbf{R}F(K_n) \overset{1-d}{\to} \bigoplus_n H^i\mathbf{R}F(K_n) \to H^i\mathbf{R}F(\varinjlim_n K_n) \to \cdots.$$

It is easy to deduce from this that, assuming condition (iii), the natural map

$$\varinjlim_n H^0\mathbf{R}F(K_n) \to H^0\mathbf{R}F(\varinjlim_n K_n)$$

is always invertible.

For an integer n, let us write $\sigma^{\geqslant n}(K)$ for the 'troncation bête', defined as $\sigma^{\geqslant n}(K)^i = K^i$ if $i \geqslant n$ and $\sigma^{\geqslant n}(K)^i = 0$ otherwise. We can then write

$$\varinjlim_n \sigma^{\geqslant m}(K) \simeq K.$$

Suppose furthermore that the complex K is degree-wise F-acyclic. Then $\sigma^{\geqslant n}(K)$ has the same property and has moreover the good taste of being bounded below. Therefore, the map

$$F(\sigma^{\geqslant n}(K)) \to \mathbf{R}F(\sigma^{\geqslant n}(K))$$

is an isomorphism for any integer n. As both the functors H^0F and $H^0\mathbf{R}F$ commutes with $\varinjlim_n$, we conclude that property (iv) is verified.

The fact that property (iv) is true whenever F is of finite cohomological dimension is well known (it is already in the book of Cartan and Eilenberg in the case where $\mathscr{A}$ is a category of modules over some ring, and a general argument may be found for instance in [SV00a, Lemma 0.4.1]). □

1.1.8. Given a topos T and a ring R, we will write $\mathrm{Sh}(T, R)$ for the category of R-modules in T (or, equivalently, the category of sheaves of R-modules over T). If $\mathscr{G}$ is a generating family of T, the category $\mathrm{C}(\mathrm{Sh}(T, R))$ is endowed with the *projective model category structure* with respect to $\mathscr{G}$ (see [CD09, Example 2.3, Theorem 2.5, Corollary 5.5]): the weak equivalences are the quasi-isomorphisms, while the fibrant objects are the complexes of sheaves of R-modules K such that, for any object U in $\mathscr{G}$, the natural map

$$H^n(\Gamma(U, K)) \to H^n(U, K)$$

is an isomorphism for any integer n (where $H^n(U, K)$ denotes the hypercohomology groups of U with coefficients in K). The fibrations (trivial fibrations) are the morphisms of shape $p : K \to L$ with the following properties.

(i) For any object U in $\mathscr{G}$, the map $p : \Gamma(U, K) \to \Gamma(U, L)$ is degree-wise surjective.
(ii) The kernel of p is fibrant (the complex $\Gamma(U, \ker(p))$ is acyclic for any U in $\mathscr{G}$, respectively).

Moreover, for any object U in $\mathscr{G}$, the object $R(U)$ (the free sheaf of R-modules generated by U), seen as a complex concentrated in degree zero, is cofibrant. We will write $\mathrm{D}(T, R)$ for the (unbounded) derived category of $\mathrm{Sh}(T, R)$.

If a topos T is canonically constructed as the category of sheaves on a Grothendieck site, the class of representable sheaves is a generating family of T, and, unless we explicitly specify another choice, the projective model structures on the categories of sheaves of R-modules over T will be considered with respect this generating family. For instance, for a scheme X, we will always understand the topos $X_{\text{ét}}$ as the category of sheaves over the small étale site of X, so that its canonical generating family is given by the collection of all étale schemes of finite presentation over X.

Proposition 1.1.9. *Consider a topos T and a ring R. We suppose that T is endowed with a generating family $\mathscr{G}$ such that any $U \in \mathscr{G}$ is coherent and of finite cohomological dimension for R-linear coefficients. Then, for any $U \in \mathscr{G}$, the functor*

$$\mathrm{C}(\mathrm{Sh}(T, R)) \to R\text{-Mod}, \quad K \mapsto \mathrm{Hom}_{\mathrm{D}(T,R)}(R(U), K) = H^0(U, K)$$

preserves small filtered colimits.

In particular, the family $\{R(U) \mid U \in \mathscr{G}\}$ form a family of compact generators of the triangulated category $\mathrm{D}(T, R)$.

Proof. This is a direct consequence of Lemma 1.1.7. □

Lemma 1.1.10. *Let T be a topos and U a coherent object of T. Consider a localization R of the ring of integers $\mathbf{Z}$. For any sheaf of abelian groups F over T, the natural map*

$$H^i(U, F) \otimes R \to H^i(U, F \otimes R)$$

is invertible for any integer i. In particular, tensoring with R preserves $\Gamma(U, -)$-acyclic sheaves

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over T. If moreover U is of finite cohomological dimension with rational coefficients, then, for any complex of sheaves of abelian groups K over T, the canonical map

$$H^i(U, K) \otimes R \to H^i(U, K \otimes R)$$

is bijective for any integer i.

Proof. The first assertion immediately follows from the fact that the functor $H^i(U, -)$ preserves filtering colimits of sheaves. The second assertion is an immediate consequence of the first. Finally, in the case where $R = \mathbf{Q}$, the last assertion is a direct consequence of Lemma 1.1.7. To prove the general case, it is sufficient to check that the natural map

$$\mathbf{R}\Gamma(X, K) \otimes R \to \mathbf{R}\Gamma(X, K \otimes R)$$

is an isomorphism in the derived category of R-modules. As it is invertible after tensorization by $\mathbf{Q}$, it is sufficient to check that it becomes invertible after we apply the functor $C \mapsto C \otimes^{\mathbf{L}} \mathbf{Z}/p\mathbf{Z}$ for any prime number p. But such an operation commutes with the derived global section functor, and this proves the last assertion in full generality. □

Proposition 1.1.11. *Let X be a noetherian scheme of finite dimension, and R be a localization of $\mathbf{Z}$. For any complex of étale sheaves of abelian groups K over X, the natural map*

$$H^i_{\text{ét}}(X, K) \otimes R \to H^i_{\text{ét}}(X, K \otimes R)$$

is bijective for any integer i.

Proof. By virtue of Lemma 1.1.4, this obviously is a particular case of the preceding lemma. □

The following lemma is the main tool to extend results about unbounded complexes of sheaves which are known under a global finite cohomological dimension hypothesis to contexts where finite cohomological dimension is only assumed point-wise (in the topos theoretic sense). This will be used to extend to unbounded complexes of étale sheaves the smooth base change formula as well as the proper cohomological descent theorem. We will freely use the language and the results of [SGA4, Exposé VII] about coherent topoi and filtering limits of these.

Lemma 1.1.12. *Consider a ring of coefficients R and an essentially small cofiltering category I as well as a fibered topos $S \to I$. For each index i we consider a given generating family $\mathscr{G}_i$ of the topos S_i. We write $T = \varprojlim_I S$ for the limit topos, and $\pi_i : T \to S_i$ for the canonical projections. We then have a canonical generating family $\mathscr{G}$ of T, which consists of objects of the form $\pi_i^*(X_i)$, where X_i is an element of the class $\mathscr{G}_i$. Given a map $f : i \to j$ in I and a sheaf F_j over S_j, we will write F_i for the sheaf over S_i obtained by applying the pullback functor $f^* : S_j \to S_i$ to F_j. We will assume that the following properties are satisfied.*

(i) *For each index i, any object in $\mathscr{G}_i$ is coherent (in particular, the topos S_i is coherent).*

(ii) *For any map $f : i \to j$ in I, the corresponding pullback functor $f^* : S_j \to S_i$ sends any object in $\mathscr{G}_j$ to an object isomorphic to an element of $\mathscr{G}_i$ (in particular, the morphism of topoi $S_i \to S_j$ is coherent).*

(iii) *For any map $f : i \to j$ in I, the pullback functor $f^* : S_j \to S_i$ has a left adjoint $f_\sharp : S_i \to S_j$ which sends any object in $\mathscr{G}_i$ to an object isomorphic to an element of $\mathscr{G}_j$.*

(iv) *Any object in $\mathscr{G}$, has finite cohomological dimension with respect to sheaf cohomology of R-modules.*

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Then, for any index i_0, the pullback functor $\pi_{i_0}^* : \mathrm{C}(\mathrm{Sh}(S_{i_0}, R)) \to \mathrm{C}(\mathrm{Sh}(T, R))$ preserves the fibrations of the projective model structures. Moreover, for any object U_{i_0} of $\mathscr{G}_{i_0}$, and for any complex K_{i_0} of $\mathrm{Sh}(S_{i_0}, R)$, if $U = \pi_{i_0}^*(U_{i_0})$ and $K = \pi_{i_0}^*(K_{i_0})$, then the canonical map

$$\varinjlim_{i \to i_0} H^n(U_i, K_i) \to H^n(U, K) \tag{1.1.12.a}$$

is bijective for any integer n.

Proof. Note that (1.1.12.a) is known to hold whenever K_{i_0} is concentrated in degree zero and $n = 0$; see [SGA4, Exposé VII, Corollaire 8.5.7]. This shows that condition (i) of Paragraph 1.1.8 is preserved by the functor $\pi_{i_0}^*$. Therefore, in order to prove that the functor $\pi_{i_0}^*$ preserves fibrations, it is sufficient to prove that it preserves fibrant objects. Let K_{i_0} be a fibrant object of $\mathrm{C}(\mathrm{Sh}(S_{i_0}, R))$. We have to prove that the natural map

$$H^n(\Gamma(U, K)) \to H^n(U, K) \tag{1.1.12.b}$$

is an isomorphism for any object U in $\mathscr{G}$. For any map $f : i \to j$ in I, condition (iii) above implies that the functor f^* preserves fibrations as well as trivial fibrations (whence it preserves fibrant objects as well). Possibly up to the replacement of i_0 by some other index above it, we may assume that U is the pullback of an object U_{i_0} in $\mathscr{G}_{i_0}$. Formula (1.1.12.a) in the case of complexes concentrated in degree zero then gives us a canonical isomorphism

$$H^n(\Gamma(U, K)) \simeq \varinjlim_{i \to i_0} H^n(\Gamma(U_i, K_i)). \tag{1.1.12.c}$$

As K_i is fibrant for any map $i \to i_0$, we thus get a natural identification

$$H^n(\Gamma(U, K)) \simeq \varinjlim_{i \to i_0} H^n(U_i, K_i). \tag{1.1.12.d}$$

In other words, we must prove that the natural map (1.1.12.a) is invertible for any (fibrant) unbounded complex of sheaves K_{i_0} and any object U_{i_0} in $\mathscr{G}_{i_0}$.

For this purpose, we will work with the *injective model category structure* on $\mathrm{C}(\mathrm{Sh}(S_{i_0}, R))$ (see [CD09, 2.1]), whose weak equivalences are the quasi-isomorphisms, and whose cofibrations are the monomorphisms: as any object of a model category has a fibrant resolution, it is sufficient to prove that (1.1.12.a) is invertible whenever K_{i_0} is fibrant for the injective model structure. In this case, the complex K_{i_0} is degree-wise an injective object of $\mathrm{Sh}(S_{i_0}, R)$. This implies that its image by the functor $\pi_{i_0}^*$ is a complex of $\Gamma(U, -)$-acyclic sheaves; see [SGA4, Exposé VII, Lemme 8.7.2]. Therefore, using Lemma 1.1.7 and assumption (iv), the map (1.1.12.b) is invertible for such a complex K, from which we immediately deduce that (1.1.12.a) is invertible. □

Remark 1.1.13. With the same assumptions as in the preceding lemma, in the case $R = \mathbf{Q}$, for any complex of sheaves of abelian groups K_{i_0} over S_{i_0} and any object U_{i_0} in $\mathscr{G}_{i_0}$, the natural maps

$$\varinjlim_{i \to i_0} H^n(U_i, K_i) \otimes \mathbf{Q} \to H^n(U, K \otimes \mathbf{Q})$$

are isomorphism. Indeed, we know from Lemma 1.1.10 that tensoring with $\mathbf{Q}$ preserves $\Gamma(U, -)$-acyclic sheaves of abelian groups over T for any object U in $\mathscr{G}$. Therefore, as we may assume that K_{i_0} is fibrant for the injective model structure, which implies, by [SGA4, Exposé VII, Lemme 8.7.2], that K is degree-wise $\Gamma(U, -)$-acyclic, the complex $K \otimes \mathbf{Q}$ has the same property. As the functors $\Gamma(V, -)$ commute with $(-) \otimes \mathbf{Q}$ for any coherent sheaf of sets V, we conclude as in the proof of the preceding lemma.

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Theorem 1.1.14. *Consider a cartesian square of locally noetherian schemes*

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

with the following properties.

(a) *The scheme S' is the limit of a projective system of étale schemes of finite type over S, with affine transition morphisms.*

(b) *The morphism f is of finite type.*

Then, for any object K of $\mathrm{D}(X_{\text{ét}}, \mathbf{Z})$, the base change map

$$g^* \, \mathbf{R}f_*(K) \to \mathbf{R}f'_* \, h^*(K)$$

is an isomorphism in $\mathrm{D}(S'_{\text{ét}}, \mathbf{Z})$.

Proof. Let us first prove the theorem under the additional assumption that the scheme S' is strictly local. By virtue of Theorem 1.1.5, any scheme of finite type over S' is of finite étale cohomological dimension. If $S' = \varprojlim_i S_i$, where $\{S_i\}$ is a projective system of étale S-schemes with affine transition maps, then the topos $S'_{\text{ét}}$ is canonically equivalent to the projective limit of topoi $\varprojlim_i S_{i,\text{ét}}$; see [SGA4, Exposé VII, Theorem 5.7]. Similarly, if we write $X_i = S_i \times_S X$, we have $X' \simeq \varprojlim_i X_i$ and $X' \simeq \varprojlim_i X_{i,\text{ét}}$. Note that, for any étale map $u : T' \to T$, the pullback functor $u^* : T_{\text{ét}} \to T'_{\text{ét}}$ has a left adjoint (because the category $T'_{\text{ét}}$ is naturally equivalent to the category $T_{\text{ét}}/T'$, where T' is seen as a sheaf over $T_{\text{ét}}$), and that any map between étale schemes is itself étale, from which one deduces that condition (iii) of Lemma 1.1.12 is satisfied for both projective systems $\{S_i\}$ and $\{X_i\}$. As the other assumptions of this lemma are also verified, we see that the functors g^* and h^* preserve finite limits, weak equivalences, as well as fibrations of the projective model structures. On the other hand, the functors f_* and f'_* are always right Quillen functors for the projective model structures. We deduce from this that we have natural isomorphism as the level of total right derived functors:

$$\mathbf{R}(g^* \, f_*) \simeq \mathbf{R}g^* \, \mathbf{R}f_* = g^* \, \mathbf{R}f_* \quad \text{and} \quad \mathbf{R}(f'_* \, h^*) \simeq \mathbf{R}f'_* \, \mathbf{R}h^* = \mathbf{R}f'_* \, h^*.$$

As the natural map $g^* \, f_*(F) \to f'_* \, h^*(F)$ is an isomorphism for any sheaf F over $X_{\text{ét}}$ (one checks this by first replacing S' by each of the S_i's and X' by the X_i's, and then proceed to the limit), this proves that, under our additional assumptions, the natural transformation $g^* \, \mathbf{R}f_* \to \mathbf{R}f'_* \, h^*$ is invertible.

The general case can now be proven as follows. It is sufficient to prove that, for any geometric point ξ' of S', if S'' denotes the spectrum of the strict henselization of the local ring $\mathscr{O}_{S',\xi'}$, and if $g' : S'' \to S'$ is the natural map, then the morphism

$$g'^* \, g^* \, \mathbf{R}f_*(K) \to g'^* \, \mathbf{R}f'_* \, h^*(K)$$

is invertible for any object K of $\mathrm{D}(X_{\text{ét}}, \mathbf{Z})$. We then have the following pullback squares.

$$\begin{tikzcd} X'' \arrow[r, "h'"] \arrow[d, "f''"'] & X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S'' \arrow[r, "g'"] & S' \arrow[r, "g"] & S \end{tikzcd}$$

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Therefore, applying twice the first part of this proof, we obtain two canonical isomorphisms

$$g'^*\,\mathbf{R}f'_*\,h^*(K) \to \mathbf{R}f''_*\,h'^*\,h^*(K) \quad \text{and} \quad g'^*\,g^*\,\mathbf{R}f_*(K) \to \mathbf{R}f''_*\,h'^*\,h^*(K).$$

As we have a commutative triangle

$$\begin{tikzcd} g'^*\,g^*\,\mathbf{R}f_*(K) \arrow[rr] \arrow[dr, "\simeq"'] & & g'^*\,\mathbf{R}f'_*\,h^*(K) \arrow[dl, "\simeq"] \\ & \mathbf{R}f''_*\,h'^*\,h^*(K) & \end{tikzcd}$$

this shows that the map $g^*\,\mathbf{R}f_*(K) \to \mathbf{R}f'_*\,h^*(K)$ is invertible. □

Corollary 1.1.15. *Let $f : X \to S$ be a morphism between locally noetherian schemes. We assume that, either f is of finite type, or X is the projective limit of quasi-finite S-schemes with affine transition maps. Then the induced derived direct image functor*

$$\mathbf{R}f_* : \mathrm{D}(X_{\text{ét}}, \mathbf{Z}) \to \mathrm{D}(S_{\text{ét}}, \mathbf{Z})$$

preserves small sums.

Proof. By virtue of the preceding theorem, we may assume that S is strictly local. Then, any quasi-compact separated étale scheme over X or S is of finite étale cohomological dimension: in the case where f is of finite type, this follows from Theorem 1.1.5. Otherwise, the proof of Theorem 1.1.5 shows that the étale cohomological dimension of quasi-finite affine S-schemes is uniformly bounded, so that, by an easy limit argument, we see that any quasi-compact quasi-finite separated X-scheme if of finite étale cohomological dimension. In any case, Proposition 1.1.9 tells us that both $\mathrm{D}(S_{\text{ét}}, \mathbf{Z})$ and $\mathrm{D}(X_{\text{ét}}, \mathbf{Z})$ are compactly generated triangulated categories (with canonical families of compact generators given by sheaves of shape $\mathbf{Z}(U)$ for U quasi-compact, separated, and étale over the base). Therefore, the functor $f^* : \mathrm{D}(S_{\text{ét}}, \mathbf{Z}) \to \mathrm{D}(X_{\text{ét}}, \mathbf{Z})$ preserves compact objects (because it sends a generating family of compact objects into another). This immediately implies that its right adjoint $\mathbf{R}f_*$ commutes with small sums. □

1.2 Proper base change isomorphism

Theorem 1.2.1. *Consider a cartesian square of schemes*

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

with f proper. Then, for any ring R of positive characteristic, and for any object K of $\mathrm{D}(X_{\text{ét}}, R)$, the canonical map

$$g^*\,\mathbf{R}f_*(K) \to \mathbf{R}f'_*\,h^*(K)$$

is an isomorphism in $\mathrm{D}(S'_{\text{ét}}, R)$.

Corollary 1.2.2. *Let $f : X \to S$ be a proper morphism of schemes, and let ξ be a geometric point of S. Let us denote by X_ξ the fiber of X over ξ. Then, for any ring R of positive characteristic, and for any object K of $\mathrm{D}(X_{\text{ét}}, R)$, the natural map*

$$\mathbf{R}f_*(K)_\xi \to \mathbf{R}\Gamma(X_\xi, K|_{X_\xi})$$

is an isomorphism in the derived category of the category of R-modules.

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Let us see that Corollary 1.2.2 implies Theorem 1.2.1.

In order to prove that the map $g^* \mathbf{R}f_*(K) \to \mathbf{R}f'_* h^*(K)$ is invertible, it is sufficient to prove that, for any geometric point ξ' of S', if we write $\xi = g(\xi')$, the induced map

$$(g^* \mathbf{R}f_*(K))_{\xi'} = \mathbf{R}f_*(K)_\xi \to \mathbf{R}f'_*(h^*(K))_{\xi'}$$

is an isomorphism. If X_ξ and $X'_{\xi'}$ denote the fiber of X over ξ and of X' over ξ' respectively, as the commutative square of Theorem 1.2.1 is cartesian, the natural map $X'_{\xi'} \to X_\xi$ is an isomorphism. Moreover, applying twice Corollary 1.2.2 gives canonical isomorphisms

$$\mathbf{R}f_*(K)_\xi \simeq \mathbf{R}\Gamma(X_\xi, K|_{X_\xi}) \quad \text{and} \quad \mathbf{R}f'_*(h^*(K))_{\xi'} \simeq \mathbf{R}\Gamma(X'_{\xi'}, h^*(K)|_{X'_{\xi'}}).$$

As the square

$$\begin{tikzcd}
\mathbf{R}f_*(K)_\xi \arrow[r] \arrow[d, "\wr"] & \mathbf{R}f'_*(h^*(K))_{\xi'} \arrow[d, "\wr"] \\
\mathbf{R}\Gamma(X_\xi, K|_{X_\xi}) \arrow[r, "\sim"] & \mathbf{R}\Gamma(X'_{\xi'}, h^*(K)|_{X'_{\xi'}})
\end{tikzcd}$$

commutes, this proves the theorem.

Proof of Corollary 1.2.2. By virtue of [SGA4, Exposé XII, Corollaire 5.2], we already know this corollary is true whenever K is actually a sheaf of R-modules over $X_{\text{ét}}$, from which we easily deduce that this is an isomorphism for K a bounded complex of sheaves of R-modules. Note that X_ξ is of finite cohomological dimension (by Theorem 1.1.5, although this is here much more elementary, as this readily follows from [SGA4, Exposé X, 4.3 and 5.2]). Moreover, as the fiber functor

$$\mathrm{Sh}(S_{\text{ét}}, R) \to R\text{-Mod}, \quad F \mapsto F_\xi$$

is exact, the functor $K \mapsto \mathbf{R}f_*(K)_\xi$ is the total right derived functor of the left exact functor $F \mapsto f_*(F)_\xi \simeq \Gamma(X_\xi, F|_{X_\xi})$, which is thus of finite cohomological dimension; see [SGA4, Exposé XII, 5.2 and 5.3]. Therefore, by virtue of Lemma 1.1.7, the map $H^i(\mathbf{R}f_*(K)_\xi) \to H^i_{\text{ét}}(X_\xi, K|_{X_\xi})$ is a natural transformation between functors which preserve small filtering colimits of complexes of sheaves. As any complex is a filtered colimit of bounded complexes, this ends the proof. $\square$

COROLLARY 1.2.3. *For any proper morphism $f : X \to S$, and for any ring R of positive characteristic, the functor*

$$\mathbf{R}f_* : \mathrm{D}(X_{\text{ét}}, R) \to \mathrm{D}(S_{\text{ét}}, R)$$

has a right adjoint

$$f^! : \mathrm{D}(S_{\text{ét}}, R) \to \mathrm{D}(X_{\text{ét}}, R).$$

Proof. By virtue of the Brown representability theorem, it is sufficient to prove that $\mathbf{R}f_*$ preserves small sums. For this purpose, it is sufficient to prove that, for any geometric point ξ of S, the functor $\mathbf{R}f_*(-)_\xi : \mathrm{D}(X_{\text{ét}}, R) \to \mathrm{D}(R\text{-Mod})$ preserves small sums. This readily follows from Corollaries 1.2.2 and 1.1.15. $\square$

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1.3 Smooth base change isomorphism and homotopy invariance

Theorem 1.3.1. *Consider the cartesian square of locally noetherian schemes below, with g a smooth morphism, and f of finite type.*

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Consider a ring R of positive characteristic which is prime to the residue characteristics of S. Then, for any object K of $\mathrm{D}(X_{\text{ét}}, R)$, the map

$$g^*\,\mathbf{R}f_*(K) \to \mathbf{R}f'_*\,h^*(K)$$

is an isomorphism in $\mathrm{D}(S'_{\text{ét}}, R)$.

Proof. The smallest triangulated full subcategory of $\mathrm{D}(X_{\text{ét}}, R)$ which is closed under small sums, and which contains sheaves of R-modules over $X_{\text{ét}}$, is the whole category $\mathrm{D}(X_{\text{ét}}, R)$. Therefore, by virtue of Corollary 1.1.15, it is sufficient to prove that, for any sheaf of R-modules F over $X_{\text{ét}}$, the map

$$g^*\,\mathbf{R}f_*(F) \to \mathbf{R}f'_*\,h^*(F)$$

is an isomorphism. This follows from [SGA4, Exposé XVI, Corollaire 1.2]. □

Theorem 1.3.2. *Let S be a locally noetherian scheme and $p : V \to S$ be a vector bundle. Consider a ring R of positive characteristic which is prime to the residue characteristics of S. Then the pullback functor $p^* : \mathrm{D}(S_{\text{ét}}, R) \to \mathrm{D}(V_{\text{ét}}, R)$ is fully faithful.*

Proof. The property that p^* is fully faithful is local over S for the Zariski topology, so that may assume that $V = \mathbf{A}^n_S$, and even that $n = 1$. We have to check that, for any complex K of sheaves of R-modules over $S_{\text{ét}}$, the unit map $K \to \mathbf{R}p_*p^*(K)$ is an isomorphism in $\mathrm{D}(S_{\text{ét}}, R)$. By Corollary 1.1.15, the functor $\mathbf{R}p_*$ preserves small sums, so that we may assume that K is concentrated in degree zero (by the same argument as in the preceding proof). This follows then from [SGA4, Exposé XV, Corollaire 2.2]. □

2. The premotivic étale category

The category Sm_S of smooth (and separated of finite type) S-schemes, endowed with the étale topology, is called the *smooth-étale site*. We denote by $\mathrm{Sh}_{\text{ét}}(S, R)$ the category of sheaves of R-modules on this site (this has to be distinguished from the category of sheaves on the small site; see Paragraph 1.1.1).

2.1 Étale sheaves with transfers

2.1.1. We recall here the theory of finite correspondences and of sheaves with transfers introduced by Suslin and Voevodsky [SV00b]. The precise definitions and conventions can be found in [CD12, § 9].

Given any S-scheme X, we denote by

$$c_0(X/S)$$

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the module of cycles α in X with coefficients in Λ such that α is finite and Λ-universal over S (i.e. the support of α is finite over S and α/S satisfies the definition [CD12, 9.1.1]).

Given any S-schemes X and Y, we put

$$c_S(X,Y) := c_0(X \times_S Y/X)$$

and call its elements the *finite S-correspondences from X to Y* (cf. [CD12, 9.1.2]). Beware that the coefficients ring of cycles do not appear in our notation, contrary to the case of [CD12, 9.1.2]. Indeed, we will always assume (relative) cycles and finite correspondences have coefficients in Λ so that we can allow this abuse of notation.

These correspondences can be composed and we denote by $\mathrm{Sm}^{\mathrm{cor}}_{\Lambda,S}$ the category whose objects are smooth S-schemes and morphisms are finite S-correspondences (see [CD12, 9.1.8] for $\mathscr{P}$ the class of smooth separated morphisms of finite type).

We can define a functor

$$\gamma_S : \mathrm{Sm}_S \to \mathrm{Sm}^{\mathrm{cor}}_{\Lambda,S} \tag{2.1.1.a}$$

which is the identity on objects and associates with an S-morphism its graph seen as a finite S-correspondence [CD12, 9.1.8.1].

Definition 2.1.2 (see [CD12, 10.1.1 and 10.2.1]). An *R-presheaf with transfers over S* is an additive presheaf of R-modules on $\mathrm{Sm}^{\mathrm{cor}}_S$. We denote by $\mathrm{PSh}^{\mathrm{tr}}(S,R)$ the corresponding category.

An *étale R-sheaf with transfers over S* is an R-presheaf with transfers F such that $F \circ \gamma_S$ is a sheaf for the étale topology. We denote by $\mathrm{Sh}^{\mathrm{tr}}_{\acute{e}t}(S,R)$ the corresponding full subcategory of $\mathrm{PSh}^{\mathrm{tr}}(S,R)$.

Thus, by definition, we have an obvious functor:

$$\gamma_* : \mathrm{Sh}^{\mathrm{tr}}_{\acute{e}t}(S,R) \to \mathrm{Sh}_{\acute{e}t}(S,R), F \mapsto F \circ \gamma. \tag{2.1.2.a}$$

2.1.3. Given any S-scheme X, we let $R^{\mathrm{tr}}_S(X)$ be the following R-presheaf with transfers:

$$Y \mapsto c_S(Y,X) \otimes_\Lambda R.$$

Proposition 2.1.4. *The presheaf $R^{\mathrm{tr}}_S(X)$ is an étale R-sheaf with transfers.*

Proof. In the case where $R = \Lambda$ this is [CD12, Proposition 10.2.4]. For the general case, we observe that for any smooth S-scheme Y, $c_S(Y,X)$ is a free Λ-module. Indeed, it is a sub-Λ-module of the free Λ-module of cycles in $Y \times_S X$. Thus, we have

$$\mathrm{Tor}^1_\Lambda(c_S(Y,X),R) = 0, \tag{2.1.4.a}$$

and the general case follows from the case $R = \Lambda$. $\square$

2.1.5. Let $Y_\bullet$ be a simplicial S-scheme. If we apply R^{tr}_S point-wise, we obtain a simplicial object of the additive category $\mathrm{Sh}^{\mathrm{tr}}_{\acute{e}t}(S,R)$. We denote by $R^{\mathrm{tr}}_S(Y_\bullet)$ the complex associated with this simplicial object. This is obviously functorial in $Y_\bullet$.

The following proposition is the main technical point of this section.

Proposition 2.1.6. *Let $p : Y_\bullet \to X$ be an étale hypercover of X in the category of S-schemes. Then the induced map*

$$p_* : \gamma_* R^{\mathrm{tr}}_S(Y_\bullet) \to \gamma_* R^{\mathrm{tr}}_S(X)$$

is a quasi-isomorphism of complexes of étale R-sheaves.

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Proof. The general case follows from the case $R = \Lambda$: use the argument (2.1.4.a). In the proof, a *geometric point* will mean a point with coefficients in an algebraically closed field, not only separably closed.[6] We will use the Λ-module $c_0(Z/S)$ defined for any S-scheme Z in Paragraph 2.1.1. Remember that it is covariantly functorial in Z; see [CD12, 9.1.1].

First step. We reduce to the case where S is strictly local and to prove that the canonical map of complexes of Λ-modules

$$p_* : c_0(Y_\bullet/S) \to c_0(X/S) \tag{2.1.6.a}$$

is a quasi-isomorphism.

Indeed, to check that p_* is a quasi-isomorphism, it is sufficient to look at fibers over a point of the smooth-étale site. Such a point corresponds to a smooth S-scheme T with a geometric point $\bar{t}$; we have to show that the map of complexes of Λ-modules,

$$\varinjlim_{V \in \mathscr{V}_{\bar{t}}(T)} c_S(V, Y_\bullet) \to \varinjlim_{V \in \mathscr{V}_{\bar{t}}(T)} c_S(V, X),$$

is a quasi-isomorphism.

Let T_0 be the strict local scheme of T at $\bar{t}$. By virtue of [CD12, 8.3.9], for any smooth S-scheme W, the canonical map,

$$\varinjlim_{V \in \mathscr{V}_{\bar{t}}(T)} c_S(V, W) \to c_0(Z \times_S T_0/T_0) = c_{T_0}(T_0, W \times_S T_0),$$

is an isomorphism. This completes the first step as we may replace S by T_0 as well as p by $p \times_S T_0$.

Second step. We reduce to prove that (2.1.6.a) is a quasi-isomorphism in the case where X is connected and finite over S.

Let $\mathscr{Z}$ be the set of closed subschemes Z of X which are finite over S, ordered by inclusion. Given such a Z, we consider the canonical immersion $i : Z \to X$ and the pullback square.

$$\begin{tikzcd} Z \times_S Y_\bullet \arrow[r, "p_Z"] \arrow[d, "k"'] & Z \arrow[d, "i"] \\ Y_\bullet \arrow[r, "p"] & X \end{tikzcd}$$

We thus obtain a commutative diagram.

$$\begin{tikzcd} c_0(Z \times_X Y_\bullet/S) \arrow[r, "p_{Z*}"] \arrow[d] & c_0(Z/S) \arrow[d] \\ c_0(Y_\bullet/S) \arrow[r, "p_*"] & c_0(X/S) \end{tikzcd}$$

In this diagram, the vertical maps are injective and we can check that p_* is the colimit of the morphism p_{Z*} as Z runs over $\mathscr{Z}$. In fact, taking any cycle α in $c_0(Y_n/S)$, its support T is finite over S; as $p_n : Y_n \to X$ is separated, $Z = p_n(T)$ is a closed subscheme of X which is finite over S. Obviously, α belongs to $c_0(Z \times_X Y_n/S)$.

[6] In the proof, we will only use the fact that *any* surjective family of geometric points on a scheme X gives a conservative family of points of the small étale site of X; see [SGA4, VIII, 3.5].

Because $\mathscr{X}$ is a filtering ordered set, it is sufficient to consider the case where p is p_Z and X is Z. Because $c_0(Z/S)$ is additive with respect to Z, we can assume in addition that Z is connected, which finishes the reduction of the second step.

Final step. Now, S is strictly local and X is finite and connected over S. In particular, X is a strictly local scheme. Let x and s be the closed points of X and S, respectively. Under these assumptions, we have the following lemma (whose proof is given below).

LEMMA 2.1.7. *For any S-scheme U and any étale S-morphism $f : U \to X$, the canonical morphism*

$$\begin{array}{rcl} \varphi_U : \mathbf{Z}\langle \mathrm{Hom}_X(X,U)\rangle \otimes c_0(X/S) & \longrightarrow & c_0(U/S) \\ (i : X \to U) \otimes \beta & \longmapsto & i_*(\beta) \end{array}$$

is an isomorphism.

Thus, according to the lemma above, the map (2.1.6.a) is isomorphic to

$$p_* : \mathbf{Z}\langle \mathrm{Hom}_X(X,Y_\bullet)\rangle \otimes c_0(X/S) \to \mathbf{Z}\langle \mathrm{Hom}_X(X,X)\rangle \otimes c_0(X/S).$$

As p is an étale hypercovering and X is a strictly local scheme, the simplicial set $\mathrm{Hom}_X(X,Y_\bullet)$ is contractible. This readily implies that p_* is a chain homotopy equivalence, which achieves the proof of the proposition. □

Proof of Lemma 2.1.7. We construct an inverse ψ_U to φ_U. Because $c_0(-/S)$ is additive, the (free) Λ-module $c_0(U/S)$ is generated by cycles α whose support is connected. Thus it is enough to define ψ_U on cycles $\alpha \in c_0(U/S)$ whose support T is connected.

By definition, T is finite over S. As f is separated, $f(T)$ is closed in X and the induced map $T \to f(T)$ is finite. In particular, the closed point x of X belongs to $f(T)$: we fix a point $t \in T$ such that $f(t) = x$. Then the residual extension $\kappa(t)/\kappa(x)$ is finite. This implies $\kappa(t) \simeq \kappa(x)$ as $\kappa(x)$ is algebraically closed (according to convention at the beginning of the proof). In particular, t is a $\kappa(x)$-section of the special fiber U_x of U at x. As U/Z is étale, this section can be extended uniquely to a section $i : X \to U$ of U/X. Then $i(X)$ is a connected component, meeting T at least at t. This implies $T \subset i(X)$ as T is connected. Thus $\alpha \in c_0(U/S)$ corresponds to an element α_i in $c_0(i(X)/S) \simeq c_0(X/S)$. We put $\psi_U(\alpha) = i \otimes \alpha_i$. The map ψ_U is obviously an inverse to φ_U, and this completes the proof of the lemma. □

Remark 2.1.8. This proposition fills out a gap in the theory of motivic complexes of Voevodsky which was left open in [VSF00, ch. 5, § 3.3]: Voevodsky restricted himself to the case of a field of finite cohomological dimension.

Note also the following corollary of Lemma 2.1.7.

COROLLARY 2.1.9. *Let X be a scheme and V an étale X-scheme. Let $R_X(V)$ be the étale R-sheaf on Sm_X represented by V. Then the map*

$$R_X(V) \to R_X^{\mathrm{tr}}(V)$$

induced by the graph functor is an isomorphism.

Proof. As in the proof above, it is sufficient to treat the case $R = \Lambda$. Moreover, by looking at the toposic fibers of the above map, and by using the arguments of the first step of the proof, we are reduced to check that the map

$$\Lambda\langle \mathrm{Hom}_X(X,V)\rangle \to c_0(V/X)$$

is an isomorphism when X is strictly local with algebraically closed residue field. Then, this follows from the preceding lemma, and from the fact that, when X is connected, we have $c_0(X/X) = \Lambda$; see [CD12, Lemma 10.2.6]. $\square$

In [CD12, Proposition 10.3.3], we proved the preceding proposition in the particular case of a Čech hypercovering, i.e. the coskeleton of an étale cover. With the extension obtained in the above proposition, we can apply [CD12, Proposition 9.3.9] and get the following.

Proposition 2.1.10. *The category of étale sheaves with transfers has the following properties.*

(1) *The forgetful functor*

$$\mathscr{O}^{\mathrm{tr}}_{\text{ét}} : \mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S, R) \to \mathrm{PSh}^{\mathrm{tr}}(S, R)$$

admits an exact left adjoint $a^{\mathrm{tr}}_{\text{ét}}$ such that the following diagram commutes, where $a_{\text{ét}}$ denotes the usual sheafification functor.

$$\begin{tikzcd} \mathrm{PSh}^{\mathrm{tr}}(S, R) \arrow[r, "a^{\mathrm{tr}}_{\text{ét}}"] \arrow[d, "\hat{\gamma}_*"'] & \mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S, R) \arrow[d, "\gamma_*"] \\ \mathrm{PSh}(S, R) \arrow[r, "a_{\text{ét}}"] & \mathrm{Sh}_{\text{ét}}(S, R) \end{tikzcd}$$

(2) *The category $\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S, R)$ is a Grothendieck abelian category generated by the sheaves of shape $R^{\mathrm{tr}}_S(X)$, for any smooth S-scheme X.*

(3) *The functor γ_* is conservative and commutes with every small limits and colimits.*

2.1.11. We deduce immediately from that proposition that the functor γ_* admits a left adjoint γ^*.

As in [CD12, Corollary 10.3.11], we get the following corollary of the above proposition; see § A.1 for explanation on premotivic categories which where defined in [CD09].

Corollary 2.1.12. *The category $\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(-, R)$ has a canonical structure of an abelian premotivic category. Moreover, the adjunction,*

$$\gamma^* : \mathrm{Sh}_{\text{ét}}(-, R) \rightleftarrows \mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(-, R) : \gamma_*, \tag{2.1.12.a}$$

is an adjunction of abelian premotivic categories.

2.1.13. Remember that the category of (Nisnevich) sheaves with transfers $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S, R)$ is defined as the category of presheaves with transfers F over S such that $F \circ \gamma$ is a sheaf; see [CD12, 10.4.1]. Then $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(-, R)$ is a fibered category which is an abelian premotivic category according to [CD12, 10.4.1].

We will denote by τ the comparison functor between the Nisnevich and the étale topology on the site Sm_S. Thus, we denote by $\tau_* : \mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S, R) \to \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S, R)$ the obvious fully faithful functor. Then the functor $a^{\mathrm{tr}}_{\text{ét}} : \mathrm{PSh}^{\mathrm{tr}}(S, R) \to \mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S, R)$ obviously induces a left adjoint τ^* to the functor τ_*. Moreover, this defines an adjunction of premotivic abelian categories:

$$\tau^* : \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(-, R) \rightleftarrows \mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(-, R) : \tau_*. \tag{2.1.13.a}$$

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2.2 Derived categories

2.2.1. In [CD12, § 5], we established a theory to study derived categories such as $\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S,R))$. This category has to satisfy the technical conditions of [CD12, Definitions 5.1.3 and 5.1.9]. Let us make explicit this definition in our particular case.

Definition 2.2.2. Let K be a complex of étale R-sheaves with transfers.

(1) The complex K is said to be *local* with respect to the étale topology if, for any smooth S-scheme X and any integer $n \in \mathbf{Z}$, the canonical morphism

$$\mathrm{Hom}_{\mathrm{K}(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S,R))}(R^{\mathrm{tr}}_S(X)[n],K) \to \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S,R))}(R^{\mathrm{tr}}_S(X)[n],K)$$

is an isomorphism.

(2) The complex K is said to be *étale-flasque* if for any étale hypercover $Y_\bullet \to X$ in Sm_S and any integer $n \in \mathbf{Z}$, the canonical morphism

$$\mathrm{Hom}_{\mathrm{K}(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S,R))}(R^{\mathrm{tr}}_S(X)[n],K) \to \mathrm{Hom}_{\mathrm{K}(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S,R))}(R^{\mathrm{tr}}_S(Y_\bullet)[n],K)$$

is an isomorphism.

Proposition 2.2.3. *A complex of étale sheaves with transfers is étale-flasque if and only if it is local with respect to the étale topology. Moreover, for any complex of étale R-sheaves with transfers K over S, any smooth S-scheme X, and any integer $n \in \mathbf{Z}$, we have a natural identification:*

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(X,R))}(R^{\mathrm{tr}}_S(X),K[n]) = H^n_{\text{ét}}(X,K).$$

Proof. Note that the analogous statement is known to be true for complexes of étale sheaves without transfers (see for instance [CD09]). Therefore, the first assertion of the proposition follows from the second one, which we will now prove. Let S be a base scheme.

We consider the projective model category structure on the category $\mathrm{C}(\mathrm{Sh}_{\text{ét}}(S,R))$, that is the analog of the model structure defined in Paragraph 1.1.8: the weak equivalences are the quasi-isomorphisms, while the fibrations are the morphisms of complexes whose restriction to each of the small sites $X_{\text{ét}}$ is a fibration in the sense of Paragraph 1.1.8 for any smooth S-scheme X. On the other hand, as the category $\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S,R)$ is an abelian Grothendieck category, the category $\mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S,R))$ is endowed with the injective model category structure; see [CD09, 2.1]. By virtue of [CD09, 2.14], Proposition 2.1.6 and the last assertion of Proposition 2.1.10 imply that the functor

$$\gamma^* : \mathrm{C}(\mathrm{Sh}_{\text{ét}}(S,R)) \to \mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S,R))$$

is a left Quillen functor. As its right adjoint γ_* preserves weak equivalences, we thus get an adjunction

$$\mathbf{L}\gamma^* : \mathrm{D}(\mathrm{Sh}_{\text{ét}}(S,R)) \leftrightarrows \mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S,R)) : \gamma_*.$$

Note that, for any smooth S-scheme X, we have a natural isomorphism

$$\mathbf{L}\gamma^* R_S(X) \simeq R^{\mathrm{tr}}_S(X)$$

because $R_S(X)$ is cofibrant. Therefore, for any smooth S-scheme X and for any complex of étale

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sheaves with transfers K, we have the following identifications (compare with [VSF00, ch. 5, 3.1.9]):

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\acute{e}t}(X,R))}(R^{\mathrm{tr}}_S(X),K[n]) &\simeq \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\acute{e}t}(X,R))}(\mathbf{L}\gamma^*(R_S(X)),K[n])\\ &\simeq \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}_{\acute{e}t}(X,R))}(R_S(X),\gamma_*(K)[n])\\ &= H^n_{\acute{e}t}(X,K).\end{aligned}$$

This proves the second assertion of the proposition, and thus achieves its proof. □

2.2.4. Propositions 2.1.6 and 2.2.3 assert precisely that the premotivic abelian category $\mathrm{Sh}^{\mathrm{tr}}_{\acute{e}t}(-,R)$ is *compatible with the étale topology* in the sense of [CD12, Definition 5.1.9].

We can therefore apply the general machinery of [CD12, Definition 5.1.9] to the abelian premotivic category $\mathrm{Sh}^{\mathrm{tr}}_{\acute{e}t}(-,R)$. In particular, we get triangulated premotivic categories (again, see § A.1 for basic definitions on premotivic categories):

- [CD12, Definition 5.1.17], the associated derived category, $\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\acute{e}t}(-,R))$ whose fiber over a scheme S is $\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\acute{e}t}(S,R))$;
- [CD12, Definition 5.2.16], the associated effective $\mathbf{A}^1$-derived category,

$$\mathrm{DM}^{\mathrm{eff}}_{\acute{e}t}(-,R) := \mathrm{D}^{\mathrm{eff}}_{\mathbf{A}^1}(\mathrm{Sh}^{\mathrm{tr}}_{\acute{e}t}(-,R))$$

whose fiber over a scheme S is the $\mathbf{A}^1$-localization of the derived category $\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\acute{e}t}(S,R))$.

- [CD12, Definition 5.3.22], the associated (stable) $\mathbf{A}^1$-derived category,

$$\mathrm{DM}_{\acute{e}t}(-,R) = \mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}^{\mathrm{tr}}_{\acute{e}t}(-,R))$$

whose fiber over a scheme S is obtained from $\mathrm{D}^{\mathrm{eff}}_{\mathbf{A}^1}(\mathrm{Sh}^{\mathrm{tr}}_{\acute{e}t}(S,R))$ by $\otimes$-inverting the Tate object $R^{\mathrm{tr}}_S(1) := \tilde{R}^{\mathrm{tr}}_S(\mathbf{P}^1_S,\infty)[-2]$ (in the sense of model categories).

By construction, these categories are related by the following morphisms of premotivic triangulated categories:

$$\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\acute{e}t}(S,R)) \xrightarrow{\pi_{\mathbf{A}^1}} \mathrm{DM}^{\mathrm{eff}}_{\acute{e}t}(S,R) \xrightarrow{\Sigma^\infty} \mathrm{DM}_{\acute{e}t}(S,R). \tag{2.2.4.a}$$

Recall that the right adjoint to the functor $\pi_{\mathbf{A}^1}$ is fully faithful with essential image made by the $\mathbf{A}^1$-local complexes, in the sense of the next definition.

Remark 2.2.5. In the terminology of Voevodsky, [VSF00], the category $\mathrm{DM}^{\mathrm{eff}}_{\acute{e}t}(X,R)$ should be called the category of étale motivic complexes over X.

With a wider view, $\mathrm{DM}_{\acute{e}t}(X,R)$ could be called the category of étale motives. However, we think it deserves that name only when R has positive characteristic n invertible on X (see Theorem 5.5.3) or when X is geometrically unibranch (see Corollary 5.5.5).

Definition 2.2.6. Let K be a complex of R-sheaves with transfers over a scheme S. For any smooth S-scheme X and any integer $n \in \mathbf{Z}$, we simply denote by $H^n_{\acute{e}t}(X,K)$ the cohomology of K seen as a complex of R-sheaves over $X_{\acute{e}t}$.

We say that K is $\mathbf{A}^1$-*local* if for any smooth S-scheme X and any integer $n \in \mathbf{Z}$, the map induced by the canonical projection

$$H^n_{\acute{e}t}(X,K) \to H^n_{\acute{e}t}(\mathbf{A}^1_X,K)$$

is an isomorphism.

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2.2.7. According to [CD12, 5.1.23, 5.2.19, 5.3.28], the adjunction of abelian premotivic categories (2.1.12.a) can be derived, and it induces, over a scheme S, a commutative diagram.

$$\begin{tikzcd} \mathrm{D}(\mathrm{Sh}_{\acute{e}t}(S,R)) \arrow[r] \arrow[d, "\mathbf{L}\gamma^*"'] & \mathrm{D}^{\mathrm{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_{\acute{e}t}(S,R)) \arrow[r] \arrow[d] & \mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_{\acute{e}t}(S,R)) \arrow[d] \\ \mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\acute{e}t}(S,R)) \arrow[r] & \mathrm{DM}^{\mathrm{eff}}_{\acute{e}t}(S,R) \arrow[r] & \mathrm{DM}_{\acute{e}t}(S,R) \end{tikzcd} \tag{2.2.7.a}$$

Note that all the vertical maps are obtained by deriving (on the left) the functor γ^*. We will simply denote these maps by $\mathbf{L}\gamma^*$. By definition, they admit a right adjoint that we denote by $\mathbf{R}\gamma_*$. In fact, we will often write $\mathbf{R}\gamma_* = \gamma_*$ because of the following simple result.

Proposition 2.2.8. *The exact functor* $\gamma_* : \mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\acute{e}t}(S,R)) \to \mathrm{C}(\mathrm{Sh}_{\acute{e}t}(S,R))$ *preserves* $\mathbf{A}^1$*-equivalences.*

Proof. This follows from [CD12, Proposition 5.2.24]. □

2.2.9. Applying again [CD12, 5.1.23, 5.2.19, 5.3.28] to the adjunction (2.1.13.a), we get a commutative diagram of left derived functors

$$\begin{tikzcd} \mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S,R)) \arrow[r] \arrow[d, "\mathbf{L}\tau^*"'] & \mathrm{DM}^{\mathrm{eff}}(S,R) \arrow[r] \arrow[d] & \mathrm{DM}(S,R) \arrow[d] \\ \mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\acute{e}t}(S,R)) \arrow[r] & \mathrm{DM}^{\mathrm{eff}}_{\acute{e}t}(S,R) \arrow[r] & \mathrm{DM}_{\acute{e}t}(S,R) \end{tikzcd} \tag{2.2.9.a}$$

where $\mathrm{DM}^{\mathrm{eff}}(S,R)$ (respectively $\mathrm{DM}(S,R)$) stands for the effective category (respectively stable category) of Nisnevich motives as defined in [CD12, Definition 11.1.1].

The following proposition is a generalization of [VSF00, ch. 5, 4.1.12].

Proposition 2.2.10. *Assume R is a $\mathbf{Q}$-algebra. Then the adjunction (2.1.13.a) is an equivalence of categories. In particular, all the vertical maps of the diagram (2.2.9.a) are equivalences of categories.*

Proof. We first prove that the right adjoint τ_* of (2.1.13.a) is exact. Using the analog of Proposition 2.2.3 for the Nisnevich topology, one reduces to show that for any étale R-sheaf with transfers F over S and any local henselian scheme X over S, the cohomology group $H^1_{\acute{e}t}(X,F)$ vanishes. But, as F is rational, this last group is isomorphic to $H^1_{\mathrm{Nis}}(X,F)$; this is well known (see for example [CD12, 10.5.9]), and this group is zero.

Note also τ_* obviously commutes with filtered colimits. Being also exact, it thus commutes with arbitrary colimits.

Obviously, τ_* is fully faithful. It only remains to prove that its left adjoint τ^* is fully faithful as well. Thus, we have to prove that for any Nisnevich R-sheaf with transfers over S, the adjunction map $F \to F_{\acute{e}t} = \tau^*\tau_*(F)$ is an isomorphism. As $\tau^*\tau_*$ commutes with colimits, it is sufficient to prove this for $F = R^{\mathrm{tr}}_S(X)$ when X is an arbitrary smooth S-scheme. This is precisely Proposition 2.1.4. □

2.3 A weak localization property

Lemma 2.3.1. *Let $f : Y \to X$ be a finite morphism. Then the functor*

$$f_* : \mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{\acute{e}t}}(Y,R)) \to \mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{\acute{e}t}}(X,R))$$

preserves colimits and $\mathbf{A}^1$-equivalences.

Proof. We first check that f_* preserves colimits. By definition, $\gamma_* f_* = f_* \gamma_*$. According to point (3) of Proposition 2.1.10, we thus are reduced to prove the functor $f_* : \mathrm{Sh}(Y,R) \to \mathrm{Sh}(X,R)$ commutes with colimits. This is well known, boiling down to the fact a finite scheme over a strictly local scheme is a sum of strictly local schemes. The remaining assertion now follows from [CD12, Proposition 5.2.24]. □

Proposition 2.3.2. *Let $f : Y \to X$ be a finite morphism. Then the functor*

$$f_* = \mathbf{R}f_* : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(Y,R) \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(X,R)$$

preserves small sums, and thus, has a right adjoint $f^!$.

Proof. The fact that the functor f_* preserves small sums follows formally from the preceding lemma and from the fact that $\mathbf{A}^1$-equivalences are closed under filtered colimits; see [CD09, Proposition 4.6]. The existence of the right adjoint $f^!$ follows from the Brown representability theorem.[7] □

2.3.3. Let $i : Z \to S$ be a closed immersion and $j : U \to S$ the complementary open immersion.

Let K be a complex of étale sheaves with transfers over S. Note that the composite of the obvious adjunction maps

$$j_\sharp j^*(K) \to K \to i_* i^*(K) \tag{2.3.3.a}$$

is always 0. We will say that this sequence is *homotopy exact in* $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(S,R)$ if for any cofibrant resolution $K' \to K$ of K the canonical map

$$\mathrm{Cone}(j_\sharp j^*(K') \to K') \to i_* i^*(K')$$

is an $\mathbf{A}^1$-equivalence.

Note that given a smooth S-scheme X, $K = R^{\mathrm{tr}}_S(X)$ is cofibrant by definition and the cone appearing above is quasi-isomorphic to the cokernel of the map

$$R^{\mathrm{tr}}_S(X - X_Z) \xrightarrow{j_*} R^{\mathrm{tr}}_S(X),$$

which we will denote by $R^{\mathrm{tr}}_S(X/X - X_Z)$. Here, we put $X_Z = X \times_S Z$.

We recall the following proposition from [CD12, Corollary 2.3.17].

Proposition 2.3.4. *Consider the notations above. The following conditions are equivalent.*

(i) *The functor i_* is fully faithful and the pair of functors (i^*, j^*) is conservative for the premotivic category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(-,R)$.*

[7] One can see the existence of a right adjoint of $\mathbf{R}f_*$ in a slightly more constructive way as follows. Lemma 2.3.1 implies that the functor $f^!$ already exists at the level of étale sheaves with transfers. One can see easily from the same lemma that f_* is a left Quillen functor with respect to the $\mathbf{A}^1$-localizations of the injective model category structures, which ensures the existence of $f^!$ at the level of the homotopy categories, namely as the total right derived functor of its analog at the level of sheaves.

(ii) *For any complex K, the sequence (2.3.3.a) is homotopy exact in* $\mathrm{DM}^{\mathrm{eff}}_{\acute{\mathrm{e}}\mathrm{t}}(S,R)$.

(iii) *The functor i_* commutes with twists and for any smooth S-scheme X, the canonical map*

$$R^{\mathrm{tr}}_S(X/X-X_Z) \to i_*(R^{\mathrm{tr}}_Z(X_Z))$$

is an isomorphism in $\mathrm{DM}^{\mathrm{eff}}_{\acute{\mathrm{e}}\mathrm{t}}(S,R)$.

Moreover, when these conditions are fulfilled, for any complex K, the exchange transformation

$$(i_*(R_Z)) \otimes K \to i_*i^*(K) \tag{2.3.4.a}$$

is an isomorphism.

The equivalent conditions of the above proposition are called the *localization property with respect to i* for the premotivic triangulated category $\mathrm{DM}^{\mathrm{eff}}_{\acute{\mathrm{e}}\mathrm{t}}(-,R)$; see § A.1.11.

Proposition 2.3.5. *Let $i : Z \to S$ be a closed immersion which admits a smooth retraction $p : S \to Z$. Then $\mathrm{DM}^{\mathrm{eff}}_{\acute{\mathrm{e}}\mathrm{t}}(-,R)$ satisfies the localization property with respect to i.*

The proof of this proposition is the same than the analogous fact for the Nisnevich topology; see [CD12, Proposition 6.3.14]. As this statement plays an important role in the sequel of these notes, we will recall the essential steps of the proof. One of the main ingredients of the proof uses the following result, proved in [Ayo07, 4.5.44].

Theorem 2.3.6. *The premotivic category $\mathrm{D}^{\mathrm{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_{\acute{\mathrm{e}}\mathrm{t}}(-,R))$ satisfies localization (with respect to any closed immersion).*

Lemma 2.3.7. *For any open immersion $j : U \to S$, the exchange transformation*

$$\mathbf{L}j_\sharp\,\gamma_* \to \gamma_*\,\mathbf{L}j_\sharp$$

is an isomorphism in $\mathrm{D}^{\mathrm{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_{\acute{\mathrm{e}}\mathrm{t}}(S,R))$.

Proof. We first prove that, for any étale sheaf with transfers F over U, the map

$$j_\sharp\gamma_*(F) \to \gamma_*j_\sharp(F)$$

is an isomorphism of étale sheaves. Indeed, both in the case of étale sheaves or of étale sheaves with transfers, the sheaf $j_\sharp(F)$ is obtained as the sheaf associated with the presheaf

$$V \mapsto \begin{cases} F(V) & \text{if } V \text{ is supported over } U \text{ (i.e. if } V \times_S U \simeq V), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the functors $j_\sharp$ are exact, and they preserve $\mathbf{A}^1$-equivalences because of the projection formula $A \otimes j_\sharp(B) \simeq j_\sharp(j^*(A) \otimes B)$ (for any sheaves A and B). Using Proposition 2.2.8, this implies the lemma. □

Lemma 2.3.8. *Let $i : Z \to S$ be a closed immersion which admits a smooth retraction. Then the exchange transformation*

$$\mathbf{L}\gamma^*i_* \to i_*\mathbf{L}\gamma^*$$

is an isomorphism in $\mathrm{DM}^{\mathrm{eff}}_{\acute{\mathrm{e}}\mathrm{t}}(S,R)$.

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Proof. Let $p : S \to Z$ be a smooth morphism such that $pi = 1_Z$, and denote by $j : U \to S$ the complement of i in S. For any object M in $\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(Z, R)$, we have a natural homotopy cofiber sequence of shape

$$\mathbf{L}j_\sharp j^* p^* M \to p^* M \to i_* M \tag{2.3.8.a}$$

(note that $i_* M = i_* i^* p^* M$ because $pi = 1_Z$). Indeed, as the functor γ_* is conservative, it is sufficient to check this after applying γ_*. As the functor γ_* commutes with $\mathbf{L}j_\sharp$ (by the previous lemma) as well as with the functors j^*, p^* and i_* (because its left adjoint $\mathbf{L}\gamma^*$ commutes with the functors $\mathbf{L}j_\sharp$, $\mathbf{L}p_\sharp$ and $\mathbf{L}i^*$), it is sufficient to see that the analog of (2.3.8.a) is an homotopy cofiber sequence for any object M of $\mathrm{D}^{\mathrm{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(Z, R))$. But this latter property is a particular case of the localization property with respect to the closed immersions, which is known to hold by Theorem 2.3.6. The characterization of the functor i_* by the homotopy cofiber sequence (2.3.8.a) implies the lemma because the functor $\mathbf{L}\gamma^*$ is known to commute with the functors $\mathbf{L}j_\sharp$, j^* and p^*. □

Proof of Proposition 2.3.5. Now, the proposition can easily be deduced from the above lemma and from Theorem 2.3.6, using the fact that the functor γ_* is conservative; see the proof of [CD12, Proposition 6.3.14] for more details. □

3. The embedding theorem

3.1 Locally constant sheaves and transfers

3.1.1. Let X be a noetherian scheme.

Recall that we denote by $\mathrm{Sh}(X_{\text{ét}}, R)$ the category of R-sheaves over the small étale site $X_{\text{ét}}$. On the other hand, we also have the category $\mathrm{Sh}_{\text{ét}}(X, R)$ of R-sheaves over the smooth-étale site $\mathrm{Sm}_{X,\text{ét}}$. It is made by smooth X-schemes. The obvious inclusion of sites $\rho : X_{\text{ét}} \to \mathrm{Sm}_{X,\text{ét}}$ gives an adjunction of categories:

$$\rho_\sharp : \mathrm{Sh}(X_{\text{ét}}, R) \rightleftarrows \mathrm{Sh}_{\text{ét}}(X, R) : \rho^* \tag{3.1.1.a}$$

where $\rho^*(F) = F \circ \rho$. The following lemma is well known (see [SGA4, VII, 4.0, 4.1]).

Lemma 3.1.2. *With the above notations, the following properties hold.*

1. *The functor ρ^* commutes with arbitrary limits and colimits.*
2. *The functor $\rho_\sharp$ is exact and fully faithful.*
3. *The functor $\rho_\sharp$ is monoidal and commutes with operations f^* for any morphism of schemes f, and with $f_\sharp$, when f is étale.*

Note that point (3) can be rephrased by saying that (3.1.1.a) is an adjunction of étale-premotivic abelian categories (Definition A.1.7).

By definition, $\rho_\sharp$ sends the R-sheaf on $X_{\text{ét}}$ represented by an étale X-scheme V to the R-sheaf represented by V on Sm_X. We will denote by $R_X(V)$ both the sheaves on the small étale and on the smooth-étale site of X; the confusion here is harmless.

3.1.3. Let us denote by $\mathrm{D}(X_{\text{ét}}, R)$ the derived category of $\mathrm{Sh}(X_{\text{ét}}, R)$. As both functors $\rho_\sharp, \rho^*$ are exact, they can be derived trivially. In particular, we get a derived adjunction

$$\rho_\sharp : \mathrm{D}(X_{\text{ét}}, R) \rightleftarrows \mathrm{D}(\mathrm{Sh}_{\text{ét}}(X, R)) : \rho^* \tag{3.1.3.a}$$

in which the functor $\rho_\sharp$ is still fully faithful.

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PROPOSITION 3.1.4. *The composite functor*

$$\mathrm{Sh}(X_{\text{ét}}, R) \xrightarrow{\rho_\sharp} \mathrm{Sh}_{\text{ét}}(X, R) \xrightarrow{\gamma^*} \mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(X, R)$$

is exact and fully faithful.

Proof. As $\rho_\sharp$ is fully faithful and γ_* is exact and conservative, it is sufficient to prove that, for any R-sheaf F on $X_{\text{ét}}$, the map induced by adjunction,

$$\rho_\sharp(F) \to \gamma_*\gamma^*\rho_\sharp(F),$$

is an isomorphism of étale sheaves. Moreover, all the involved functors commute with colimits (applying in particular Proposition 2.1.10). Thus, it is sufficient to prove this in the case where $F = R_X(V)$ is representable by an étale X-scheme V. Then, the result is just a reformulation of Corollary 2.1.9. □

COROLLARY 3.1.5. *The functor*

$$\mathbf{L}\gamma^*\,\rho_\sharp = \gamma^*\rho_\sharp : \mathrm{D}(X_{\text{ét}}, R) \to \mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(X, R))$$

is fully faithful.

3.1.6. We have a composite functor

$$\rho_! : \mathrm{D}(X_{\text{ét}}, R) \to \mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(X, R)) \to \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(X, R) \tag{3.1.6.a}$$

PROPOSITION 3.1.7. *Assume that the ring R is of positive characteristic n and that the residue characteristics of X are prime to n. Then the composed functor (3.1.6.a) is fully faithful.*

Proof. Recall that the functor $\pi_{\mathbf{A}^1} : \mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(X, R)) \to \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(X, R)$ has a fully faithful right adjoint whose essential image consists of $\mathbf{A}^1$-local objects (see Definition 2.2.6). Therefore, by virtue of Proposition 2.2.3 and of Corollary 3.1.5, it is sufficient to prove that, for any complex K in $\mathrm{D}(X_{\text{ét}}, R)$, and for any étale X-scheme V, the map

$$H^i_{\text{ét}}(V, K) \to H^i_{\text{ét}}(\mathbf{A}^1 \times V, K)$$

is bijective for all i, which is Theorem 1.3.2. □

3.2 Étale motivic Tate twist

Recall from [SGA4, IX, 3.2] that, for any scheme X such that n is invertible in $\mathscr{O}_X$, the group scheme $\mu_{n,X}$ of nth roots of unity fits in the Kummer short exact sequence in $\mathrm{Sh}_{\text{ét}}(S, \mathbf{Z})$:

$$0 \to \mu_n \to \mathbf{G}_{m,X} \to \mathbf{G}_{m,X} \to 0. \tag{3.2.0.a}$$

This induces a canonical isomorphism in the derived category:

$$\mathbf{G}_{m,X}[-1] \otimes^{\mathbf{L}} \mathbf{Z}/n\mathbf{Z} \simeq \mu_{n,X}. \tag{3.2.0.b}$$

3.2.1. For any scheme S and any ring R, the Tate motive $R_S(1)$ is defined in $\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(S, R)$ as the cokernel of the split monomorphism $R^{\mathrm{tr}}_S(S)[-1] \to R^{\mathrm{tr}}_S(\mathbf{G}_{m,S})[-1]$ induced by the unit section.

As $\mathbf{G}_{m,S}$ has a natural structure of étale sheaf with transfers, there is a canonical map

$$\mathbf{Z}^{\mathrm{tr}}_S(\mathbf{G}_{m,S}) \to \mathbf{G}_{m,S}$$

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which factor through $\mathbf{Z}_S(1)[1]$. This gives a natural morphism in $\mathrm{DM}^{\mathrm{eff}}(S,R)$:

$$R_S(1)[1] \to \mathbf{G}_{m,S} \otimes^{\mathbf{L}} R. \tag{3.2.1.a}$$

In the case where R is of positive characteristic n, with n invertible in $\mathscr{O}_S$, the isomorphism (3.2.0.b) identifies the map (3.2.1.a) shifted by $[-1]$ with a morphism of shape

$$R_S(1) \to \mu_{n,S} \otimes_{\mathbf{Z}/n\mathbf{Z}} R, \tag{3.2.1.b}$$

where the locally constant étale sheaf $\mu_{n,S}$ is considered as a sheaf with transfers (according to Proposition 3.1.7). Note also that $\mu_{n,S} \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} R \simeq \mu_{n,S} \otimes_{\mathbf{Z}/n\mathbf{Z}} R$ because μ_n is a locally free sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules.

Proposition 3.2.2. *The morphism (3.2.1.a) is an isomorphism in* $\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(S,R)$ *whenever S is regular.*

Proof. The case where $R = \mathbf{Z}$ follows immediately from [CD12, Proposition 11.2.11]. In the general case, the result follows by applying the derived functor $(-) \otimes^{\mathbf{L}} R$. □

Proposition 3.2.3. *If the ring R is of positive characteristic n, with n invertible in $\mathscr{O}_S$, then the morphism (3.2.1.b) is an isomorphism in* $\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(S,R)$.

Proof. By virtue of the preceding proposition, this is true for S regular, and thus in the case where $S = \operatorname{Spec} \mathbf{Z}[1/n]$. Now, consider a morphism of schemes $f : X \to S$, with S regular (e.g. $S = \operatorname{Spec} \mathbf{Z}[1/n]$). The natural map $\mathbf{L}f^*(R_S(1)) \to R_X(1)$ is obviously an isomorphism, and, as the étale sheaf μ_n is locally constant, the canonical map $\mathbf{L}f^*(\mu_{n,S} \otimes_{\mathbf{Z}/n\mathbf{Z}} R) \to \mu_{n,X} \otimes_{\mathbf{Z}/n\mathbf{Z}} R$ is invertible as well, from which we deduce the general case. □

Corollary 3.2.4. *For any scheme X, if n is invertible in $\mathscr{O}_X$, we have a canonical identification:*

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(X,\mathbf{Z}/n\mathbf{Z})}((\mathbf{Z}/n\mathbf{Z})_X, (\mathbf{Z}/n\mathbf{Z})_X(1)[i]) = H^{i-1}_{\text{ét}}(X, \mu_n).$$

Proof. This is an immediate consequence of Propositions 3.1.7 and 3.2.3. □

Corollary 3.2.5. *If the ring R is of positive characteristic n, with n prime to the residue characteristics of X, then the Tate twist $R_X(1)$ is $\otimes$-invertible in* $\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(X,R)$. *Therefore, the infinite suspension functor (2.2.4.a)*

$$\Sigma^\infty : \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(X,R) \to \mathrm{DM}_{\text{ét}}(X,R)$$

is then an equivalence of categories.

Proof. The sheaf $\mu_{n,X}$ is locally constant: there exists an étale cover $f : Y \to X$ such that $f^*(\mu_{n,X}) = (\mathbf{Z}/n\mathbf{Z})_Y$. This implies that the sheaf $\mu_{n,X} \otimes R$ is $\otimes$-invertible in the derived category $\mathrm{D}(X_{\text{ét}}, R)$. As the canonical functor $\mathrm{D}(X_{\text{ét}}, R) \to \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(X,R)$ is symmetric monoidal, this implies that $\mu_{n,X} \otimes R$ is $\otimes$-invertible in $\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(X,R)$. The first assertion follows then from Proposition 3.2.3. The second follows from the first by the general properties of the stabilization of model categories; see [Hov01]. □

https://doi.org/10.1112/S0010437X15007459 Published online by Cambridge University Press

4. Torsion étale motives

In all this section, R is assumed to be a ring of positive characteristic n.

The aim of this section is to show that the premotivic triangulated category of R-linear étale motives $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(-,R)$ defined previously satisfies the Grothendieck's six functors formalism as well as the absolute purity property (see respectively Definitions A.1.10 and A.2.9). Then we deduce the extension of the Suslin–Voevodsky rigidity theorem [VSF00, ch. 5, 3.3.3] to arbitrary bases.

To simplify notations, we will cancel the letters $\mathbf{L}$ and $\mathbf{R}$ in front of the derived functors used in this section. Note also that we will show in Proposition 4.1.1 that

$$\Sigma^\infty : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(-,R) \to \mathrm{DM}_{\mathrm{\acute{e}t}}(-,R)$$

is an equivalence of categories. Thus we will use the simpler notation $\mathrm{DM}_{\mathrm{\acute{e}t}}(-,R)$ from § 4.2 on.

4.1 Stability and orientation

We first show that in Corollary 3.2.5 one can drop the restriction on the characteristic of the schemes we consider.

Proposition 4.1.1. *For any scheme S the Tate motive $R_S(1)$ in $\otimes$-invertible and the natural map $R_S(1)[1] \to \mathbf{G}_{m,S} \otimes^{\mathbf{L}} R$ (3.2.1.a) is an isomorphism in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(S,R)$.*

Proof. As the change of scalars functor

$$\mathrm{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(S,\mathbf{Z}/n\mathbf{Z}) \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(S,R), \quad M \mapsto R \otimes^{\mathbf{L}}_{\mathbf{Z}/n\mathbf{Z}} M$$

is symmetric monoidal, it is sufficient to prove this for $R = \mathbf{Z}/n\mathbf{Z}$. By a simple *dévissage*, we may assume that $n = p^\alpha$ is some power of a prime number p. Let $S[1/p]$ be the product $S \times \mathrm{Spec}(\mathbf{Z}[1/p])$, and let $j : S[1/p] \to S$ be the canonical open immersion. By virtue of Proposition A.3.4, the functor

$$j^* : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(S,R) \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(S[1/p],R)$$

is an equivalence of triangulated monoidal categories. Therefore, we may also assume that n is invertible in $\mathscr{O}_S$. We are thus reduced to Corollary 3.2.5. □

Corollary 4.1.2. *For any scheme S the infinite suspension functor*

$$\Sigma^\infty : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(S,R) \to \mathrm{DM}_{\mathrm{\acute{e}t}}(S,R)$$

is an equivalence of categories.

4.1.3. As a direct consequence of the preceding proposition, we have, for any scheme S, a functorial morphism of abelian groups

$$c_1^{\mathrm{\acute{e}t}} : \mathrm{Pic}(S) = \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{\acute{e}t}}(S,\mathbf{Z}))}(\mathbf{Z}_S, \mathbf{G}_{m,S}[1]) \to \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(S,R)}(R_S, R_S(1)[2])$$

which is simply induced by the canonical morphism $\mathbf{G}_{m,S} \to \mathbf{G}_{m,S} \otimes^{\mathbf{L}} R$ and the isomorphism $R_S(1)[1] \simeq \mathbf{G}_{m,S} \otimes^{\mathbf{L}} R$.

Definition 4.1.4. We call the map $c_1^{\mathrm{\acute{e}t}}$ the *étale motivic Chern class.*

We will consider this map as the canonical orientation of the triangulated premotivic category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(-,R)$.

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4.2 Purity (smooth projective case)

4.2.1. We need to simplify some of our notations which will often appear below. Given any morphism f and any smooth morphism p, we will consider the following unit and counit maps of the relevant adjunctions in $\mathrm{DM}_{\text{ét}}(-,R)$:

$$\begin{aligned} 1 \xrightarrow{\alpha_f} f_* f^*, \quad f^* f_* \xrightarrow{\alpha'_f} 1, \\ 1 \xrightarrow{\beta_p} p^* p_\sharp, \quad p_\sharp p^* \xrightarrow{\beta'_p} 1. \end{aligned} \tag{4.2.1.a}$$

Remark 4.2.2. Consider a cartesian square of schemes

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[d, "g"'] \arrow[dr, phantom, "\Delta"] & X \arrow[d, "f"] \\ T \arrow[r, "p"] & S \end{tikzcd}$$

such that p is smooth. According to property (5) of Definition A.1.1, applied to $\mathrm{DM}_{\text{ét}}(-,R)$, we associate with the square Δ the base change isomorphism

$$\mathrm{Ex}(\Delta_\sharp^*) : q_\sharp g^* \to f^* p_\sharp.$$

In what follows, the square Δ will be clear and we will put simply: $\mathrm{Ex}_\sharp^* := \mathrm{Ex}(\Delta_\sharp^*)^{-1}$.

Recall also that we associate with the square Δ another *exchange transformation* as the following composite (see [CD12, 1.1.15]):

$$\mathrm{Ex}_{\sharp *} : p_\sharp g_* \xrightarrow{\alpha_f} f_* f^* p_\sharp g_* \xrightarrow{\mathrm{Ex}_\sharp^*} f_* q_\sharp g^* g_* \xrightarrow{\alpha'_g} f_* q_\sharp. \tag{4.2.2.a}$$

4.2.3. Proposition 4.1.1, and the existence of the map $c_1^{\text{ét}}$ defined in Definition 4.1.4, show that the category $\mathrm{DM}_{\text{ét}}(S,R)$ satisfies all the assumptions of [Dég07, § 2.1]. Thus, the results of this article can be applied to that latter category. In particular, according to [Dég07, Proposition 4.3], we get the following.

Proposition 4.2.4. *Let $f : X \to S$ be a smooth morphism of pure dimension d and $s : S \to X$ be a section of f. Then, using the notation of Paragraph 2.3.3, there exists a canonical isomorphism in $\mathrm{DM}_{\text{ét}}(S,R)$:*

$$\mathfrak{p}'_{f,s} : R_S^{\mathrm{tr}}(X/X - S) \to R_S(d)[2d].$$

In particular, for any motive K in $\mathrm{DM}_{\text{ét}}(S,R)$, we get a canonical isomorphism:

$$\mathfrak{p}_{f,s} : \begin{cases} f_\sharp s_*(K) = f_\sharp s_*(s^* f^*(K) \otimes R_S) & \xrightarrow{\sim} K \otimes f_\sharp s_*(R_S) \\ & = K \otimes R_S^{\mathrm{tr}}(X/X - S) \xrightarrow{\mathfrak{p}'_{f,s}} K(d)[2d] \end{cases}$$

which is natural in K. The first isomorphism uses the projection formulas respectively for the smooth morphism f (see point (5) of Definition A.1.1) and for the immersion s (i.e. the isomorphism (2.3.4.a)).

4.2.5. Assume now that $f : X \to S$ is smooth and projective of dimension d. We consider the following diagram

$$\begin{tikzcd} X \arrow[r, "\delta"] & X \times_S X \arrow[r, "f''"] \arrow[d, "f'"'] \arrow[dr, phantom, "\Theta"] & X \arrow[d, "f"] \\ & X \arrow[r, "f"] & S \end{tikzcd}$$

where Θ is the obvious cartesian square and δ is the diagonal embedding.

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As in [CD12, 2.4.39], we introduce the following natural transformation:

$$\mathfrak{p}_f : f_\sharp = f_\sharp f''_* \delta_* \xrightarrow{\mathrm{Ex}_{\sharp *}} f_* f'_\sharp \delta_* \xrightarrow{\mathfrak{p}_{f',\delta}} f_*(d)[2d] \tag{4.2.5.a}$$

with the notation of Remark 4.2.2 with respect to the square Θ.

Theorem 4.2.6. *Under the above assumptions, the map $\mathfrak{p}_f$ is an isomorphism.*

Proof. In this proof, we put $\tau(K) = K(d)[2d]$. Note that, according to the basic properties of a premotivic category, we get the following identification of functors for $\mathrm{DM}_{\text{ét}}(-, R)$:

$$f^*\tau = \tau f^*, \quad f_\sharp\tau = \tau f_\sharp. \tag{4.2.6.a}$$

Moreover, we can define a natural *exchange transformation*:

$$\mathrm{Ex}_\tau : \tau f_* \xrightarrow{\alpha_f} f_* f^* \tau f_* = f_* \tau f^* f_* \xrightarrow{\alpha'_f} f_* \tau \tag{4.2.6.b}$$

with the notations of Paragraph 4.2.1. Using the fact τ is an equivalence of categories according to Proposition 4.1.1, we deduce easily from the identification (4.2.6.a) that τ_f is an isomorphism.

The key point of the proof is the following lemma inspired by a proof of Ayoub (see the proof of [Ayo07, 1.7.14, 1.7.15]).

Lemma 4.2.7. *To check that $\mathfrak{p}_f$ is an isomorphism, it is sufficient to prove that the natural transformation*

$$\mathfrak{p}_f.f^* : f_\sharp f^* \to f_* \tau f^*$$

is an isomorphism.

To prove the lemma we construct a right inverse ϕ_1 and a left inverse ϕ_2 to the morphism $\mathfrak{p}_f$ as the following composite maps:

$$\phi_1 : f_*\tau \xrightarrow{\alpha_f} f_* f^* f_* \tau \xrightarrow{\mathrm{Ex}_\tau^{-1}} f_* f^* \tau f_* = f_* \tau f^* f_* \xrightarrow{(\mathfrak{p}_f.f^*f_*)^{-1}} f_\sharp f^* f_* \xrightarrow{\alpha'_f} f_\sharp$$

$$\phi_2 : f_*\tau \xrightarrow{\beta_f} f_* \tau f^* f_\sharp \xrightarrow{(\mathfrak{p}_f.f^*f_\sharp)^{-1}} f_\sharp f^* f_\sharp \xrightarrow{\beta'_f} f_\sharp.$$

Let us check that $\mathfrak{p}_f \circ \phi_1 = 1$. To prove this relation, we prove that the following diagram is commutative.

$$\begin{tikzcd}
f_*\tau \arrow[r, "\alpha_f"] \arrow[ddd, equal] & f_*f^*f_*\tau \arrow[r, "\mathrm{Ex}_\tau^{-1}"] \arrow[dd, equal] & f_*\tau f^*f_* \arrow[r, "(\mathfrak{p}_f f^*f_*)^{-1}"] \arrow[d, equal] & f_\sharp f^*f_* \arrow[r, "\alpha'_f"] \arrow[d, equal] & f_\sharp \arrow[r, "\mathfrak{p}_f"] \arrow[d, phantom, "(1)"] & f_*\tau \arrow[d, equal] \\
& & f_*\tau f^*f_* \arrow[r, "(\mathfrak{p}_f f^*f_*)^{-1}"] \arrow[d, equal] & f_\sharp f^*f_* \arrow[r, "\mathfrak{p}_f f^*f_*"] \arrow[d, phantom, "(2)"] & f_*\tau f f_* \arrow[r, "\alpha'_f"] & f_*\tau \arrow[d, equal] \\
& f_*f^*f_*\tau \arrow[r, "\mathrm{Ex}_\tau^{-1}"] \arrow[d, equal] & f_*\tau f^*f_* \arrow[rrr, "\alpha'_f"] \arrow[dr, phantom, "(3)"] & & & f_*\tau \arrow[d, equal] \\
f_*\tau \arrow[r, "\alpha_f"] & f_*f^*f_*\tau \arrow[rrrr, "\alpha'_f"] & & & & f_*\tau
\end{tikzcd}$$

The commutativity of (1) and (2) is obvious and the commutativity of (3) follows from (4.2.6.b) defining Ex_τ. Then the result follows from the usual formula between the unit and counit of an adjunction. The relation $\phi_2 \circ \mathfrak{p}_f = 1$ is proved using the same kind of computations.

The end of the proof now relies on the following lemma. It relies itself on [Dég07, Theorem 5.23], which can be applied thanks to Paragraph 4.2.3.

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LEMMA 4.2.8. *Let $f : X \to S$ be smooth projective of dimension d as above, and $\delta : X \to X \times_S X$ the diagonal embedding. Then the following holds.*

- *The étale motive $R_S^{\mathrm{tr}}(X)$ is strongly dualizable in $\mathrm{DM}_{\text{ét}}(S, R)$.*
- *Consider the morphism μ defined by the following composition:*

$$R_S^{\mathrm{tr}}(X) \otimes_S R_S^{\mathrm{tr}}(X) = R_S^{\mathrm{tr}}(X \times_S X) \xrightarrow{\pi} R_S^{\mathrm{tr}}(X \times_S X / X \times_S X - \delta(X)) \xrightarrow{\mathfrak{p}'_{f',\delta}} R_S^{\mathrm{tr}}(X)(d)[2d] \xrightarrow{f_*} R_S(d)[2d]. \tag{4.2.8.a}$$

where π is the canonical map and $\mathfrak{p}'_{f',\delta}$ is the purity isomorphism of Proposition 4.2.4. Then μ induces by adjunction an isomorphism of endofunctors of $\mathrm{DM}_{\text{ét}}(S, R)$:

$$(R_S^{\mathrm{tr}}(X) \otimes_S -) \xrightarrow{d_{X/S}} \underline{\mathrm{Hom}}(R_S^{\mathrm{tr}}(X), -(d)[2d]).$$

To finish the proof, we now check that the map

$$f_\sharp f^* \xrightarrow{\mathfrak{p}_f f^*} f_* \tau f^* = f_* f^* \tau$$

is an isomorphism. Recall that, according to the smooth projection formula for the premotivic category $\mathrm{DM}_{\text{ét}}$, we get an identification of functors:

$$f_\sharp f^* = (R_S^{\mathrm{tr}}(X) \otimes -).$$

Thus the right adjoint $f_* f^*$ is identified with $\underline{\mathrm{Hom}}(R_S^{\mathrm{tr}}(X), -)$. According to the above theorem, it is sufficient to prove that the map $\mathfrak{p}_f f^*$ above coincide through these identifications with the isomorphism $d_{X/S}$ above.

According to the above definition of μ, the natural transformation of functors $(\mu \otimes -)$ can be described as the following composite:

$$\begin{aligned} f_\sharp f^* f_\sharp f^* \xrightarrow{\mathrm{Ex}_\sharp^*} f_\sharp f'_\sharp f''^* f^* = g_\sharp g^* \xrightarrow{\alpha_\delta} g_\sharp \delta_* \delta^* g^* \\ = f_\sharp f'_\sharp \delta_* f^* \xrightarrow{\mathfrak{p}_{f',\delta}} f_\sharp \tau f^* = f_\sharp f^* \tau \xrightarrow{\beta'_f} \tau \end{aligned}$$

where $g = f \circ f'' = f \circ f'$ is the projection $X \times_S X \to S$. Indeed the base change map $\mathrm{Ex}_\sharp^*$ associated with the square Θ corresponds to the first identification in (4.2.8.a) and the adjunction map α_δ corresponds to the canonical map π.

Thus, we have to prove the preceding composite map is equal to the following one, obtained by adjunction from $\mathfrak{p}_f$:

$$\begin{aligned} f_\sharp f^* f_\sharp f^* = f_\sharp f^* f_\sharp f''_* \delta_* f^* \xrightarrow{\mathrm{Ex}_{\sharp *}} f_\sharp f^* f_* f'_\sharp \delta_* f^* \\ \xrightarrow{\mathfrak{p}_{f',\delta}} f_\sharp f^* f_* \tau f^* = f_\sharp f^* f_* f^* \tau \xrightarrow{\alpha'_f} f_\sharp f^* \tau \xrightarrow{\beta'_f} \tau. \end{aligned}$$

One can check after some easy cancellation that this amounts to prove the commutativity of the following diagram.

$$\begin{tikzcd} f^* f_\sharp \arrow[r, equal] \arrow[d, "\mathrm{Ex}_\sharp^*"'] & f^* f_\sharp f''_* \delta_* \arrow[r, "\mathrm{Ex}_{\sharp *}"] & f^* f_* f''_\sharp \delta_* \arrow[d, "\alpha'_f"] \\ f'_\sharp f''^* \arrow[r, "\alpha_\delta"] & f'_\sharp \delta_* \delta^* f''^* \arrow[r, equal] & f'_\sharp \delta_* \end{tikzcd}$$

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Using (4.2.2.a), we can divide this diagram into the following pieces.

$$\begin{tikzcd}
f^*f_\sharp \arrow[r, equal] \arrow[d, "\mathrm{Ex}^*_\sharp"'] & f^*f_\sharp f''_*\delta_* \arrow[r, "\alpha_f"] \arrow[d, "\mathrm{Ex}^*_\sharp"'] & f^*f_*f^*f_\sharp f''_*\delta_* \arrow[r, "\mathrm{Ex}^*_\sharp"] & f^*f_*f'_\sharp f''^*f''_*\delta_* \arrow[r, "\alpha'_{f''}"] \arrow[d, "\alpha'_f"] & f^*f_*f''_\sharp\delta_* \arrow[d, "\alpha'_f"] \\
f'_\sharp f''^* \arrow[r, equal] \arrow[d, equal] & f'_\sharp f''^*f''_*\delta_* \arrow[rr, equal] \arrow[urr, "\alpha_f"] & & f'_\sharp f''^*f''_*\delta_* \arrow[r, "\alpha'_{f''}"] \arrow[d, phantom, "(*)"] & f'_\sharp\delta_* \arrow[d, equal] \\
f'_\sharp f''^* \arrow[rrrr, "\alpha_\delta"] & & & & f'_\sharp\delta_*
\end{tikzcd}$$

Every part of this diagram is obviously commutative except for part $(*)$. As $f''\delta = 1$, the axioms of a 2-functors (for f^* and f_* say) implies that the unit map

$$f'_\sharp f''^* \xrightarrow{\alpha_{f''\delta}} f'_\sharp f''^*(f''\delta)_*(f''\delta)^*$$

is the canonical identification that we get using $1_* = 1$ and $1^* = 1$. We can consider the following diagram

$$\begin{tikzcd}
f'_\sharp f''^* \arrow[rr, equal, "\alpha_{f''\delta}"] \arrow[d, equal] & & f'_\sharp f''^*(f''\delta)_*(f''\delta)^* \arrow[r, equal] \arrow[d, equal] & f'_\sharp f''^*f''_*\delta_* \arrow[dd, "\alpha'_{f''}"] \\
f'_\sharp f''^* \arrow[r, "\alpha_{f''}"] \arrow[d, equal] & f'_\sharp f''^*f''_\sharp f''^* \arrow[r, "\alpha_\delta"] \arrow[d, "\alpha'_{f''}"] & f'_\sharp f''^*(f''\delta)_*(f''\delta)^* \arrow[d, "\alpha'_{f''}"] & \\
f'_\sharp f''^* \arrow[r, equal] & f'_\sharp f''^* \arrow[r, "\alpha_\delta"] & f'_\sharp\delta_*\delta^*f''^* \arrow[r, equal] & f'_\sharp\delta_*
\end{tikzcd}$$

for which each part is obviously commutative. This completes the proof. $\square$

This theorem will be generalized later on (see Corollary 4.3.2, point (3)). The important fact for the time being is the following corollary.

Corollary 4.2.9. *Under the hypothesis of Remark 4.2.2, if we assume that p is projective and smooth, the morphism $\mathrm{Ex}_{\sharp *} : p_\sharp g_* \to f_*q_\sharp$ is an isomorphism.*

In fact, putting $\tau(K) = K(d)[2d]$ where d is the dimension of p, one checks easily that the following diagram is commutative

$$\begin{tikzcd}
p_\sharp g_* \arrow[rr, "\mathrm{Ex}_{\sharp *}"] \arrow[d, "\mathfrak{p}_p"'] & & f_*q_\sharp \arrow[d, "\mathfrak{p}_q"] \\
p_*\tau g_* & p_*g_*\tau \arrow[l, "\mathrm{Ex}_\tau"'] \arrow[r, equal] & f_*q_*\tau
\end{tikzcd}$$

where we use formula (4.2.6.b) for the isomorphism Ex_τ.

4.3 Localization

Theorem 4.3.1. *For any ring of positive characteristic R, the triangulated premotivic category $\mathrm{DM}_{\acute{e}t}(-, R)$ satisfies the localization property (see Definition A.1.12).*

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Proof. We will prove that condition (iii) of Proposition 2.3.4 is satisfied. Note that, according to Proposition 4.1.1, i_* commutes with twists.[8] Thus it remains to prove that for any smooth S-scheme X, the canonical morphism

$$\epsilon_{X/S} : R^{\mathrm{tr}}_S(X/X - X_Z) \to i_* R^{\mathrm{tr}}_Z(X_Z)$$

is an isomorphism in $\mathrm{DM}_{\text{ét}}(S, R)$ (recall that $i_* = \mathbf{R}i_*$ according to Lemma 2.3.1).

Let us first consider the case where X is étale. Then according to Corollary 2.1.9, the sequence of sheaves with transfers

$$0 \to R^{\mathrm{tr}}_S(X - X_Z) \xrightarrow{j_*} R^{\mathrm{tr}}_S(X) \xrightarrow{i^*} i_* R^{\mathrm{tr}}_Z(X_Z) \to 0 \tag{4.3.1.a}$$

is isomorphic after applying the functor γ_* to the sequence

$$0 \to R_S(X - X_Z) \xrightarrow{j_*} R_S(X) \xrightarrow{i^*} i_* R_Z(X_Z) \to 0.$$

This sequence of sheaves is obviously exact (we can easily check this on the fibers). As γ_* is conservative and exact, the sequence (4.3.1.a) is exact. Thus the canonical map

$$R^{\mathrm{tr}}_S(X/X - X_Z) := \mathrm{coker}(j_*) \to i_* R^{\mathrm{tr}}_Z(X_Z)$$

is an isomorphism in $\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(X, R)$ and *a fortiori* in $\mathrm{DM}_{\text{ét}}(S, R)$.

We now turn to the general case. For any open cover $X = U \cup V$, we easily get the usual Mayer–Vietoris short exact sequence in $\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S, R)$:

$$0 \to R^{\mathrm{tr}}_S(U \cap V) \to R^{\mathrm{tr}}_S(U) \oplus R^{\mathrm{tr}}_S(V) \to R^{\mathrm{tr}}_S(X) \to 0.$$

Thus the assertion is local on X for the Zariski topology. In particular, as X/S is smooth, we can assume there exists an étale map $X \to \mathbf{A}^n_S$. Therefore, by composing with any open immersion $\mathbf{A}^n_S \to \mathbf{P}^n_S$, we get an étale S-morphism $f : X \to \mathbf{P}^n_S$. Consider the following cartesian square

$$\begin{tikzcd} \mathbf{P}^n_Z \arrow[r, "k"] \arrow[d, "q"'] & \mathbf{P}^n_S \arrow[d, "p"] \\ Z \arrow[r, "i"] & S \end{tikzcd}$$

where p is the canonical projection. If we consider the notations of Paragraph 4.2.1 and Remark 4.2.2 relative to this square, then the following diagram

$$\begin{tikzcd} p_\sharp \arrow[rr, "p_\sharp(\alpha_k)"] \arrow[d, equal] & & p_\sharp k_* k^* \arrow[d, "\mathrm{Ex}_{\sharp *}"] \\ p_\sharp \arrow[r, "\alpha_i"] & i_* i^* p_\sharp \arrow[r, "\mathrm{Ex}^*_\sharp"] & i_* q_\sharp k^* \end{tikzcd}$$

is commutative; this can be easily checked using (4.2.2.a).

[8] Essentially because it is true for its left adjoint i^*. This fact was already remarked at the beginning of the proof of Theorem 4.2.6.

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If we apply the preceding commutative diagram to the object $R^{\mathrm{tr}}_S(X/X-X_Z)$, we get the following commutative diagram in $\mathrm{DM}_{\mathrm{\acute{e}t}}(S,R)$.

$$\begin{tikzcd} p_\sharp R^{\mathrm{tr}}_{\mathbf{P}^n_S}(X/X-X_Z) \arrow[rr, "p_\sharp(\epsilon_{X/\mathbf{P}^n_S})"] \arrow[d, equal] & & p_\sharp k_* R^{\mathrm{tr}}_{\mathbf{P}^n_Z}(X_Z) \arrow[d, "\mathrm{Ex}_{\sharp *}"] \\ R^{\mathrm{tr}}_S(X/X-X_Z) \arrow[r, "\epsilon_{X/S}"] & i_* q_\sharp R^{\mathrm{tr}}_Z(X_Z) \arrow[r, equal] & i_* q_\sharp R^{\mathrm{tr}}_{\mathbf{P}^n_Z}(X_Z) \end{tikzcd}$$

The conclusion follows from the case treated above and from Corollary 4.2.9. □

As the premotivic triangulated category $\mathrm{DM}_{\mathrm{\acute{e}t}}(-,R)$ satisfies the stability property (Proposition 4.1.1) and the weak purity property (Theorem 4.2.6) the previous result allows to apply Theorem A.1.13 to $\mathrm{DM}_{\mathrm{\acute{e}t}}(-,R)$.

Corollary 4.3.2. *For any ring R of positive characteristic, the oriented triangulated premotivic category $\mathrm{DM}_{\mathrm{\acute{e}t}}(-,R)$ satisfies Grothendieck's six functors formalism (Definition A.1.10).*

In other words, $\mathrm{DM}_{\mathrm{\acute{e}t}}(-,R)$ is an oriented motivic triangulated category over the category of noetherian schemes.

4.4 Compatibility with direct image

4.4.1. According to Example A.1.3, the categories $\mathrm{D}(X_{\mathrm{\acute{e}t}},R)$ are the fibers of an Ét-premotivic triangulated category over the category of noetherian schemes.

Recall that the derived tensor product $\otimes^{\mathbf{L}}$ is essentially characterized by the property that for any étale X-schemes U and V, $R_X(U)\otimes^{\mathbf{L}} R_X(V)=R_X(U\times_X V)$ in $D(X_{\mathrm{\acute{e}t}},R)$.

Similarly, for any étale morphism $p:V\to X$, the operation $\mathbf{L}p_\sharp$ is characterized by the property that for any étale V-scheme W, $\mathbf{L}p_\sharp(R_V(W))=R_X(W)$.

4.4.2. (Following the abuse of this section we drop again the letters $\mathbf{L}$ and $\mathbf{R}$ in front of derived functors to simplify notations.) Due to the properties of the functors involved in the construction of

$$\rho_!:\mathrm{D}(-_{\mathrm{\acute{e}t}},R)\to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(-,R)$$

we get the following compatibility properties.

(1) The functor $\rho_!$ is monoidal.

(2) For any morphism $f:Y\to X$ of schemes, there exists a canonical isomorphism:

$$\mathrm{Ex}(f^*,\rho_!):f^*\rho_!\to\rho_! f^*.$$

(3) For any étale morphism $p:V\to X$, there exists a canonical isomorphism:

$$\rho_! p_\sharp\to p_\sharp\rho_!.$$

Assume that R is of positive characteristic n, and consider now a proper morphism $f:Y\to X$ between schemes whose residue characteristics are prime to n. Then, we can form the following natural transformation:

$$\mathrm{Ex}(\rho_!,f_*):\rho_! f_*\xrightarrow{\alpha_f} f_*f^*\rho_! f_*\xrightarrow{\mathrm{Ex}(f^*,\rho_!)} f_*\rho_! f^* f_*\xrightarrow{\alpha'_f} f_*\rho_!.$$

Proposition 4.4.3. *Using the assumptions and notations above, the map*

$$\mathrm{Ex}(\rho_!, f_*) : \rho_! f_*(K) \to f_* \rho_!(K)$$

is an isomorphism for any object K of $\mathrm{D}(Y_{\text{ét}}, R)$.

Proof. Recall the triangulated category $\mathrm{DM}_{\text{ét}}(X,R) = \mathrm{DM}^{\text{eff}}_{\text{ét}}(X,R)$ is generated by objects of the form $R^{\text{tr}}_X(W) = p_\sharp(\mathbb{1}_W)$ where $p : W \to X$ is a smooth morphism. Thus, we have to prove that for any integer $n \in \mathbf{Z}$, the induced map

$$\mathrm{Hom}_{\mathrm{DM}^{\text{eff}}_{\text{ét}}(X,R)}(p_\sharp(R_W)[n], \rho_! f_*(K)) \to \mathrm{Hom}_{\mathrm{DM}^{\text{eff}}_{\text{ét}}(X,R)}(p_\sharp(R_W)[n], f_* \rho_!(K)) \tag{4.4.3.a}$$

is invertible. Consider the following cartesian square.

$$\begin{tikzcd} W' \arrow[r, "q"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ W \arrow[r, "p"] & X \end{tikzcd}$$

Then we get canonical isomorphisms

$$\mathrm{Ex}^*_* : p^* f_* \to g_* q^*$$

both in $D(-_{\text{ét}}, R)$ and in the premotivic triangulated category $\mathrm{DM}_{\text{ét}}(-,R)$, by the proper base change theorem; see Theorem 1.2.1 and respectively Corollary 4.3.2, Definition A.1.10(4).

On the other hand, the following diagram is commutative.

$$\begin{tikzcd} p^* \rho_! f_* \arrow[r, "\mathrm{Ex}(\rho_!, f_*)"] \arrow[d, "\mathrm{Ex}(p^*, \rho_!)"'] & p^* f_* \rho_! \arrow[d, "\mathrm{Ex}^*_*"] \\ \rho_! p^* f_* \arrow[d, "\mathrm{Ex}^*_*"'] & g_* q^* \rho_! \arrow[d, "\mathrm{Ex}(q^*, \rho_!)"] \\ \rho_! g_* q^* \arrow[r, "\mathrm{Ex}(\rho_!, g_*)"] & g_* \rho_! q^* \end{tikzcd}$$

Thus, using the adjunction $(p_\sharp, p^*)$ and replacing K by $g^*(K)[-n]$, we reduce to prove that the map (4.4.3.a) is an isomorphism for any complex K when $p = 1_X$ and $n = 0$. We have to prove that the map

$$\mathrm{Ex}(\rho_!, f_*)_* : \mathrm{Hom}_{\mathrm{DM}^{\text{eff}}_{\text{ét}}(X,R)}(R_X, \rho_! f_*(K)) \to \mathrm{Hom}_{\mathrm{DM}^{\text{eff}}_{\text{ét}}(X,R)}(R_X, f_* \rho_!(K))$$

is an isomorphism.

However, using the fact $\rho_!(R_X) = R_X$, Proposition 3.1.7, as well as the adjunction (f^*, f_*), the source and target of this map can be identified to $H^0_{\text{ét}}(Y, K)$ and this completes the proof. For the cautious reader, let us say more precisely that this follows from the commutativity of the following diagram.

$$\begin{tikzcd} \mathrm{Hom}(R_X, f_*(K)) \arrow[r, "\text{adj.}"] \arrow[d, "\rho_!"'] & \mathrm{Hom}(f^*(R_X), K) \arrow[d, "\rho_!"] & \\ \mathrm{Hom}(\rho_!(R_X), \rho_! f_*(K)) \arrow[d, equal] & \mathrm{Hom}(\rho_! f^*(R_X), \rho_!(K)) \arrow[r, "\mathrm{Ex}(f^*, \rho_!)^*"] & \mathrm{Hom}(f^* \rho_!(R_X), \rho_!(K)) \arrow[d, "\text{adj.}"] \\ \mathrm{Hom}(\rho_!(R_X), \rho_! f_*(K)) \arrow[rr, "\mathrm{Ex}(\rho_!, f_*)_*"] & & \mathrm{Hom}(\rho_!(R_X), f_* \rho_!(K)) \end{tikzcd}$$

□

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4.5 The rigidity theorem

PROPOSITION 4.5.1. *The category* $\mathrm{DM}_{\text{ét}}(X,R)$ *is the localizing subcategory of the triangulated category* $\mathrm{DM}_{\text{ét}}(X,R)$ *generated by objects of the form* $f_*(R_Y)(n)$ *for any projective morphism* $f : Y \to X$ *and any integer* $n \in \mathbf{Z}$.

Proof. The category $\mathrm{DM}_{\text{ét}}(X,R)$ is the localizing subcategory of $\mathrm{DM}_{\text{ét}}(X,R)$ generated by objects of the form $R_X(Y)(n)$ for any smooth X-scheme Y and any integer $n \in \mathbf{Z}$. But such objects belong to the thick subcategory generated by objects of the form $f_*(R_Y)(n)$ for any projective morphism $f : Y \to X$ and any integer $n \in \mathbf{Z}$: see [Ayo07, Lemma 2.2.23] or [CD12, Proposition 4.2.13], which is meaningful thanks to Theorem 4.3.1 above. □

The following theorem is a generalization of the rigidity theorem of Suslin and Voevodsky ([Voe96, 4.1.9] or [VSF00, ch. 5, 3.3.3]) when the base is of positive dimension.

THEOREM 4.5.2. *Assume that* R *is a ring of positive characteristic* n*, and consider a noetherian* $\mathbf{Z}[1/n]$*-scheme* X*. Then the functor*

$$\rho_! : \mathrm{D}(X_{\text{ét}}, R) \to \mathrm{DM}^{\text{eff}}_{\text{ét}}(X,R) \simeq \mathrm{DM}_{\text{ét}}(X,R)$$

is an equivalence of symmetric monoidal triangulated categories, whose quasi-inverse is induced by the restriction functor on the small étale site (for $\mathbf{A}^1$*-local complexes of étale sheaves with transfers).*

Proof. The fully faithfulness of the functor $\rho_!$ has been established in Proposition 3.1.7. As the functor $\rho_!$ commutes with small sums, it identifies $\mathrm{D}(X_{\text{ét}}, R)$ with a localizing subcategory of $\mathrm{DM}_{\text{ét}}(X,R)$. Therefore, the essential surjectivity of the functor $\rho_!$ readily follows from Propositions 4.4.3 and 4.5.1. □

We can extend these results in the case of p-torsion coefficients as follows.

COROLLARY 4.5.3. *Assume that* R *is of characteristic* p^r *for a prime* p *and an integer* $r \geqslant 1$. *Let* X *be any noetherian scheme, and* $X[1/p] = X \times \mathrm{Spec}(\mathbf{Z}[1/p])$*. Then there is a canonical equivalence of categories*

$$\mathrm{DM}_{\text{ét}}(X,R) \simeq \mathrm{D}(X[1/p]_{\text{ét}}, R).$$

Proof. This follows from Theorem 4.5.2 and from Proposition A.3.4. □

COROLLARY 4.5.4. *Under the assumptions of Theorem 4.5.2, for any complex of étale sheaves with transfers of* R*-modules* C *over* X*, the following conditions are equivalent.*

(i) *The complex* C *is* $\mathbf{A}^1$*-local.*
(ii) *For any integer* n*, the étale sheaf* $H^n(C)$ *(seen as a complex concentrated in degree zero) is* $\mathbf{A}^1$*-local.*
(iii) *The map* $\rho_!\rho^*C \to C$ *is a quasi-isomorphism of complexes of étale sheaves.*
(iv) *For any integer* n*, the map* $\rho_!\rho^*H^n(C) \to H^n(C)$ *is invertible.*

Proof. The equivalence between conditions (i) and (iii) follows immediately from Theorem 4.5.2, from which we deduce the equivalence between conditions (ii) and (iv). The equivalence between conditions (iii) and (iv) comes from the fact that both $\rho_!$ and ρ^* are exact functors. □

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4.6 Absolute purity with torsion coefficients

Theorem 4.6.1. *The oriented triangulated premotivic category* $\mathrm{DM}_{\text{ét}}(-, R)$ *satisfies the absolute purity property (Definition A.2.9).*

This means in particular that for any closed immersion $i : Z \to S$ between regular schemes, one has a canonical isomorphism in $\mathrm{DM}_{\text{ét}}(S, R)$:

$$\eta_X(Z) : R_Z \to i^!(R_S)(c)[2c].$$

Proof. For any closed immersion $i : Z \to S$, we define a complex of R-modules using the dg-enrichment of $\mathrm{DM}_{\text{ét}}(S, R)$:

$$\mathbf{R}\Gamma_Z(X) = \mathbf{R}\mathrm{Hom}(i_*(R_Z), R_S).$$

This complex is contravariant in (X, Z); see § A.2.1 for morphisms of closed pairs. We have to prove that whenever S and R are regular, the maps induced by the deformation diagram (A.2.7.a),

$$\mathbf{R}\Gamma_Z(X) \xleftarrow{d_1^*} \mathbf{R}\Gamma_{\mathbf{A}^1_Z}(D_Z X) \xrightarrow{d_0^*} \mathbf{R}\Gamma_Z(N_Z X)$$

are quasi-isomorphisms. We may assume that $R = \mathbf{Z}/n\mathbf{Z}$ for some natural number $n > 0$. By a simple *dévissage*, we may as well assume that n is a power of some prime p. By virtue of Corollary 4.5.3, we see that all this is a reformulation of the analogous property in the setting of classical étale cohomology, with coefficients prime to the residue characteristics. We conclude with Gabber's absolute purity theorem (see [Fuj02]). □

5. Motives and h-descent

5.1 h-Motives

5.1.1. Recall that Voevodsky has defined the h-topology on the category of noetherian schemes as the topology whose covers are the universal topological epimorphisms; see [Voe96, 3.1.2]. Given a noetherian scheme S as well as a ring R, we will denote by $\mathrm{Sh}_{\mathrm{h}}(S, R)$ the category of h-sheaves of R-modules on the category $\mathscr{S}^{\mathrm{ft}}_S$. Given any S-scheme X of finite type, we will denote by $\underline{R}^{\mathrm{h}}_S(X)$ the free h-sheaf or R-modules represented by X. As proved in [CD12, Example 5.1.4], the Sch-fibered category $\mathrm{Sh}_{\mathrm{h}}(-, R)$ is an abelian $\mathscr{S}^{\mathrm{ft}}$-premotivic category in the sense of Definition A.1.1.

The following definition, although using the theory of [CD12] for the existence of derived functors, follows the original idea of Voevodsky in [Voe96].

Definition 5.1.2. Using the notations above, we define the $\mathscr{S}^{\mathrm{ft}}$-premotivic *category of effective* h*-motives* (respectively of h*-motives*) with R-linear coefficients

$$\underline{\mathrm{DM}}^{\mathrm{eff}}_{\mathrm{h}}(-, R) \quad (\text{respectively } \underline{\mathrm{DM}}_h(-, R))$$

as the $\mathbf{A}^1$-derived category (respectively stable $\mathbf{A}^1$-derived category) associated with the fibered category $\mathrm{Sh}_{\mathrm{h}}(-, R)$ over noetherian schemes.

In other words, the triangulated monoidal category $\underline{\mathrm{DM}}^{\mathrm{eff}}_{\mathrm{h}}(S, R)$ is the $\mathbf{A}^1$-localization of the derived category $\mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(S, R))$; this is precisely the original definition of Voevodsky [Voe96, § 4]. This category is completely analogous to the case of the étale topology (2.2.4). Similarly, the category $\underline{\mathrm{DM}}_{\mathrm{h}}(S, R)$ is obtained from $\underline{\mathrm{DM}}^{\mathrm{eff}}_{\mathrm{h}}(S, R)$ by $\otimes$-inverting the Tate h-motive in the sense of model categories. We get functors as in (2.2.4.a):

$$\mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(S, R)) \xrightarrow{\pi_{\mathbf{A}^1}} \underline{\mathrm{DM}}^{\mathrm{eff}}_{\mathrm{h}}(S, R) \xrightarrow{\Sigma^\infty} \underline{\mathrm{DM}}_{\mathrm{h}}(S, R). \tag{5.1.2.a}$$

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Note however that the category $\underline{\mathrm{DM}}^{\mathrm{eff}}_{\mathrm{h}}(S,R)$ ($\underline{\mathrm{DM}}_{\mathrm{h}}(S,R)$) is generated by objects of the form $\underline{R}^{\mathrm{h}}_S(X)$ ($\Sigma^\infty \underline{R}^{\mathrm{h}}_S(X)(n)$) for any S-scheme of finite type X (for any S-scheme of finite type X and any integer $n \in, \mathbf{Z}$, respectively). These categories are too big to satisfy the six functors formalism (the drawback is about the localization property with respect to closed immersions, which means that there is no good theory of support).

This is why we introduce the following definition (following [CD12, Example 5.3.31]).

Definition 5.1.3. The category of *effective* h-*motives* (respectively of h-*motives*)

$$\mathrm{DM}^{\mathrm{eff}}_{\mathrm{h}}(X,R) \quad (\text{respectively } \mathrm{DM}_{\mathrm{h}}(X,R))$$

is the smallest full subcategory of $\underline{\mathrm{DM}}^{\mathrm{eff}}_{\mathrm{h}}(S,R)$ (respectively of $\underline{\mathrm{DM}}_{\mathrm{h}}(X,R)$) closed under arbitrary small sums and containing the objects of the form $\underline{R}^{\mathrm{h}}_S(X)$ (respectively $\Sigma^\infty \underline{R}^{\mathrm{h}}_S(X)(n)$) for X/S smooth (respectively for X/S smooth and $n \in \mathbf{Z}$).

The category of *constructible effective* (respectively of *constructible*) h-*motives of geometric origin*

$$\mathrm{DM}^{\mathrm{eff}}_{\mathrm{h},c}(X,R) \quad (\text{respectively } \mathrm{DM}_{\mathrm{h},c}(X,R))$$

is the thick triangulated subcategory of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{h}}(S,R)$ (respectively $\mathrm{DM}_{\mathrm{h}}(X,R)$) generated by objects of the form $\underline{R}^{\mathrm{h}}_S(X)$(respectively $\Sigma^\infty \underline{R}^{\mathrm{h}}_S(X)(n)$) for X/S smooth (respectively for X/S smooth and $n \in \mathbf{Z}$).

We will sometimes simplify the notations and write $R(X) := \Sigma^\infty \underline{R}^{\mathrm{h}}_S(X)$, as an object of $\mathrm{DM}_{\mathrm{h}}(X,R)$ (for a smooth S-scheme X).

Remark 5.1.4. The objects of $\mathrm{DM}_{\mathrm{h},c}(X,R)$ will often simply be called *constructible* following the terminology of [Ayo07, CD12]. However, it should be pointed out that this finiteness assumption corresponds rather to what is usually called 'geometric' or 'of geometric origin' in the theories of Galois representations, or D-modules (this fits well with the terminology 'geometric' chosen by Voevodsky for motivic complexes in [VSF00, ch. 5]).

Moreover, if R is a ring of positive characteristic n, with n invertible in $\mathscr{O}_X$, we will see later (Corollary 5.5.4) that we have a canonical equivalence of categories: $\mathrm{D}(X_{\mathrm{\acute{e}t}},R) \simeq \mathrm{DM}_{\mathrm{h}}(X,R)$. There is two classical finiteness conditions on the left-hand side, given by the subcategories:

- $\mathrm{D}^b_c(X_{\mathrm{\acute{e}t}},R)$, complexes with bounded and constructible cohomology sheaves;
- $\mathrm{D}^b_{\mathrm{ctf}}(X_{\mathrm{\acute{e}t}},R)$, complexes in $\mathrm{D}^b_c(X_{\mathrm{\acute{e}t}},R)$ which have of finite Tor-dimension (or, equivalently, by virtue of [SGA4$\frac{1}{2}$, Rapport, 4.6], which are isomorphic in $\mathrm{D}(X_{\mathrm{\acute{e}t}},R)$ to bounded complexes whose components are flat and constructible).

Then through the previous equivalence of categories, constructible h-motives of geometric origin forms a full subcategory of $\mathrm{D}^b_{\mathrm{ctf}}(X_{\mathrm{\acute{e}t}},R)$ (see again Corollary 5.5.4).

These issues will be thoroughly studied in § 6.3. In particular, we will see in Proposition 6.3.10 that constructible h-motives are equivalent to the whole of $\mathrm{D}^b_{\mathrm{ctf}}(X_{\mathrm{\acute{e}t}},R)$ whenever the étale R-cohomological dimension of the residue fields of X is uniformly bounded (in which case they are also characterized by the property of being compact). In general, we will characterize the objects of $\mathrm{D}^b_{\mathrm{ctf}}(X_{\mathrm{\acute{e}t}},R)$ by introducing a stronger version of constructibility for h-motives: see Theorem 6.3.11.

It is obvious that the subcategory $\mathrm{DM}_{\mathrm{h}}(-,R)$ is stable by the operations f^* for any morphism f, by the operation $f_\sharp$ for any *smooth* morphism f, and by the operation $\otimes^{\mathbf{L}}$. The Brown representability theorem implies that the inclusion functor $\nu_\sharp$ admits a right adjoint ν^*, so that

$\mathrm{DM}_{\mathrm{h}}(-,R)$ is in fact a premotivic triangulated category, and we get an enlargement of premotivic triangulated category,

$$\nu_\sharp : \mathrm{DM}_{\mathrm{h}}(X,R) \rightleftarrows \underline{\mathrm{DM}}_{\mathrm{h}}(S,R) : \nu^* \tag{5.1.4.a}$$

(see [CD12, Example 5.3.31(2)]). More precisely, for any morphism of schemes $f : X \to Y$, the functor

$$\mathbf{L}f^* : \mathrm{DM}_{\mathrm{h}}(Y,R) \to \mathrm{DM}_{\mathrm{h}}(X,R)$$

admits a right adjoint

$$\mathbf{R}f_* : \mathrm{DM}_{\mathrm{h}}(X,R) \to \mathrm{DM}_{\mathrm{h}}(Y,R)$$

defined by the formula

$$\mathbf{R}f_*(M) = \nu^*(\mathbf{R}f_*(\nu_\sharp(M))).$$

Similarly, the (derived) internal Hom of $\mathrm{DM}_{\mathrm{h}}(X,R)$ is defined by the formula

$$\mathbf{R}\underline{\mathrm{Hom}}_R(M,N) = \nu^*(\mathbf{R}\underline{\mathrm{Hom}}_R(\nu_\sharp(M),\nu_\sharp(N))).$$

We will sometimes write $\mathbf{R}\underline{\mathrm{Hom}}_R(M,N) = \mathbf{R}\underline{\mathrm{Hom}}(M,N)$ when the coefficients are understood from the context. Also, when it is clear that we work with derived functors only, it might happen that we drop the thick letters $\mathbf{L}$ and $\mathbf{R}$ from the notations. The unit object of the monoidal category $\mathrm{DM}_{\mathrm{h}}(X,R)$ will be written $\mathbb{1}_X$ or R_X, depending on the emphasis we want to put on the coefficients.

Remark 5.1.5. The category $\underline{\mathrm{DM}}^{\mathrm{eff}}_{\mathrm{h}}(X,\mathbf{Z})$ is nothing else than the category introduced by Voevodsky in [Voe96] under the notation $\mathrm{DM}(S)$. The fact it corresponds to the 'étale version of mixed motives' is clearly envisioned in [Voe96] (see the end of the introduction of [Voe96]).

5.2 Comparison with Beilinson motives

5.2.1. Recall from [CD12, Paragraph 14.2.20] the category $\mathrm{DM}_{\mathrm{B}}(X)$ of Beilinson motives. The following theorem was proved in [CD12, Theorem 16.1.2] in the case of quasi-excellent schemes.

Theorem 5.2.2. *There exists a canonical equivalence*

$$\mathrm{DM}_{\mathrm{B}} \simeq \mathrm{DM}_{\mathrm{h}}(-,\mathbf{Q})$$

of premotivic triangulated categories over the category of noetherian finite-dimensional schemes. In particular, given such a scheme X, assuming in addition it is regular, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Q})}(\mathbf{Q}_X, \mathbf{Q}_X(p)[q]) \simeq \mathrm{Gr}^p_\gamma\, K_{2p-q}(X) \otimes \mathbf{Q}, \tag{5.2.2.a}$$

where the second term stands for the graded pieces of algebraic K-theory with respect to the γ-filtration.[9]

The proof of this theorem is the main goal of this section. It will be by reduction to the case of separated schemes of finite type over $\mathbf{Z}$. This will require a few intermediate steps which will also be useful later on.

Remark 5.2.3. Note that this theorem obviously extends to the case of coefficients in an arbitrary $\mathbf{Q}$-algebra R where the left-hand side is defined in [CD12, Paragraph 14.2.20].

[9] Recall that, according to [Cis13] and [CD12, 14.1.1], the regularity assumption can be dropped if we replace K-theory by its homotopy invariant version in the sense of Weibel.

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Theorem 5.2.4. *Consider a noetherian scheme X of finite dimension. Assume that the étale cohomological dimension of any residue field of X is uniformly bounded for R-linear coefficients. Then, for an object M of $\mathrm{DM_h}(X,R)$, the following conditions are equivalent.*

(a) *The motive M is constructible.*
(b) *There exists an étale covering $\{u_i : X_i \to X\}_{i\in I}$ such that, for any $i \in I$, the object $u_i^*(M)$ is constructible.*
(c) *The motive M is a compact object of $\mathrm{DM_h}(X,R)$.*
(d) *The functor $\mathbf{R}\underline{\mathrm{Hom}}(M,-) : \mathrm{DM_h}(X,R) \to \mathrm{DM_h}(X,R)$ commutes with small sums.*

Proof. We first prove that conditions (a) and (c) are equivalent. Under our assumptions, by virtue of the Goodwillie and Lichtenbaum Theorem [GL01b], any X-scheme of finite type is of finite cohomological dimension with respect to the h-topology for R-linear coefficients. Therefore, Proposition 1.1.9 shows that, for any scheme Y of finite type over X, the representable sheaf $\underline{R}^{\mathrm{h}}_X(Y)$ is a compact object of the derived category of h-sheaves of R-modules over X. As this class of h-sheaves is closed by (derived) tensor product, this implies that the functor $\mathbf{R}\underline{\mathrm{Hom}}_R(\underline{R}^{\mathrm{h}}_X(Y),-)$ preserves small sums in the derived category of h-sheaves of R-modules over X. It is easy to deduce from this property (by inspection of the definition) that the class of $\mathbf{A}^1$-local objects is closed under small sums and that Ω-spectra are closed under small sums in the derived category of Tate spectra in the category of h-sheaves of R-modules (in the sense of [CD12, Definitions 5.5.16 and 5.3.24]). Thus the objects of the form $R(Y)(n)$, for Y of finite over X and any integer n, form a generating family of compact generators in $\underline{\mathrm{DM}}_{\mathrm{h}}(X,R)$. Therefore, the family of objects $R(Y)(n)$, for Y smooth of finite over X and any integer n, form a generating family of compact generators of $\mathrm{DM_h}(X,R)$. This implies that the subcategory of compact objects of $\mathrm{DM_h}(X,R)$ is precisely $\mathrm{DM}_{\mathrm{h},c}(X,R)$.

The fact that conditions (c) and (d) are equivalent readily follows from formula

$$\mathbf{R}\mathrm{Hom}(R(Y)(n)\otimes^{\mathbf{L}}_R M,N) \simeq \mathbf{R}\mathrm{Hom}(R(Y)(n),\mathbf{R}\underline{\mathrm{Hom}}_R(M,N))$$

(for any object N), and the fact that $R(Y)(n)$ is always compact in $\mathrm{DM_h}(X,R)$ (with Y smooth over X and $n \in \mathbf{Z}$).

It is now sufficient to check that condition (b) implies condition (d). Let $\{u_i : X_i \to X\}_{i\in I}$ be an étale covering such that, for any $i \in I$, the object $u_i^*(M)$ is constructible. As the functors u_i^* form a conservative family of functors which preserve small sums (by étale descent, see [CD12, Proposition 3.2.8], and because they have right adjoints, respectively), formula

$$u_i^*(\mathbf{R}\underline{\mathrm{Hom}}_R(M,N)) \simeq \mathbf{R}\underline{\mathrm{Hom}}_R(u_i^*(M),u_i^*(N))$$

readily implies that M satisfies condition (d). □

Proposition 5.2.5. *Here, all schemes are assumed to be noetherian of finite dimension. Consider a scheme X which is the limit of a projective system $\{X_i\}_{i\in I}$ with affine transition maps. Let $\{M_i\}$ and $\{N_i\}_{i\in I}$ be two cartesian sections of the fibered category $\mathrm{DM_h}(-,\mathbf{Q})$ over the diagram of schemes $\{X_i\}_{i\in I}$, and denote by M and N the respective pullback of M_i and N_i along the projection $X \to X_i$. If each M_i is constructible, then the canonical map*

$$\varinjlim_i \mathrm{Hom}_{\mathrm{DM_h}(X_i,\mathbf{Q})}(M_i,N_i) \to \mathrm{Hom}_{\mathrm{DM_h}(X,\mathbf{Q})}(M,N)$$

is an isomorphism.

Proof. It is sufficient to prove the analogous property in $\underline{\mathrm{DM}}_{\mathrm{h}}(X, \mathbf{Q})$. The property of continuity is known to hold if we replace $\underline{\mathrm{DM}}_{\mathrm{h}}(X, \mathbf{Q})$ by the triangulated category $\mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(X, \mathbf{Q}))$ (because the representable sheaves are of finite cohomological dimension with respect to the h-topology with $\mathbf{Q}$-linear coefficients, so that we are essentially reduced to classical formulas such as [SGA4, Exposé VII, Corollaire 8.5.7]). On the other hand, we have a canonical adjunction for any (diagram of) scheme(s) S

$$a^* : \mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(X, \mathbf{Q})) \rightleftarrows \underline{\mathrm{DM}}_{\mathrm{h}}(S, \mathbf{Q}) : a_* \tag{5.2.5.a}$$

in which a^* is the composition of the $\mathbf{A}^1$-localization functor and of the infinity loop space functor Σ^∞.

By virtue of Lemma 1.1.4, the proof of the preceding theorem ensures that, for any scheme S, the family of h-motives $\mathbf{Q}(U)(n)$, for U separated of finite type over S and n any integer, form a family of compact generators of the triangulated category $\underline{\mathrm{DM}}_{\mathrm{h}}(S, \mathbf{Q})$. This implies that the functor a_* commutes with small sums (whence with arbitrary small homotopy colimits) and that the family of functors $E \mapsto a_*(E(n))$, $n \geqslant 0$, is conservative. This description of compact objects also implies the following computation. An object E of $\underline{\mathrm{DM}}_{\mathrm{h}}(X, \mathbf{Q})$ is a collection of complexes of h-sheaves of $\mathbf{Q}$-vector spaces E_n, $n \geqslant 0$, together with maps $E_n(1) \to E_{n+1}$. One then has this canonical identification:

$$a_*(E(n)) \simeq \mathbf{L}\varinjlim_{i \geqslant 0} \mathbf{R}\underline{\mathrm{Hom}}_{\mathbf{Q}}(\mathbf{Q}(i), E_{n+i}) \tag{5.2.5.b}$$

(here the internal Hom $\underline{\mathrm{Hom}}_{\mathbf{Q}}$ is the one of $\underline{\mathrm{DM}}^{\mathrm{eff}}_{\mathrm{h}}(X, \mathbf{Q})$, but it can be understood as the one of $\mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(X, \mathbf{Q}))$ whenever each E_n is $\mathbf{A}^1$-local as an object of $\mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(X, \mathbf{Q}))$). We want to prove that, the map

$$\mathbf{L}\varinjlim_{i} \mathbf{R}\mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(X_i, \mathbf{Q})}(M_i, N_i) \to \mathbf{R}\mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(X, \mathbf{Q})}(M, N) \tag{5.2.5.c}$$

is an isomorphism in the derived category of $\mathbf{Q}$-vector spaces. We can replace the indexing category I by $\{i \geqslant j\}$ for an arbitrary index $j \in I$, and, as $\mathbf{Q}(U)$ is compact in $\mathrm{DM}_{\mathrm{h}}(X_j, \mathbf{Q})$ for any separated X_j-scheme of finite type U, we easily see that it is equivalent to prove that the canonical map

$$\mathbf{L}\varinjlim_{i \geqslant j} \mathbf{R}p_{i,*}\mathbf{R}\underline{\mathrm{Hom}}_{\mathbf{Q}}(M_i, N_i) \to \mathbf{R}p_*\mathbf{R}\underline{\mathrm{Hom}}_{\mathbf{Q}}(M, N) \tag{5.2.5.d}$$

is an isomorphism in $\mathrm{DM}_{\mathrm{h}}(X_j, \mathbf{Q})$, where $p_j : X_i \to X_j$ and $X \to X_j$ denote the structural maps for $i \geqslant j$. Moreover, we may assume that $M_j = \mathbf{Q}(U)$. Replacing X_j by U (and each X_i as well as X by their pullbacks along the structural map $U \to X_j$), we may assume that $M_j = \mathbf{Q}$ is the unit object, so that the map (5.2.5.d) now has the following form:

$$\mathbf{L}\varinjlim_{i \geqslant j} \mathbf{R}p_{i,*}(N_i) \to \mathbf{R}p_*(N). \tag{5.2.5.e}$$

Remark that the functor $\mathbf{R}q_*$ preserves small homotopy colimits for any morphism of schemes q because its left adjoint $\mathbf{L}q^*$ preserves compact objects. Formula (5.2.5.b) thus implies that the image of the map (5.2.5.e) by a_* is isomorphic to an homotopy colimit of images by the functors $\underline{\mathrm{Hom}}_{\mathbf{Q}}(\mathbf{Q}(i), -)$ of analogous maps in $\mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(X, \mathbf{Q}))$. Therefore, we are reduced to prove the analog of this proposition in the premotivic category $\mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(-, \mathbf{Q}))$ instead of $\mathrm{DM}_{\mathrm{h}}(-, \mathbf{Q})$, and this ends the proof. $\square$

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Proof of Theorem 5.2.2. We first remark that the premotivic category $\mathrm{DM_h}(-,\mathbf{Q})$ is oriented in the sense of Definition A.1.5(3): this follows from [Voe96, Theorem 4.2.5 and Definition 4.2.1] which implies that for any noetherian finite-dimensional scheme X, there is a map:

$$\mathrm{Pic}(X) \simeq H^1_{\text{ét}}(X, \mathbf{G}_m) \to \mathrm{Hom}_{\mathrm{DM_h}(X)}(\mathbf{Q}_X, \mathbf{Q}_X(1)[2]).$$

Therefore, the spectrum $a_*(\mathbf{Q}_X)$ is orientable: according to [CD12, 14.2.16], it admits a unique structure of H_{B}-algebra, where H_{B} denotes the Beilinson motivic cohomology spectrum [CD12, 14.1.2]. In particular, the image of the weakly monoidal functor a_* of (5.2.5.a) is contained in the category of H_{B}-modules, which coincide with the category $\mathrm{DM_B}(X)$ applying again [CD12, 14.2.16]. This implies that the premotivic adjunction (5.2.5.a) induces a unique premotivic adjunction (X varying in the category of noetherian finite-dimensional schemes):

$$\alpha^* : \mathrm{DM_B}(X) \rightleftarrows \mathrm{DM_h}(X, \mathbf{Q}) : \alpha_*$$

such that $\alpha^*(H_{\mathrm{B}} \otimes M) = a^*(M)$ for any object M of $\mathrm{D}_{\mathbf{A}^1}(X, \mathbf{Q})$. In particular, the functor α_* is conservative and preserves small sums: it is the composition of the functor a_* (which commutes with small sums and is conservative, as recalled in the proof of Proposition 5.2.5) and of the forgetful functor from $\mathrm{DM_B}(X)$ to $\mathrm{D}_{\mathbf{A}^1}(X, \mathbf{Q})$ (which commutes with small sums as well and is fully faithful: this readily follows from [CD12, Proposition 14.2.3 and Corollary 14.2.16]).

It is sufficient to prove that the functor α^* is fully faithful on compact objects for any noetherian scheme of finite dimension X. Indeed, if this is the case, then the class of objects M such that the unit $M \to \alpha_* \alpha^*(M)$ is invertible forms a localizing subcategory of the compactly generated triangulated category $\mathrm{DM_B}(X)$ which contains all compact objects, hence is the class of all the Beilinson motives. But then, the functor α^* is fully faithful with conservative right adjoint, hence an equivalence of categories.

It is sufficient to prove that the functor α^* is fully faithful on constructible objects when X is affine. Indeed, we have to prove that the unit map $M \to \alpha_* \alpha^*(M)$ is invertible whenever M is a compact object of $\mathrm{DM_B}(X)$. As both operations α^* and α_* commute with functors of the form j^* for any open immersion j, a simple descent argument (namely [CD12, Proposition 8.2.8 and Theorem 14.3.4 (1)]) shows that we are looking at a property which is local on X for the Zariski topology. In other words, we may assume that X is the limit of a projective system $\{X_i\}$ of schemes of finite type over $\mathbf{Z}$, with affine transition maps. Using [CD12, Proposition 15.1.6] as well as Proposition 5.2.5, we are thus reduced to prove this proposition in the case where X is of finite type over $\mathbf{Z}$, whence excellent, in which case this is already known; see [CD12, Theorem 16.1.2]. □

5.3 h-Descent for torsion étale sheaves

5.3.1. Given any noetherian scheme S and any ring R, proceeding as in Paragraph 3.1.1, there is an exact fully faithful embedding of the category $\mathrm{Sh}(S_{\text{ét}}, R)$ in the category of étale sheaves of R-modules over the big étale site of S-schemes of finite type. Composing this embedding with the h-sheafification functor leads to an exact functor

$$\alpha^* : \mathrm{Sh}(S_{\text{ét}}, R) \to \mathrm{Sh_h}(S, R), \quad F \mapsto \alpha^*(F) = F_h. \tag{5.3.1.a}$$

This functor has a right adjoint

$$\alpha_* : \mathrm{Sh_h}(S, R) \to \mathrm{Sh}(S_{\text{ét}}, R). \tag{5.3.1.b}$$

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which is defined by $\alpha_*(F) = F|_{S_{\text{ét}}}$. The functor (5.3.1.a) induces a functor

$$\alpha^* : \mathrm{D}(S_{\text{ét}}, R) \to \mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(S, R)). \tag{5.3.1.c}$$

which has a right adjoint

$$\mathbf{R}\alpha_* : \mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(S, R)) \to \mathrm{D}(S_{\text{ét}}, R). \tag{5.3.1.d}$$

Lemma 5.3.2. *For any ring R and any noetherian scheme S, the derived restriction functor (5.3.1.d) preserves small sums.*

Proof. Let us prove first the lemma in the case where S is of finite dimension and where all the residue fields of S are uniformly of finite étale cohomological dimension. Then any S-scheme of finite type has the same property; see [SGA4, Exposé X, Théorème 2.1]. Moreover, by virtue of a theorem of Goodwillie and Lichtenbaum [GL01b], any S-scheme of finite type has finite h-cohomological dimension as well. For a complex C of h-sheaves of R-modules over S, the sheaf cohomology $H^i(\mathbf{R}\alpha_*(C))$ is the étale sheaf associated with the presheaf

$$V \mapsto H^i_{\mathrm{h}}(V, C).$$

It follows from Proposition 1.1.9 that the functors $H^i_{\mathrm{h}}(V, -)$ preserve small sums, which implies that the functor $\mathbf{R}\alpha_*$ has the same property.

We now can deal with the general case as follows. Let ξ be a geometric point of S, and write $u : S_\xi \to S$ for the canonical map from the strict henselization of S at ξ. Then S_ξ is of finite dimension and its residue fields are uniformly of finite étale cohomological dimension; see Theorem 1.1.5. We then have pullback functors

$$u^* : \mathrm{D}(S_{\text{ét}}, R) \to \mathrm{D}(S_{\xi,\text{ét}}, R) \quad \text{and} \quad u^* : \mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(S, R)) \to \mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(S_\xi, R)).$$

The family of functors u^* form a conservative family of functors which commutes with sums (when ξ runs over all geometric points of S). Therefore, it is sufficient to prove that the functor $u^*\mathbf{R}\alpha_*$ commutes with sums. Let V be an affine étale scheme over S_ξ. There exists a projective system of étale S-schemes $\{V_i\}$ with affine transition maps such that $V = \varprojlim_i V_i$. Note that any S_ξ-scheme of finite type is of finite étale cohomological dimension (see Gabber's theorem 1.1.5), so that, by virtue of Lemma 1.1.12, for any complex of sheaves of R-modules K over $S_{\text{ét}}$, one has

$$\varinjlim_i H^n_{\text{ét}}(V_i, K) \simeq H^n_{\text{ét}}(V, u^*(K)).$$

Similarly, applying Lemma 1.1.12 to the h-sites, for any complex of h-sheaves of R-modules L over S, we have

$$\varinjlim_i H^n_{\mathrm{h}}(V_i, L) \simeq H^n_{\mathrm{h}}(V, u^*(L)).$$

Note that, for any étale map $w : W \to S$, the natural map $w^*\mathbf{R}\alpha_*(C) \to \mathbf{R}\alpha_* w^*(C)$ is invertible. Therefore, for any complex of h-sheaves of R-modules C over S, we have natural isomorphisms

$$\begin{aligned} H^n_{\text{ét}}(V, u^*\mathbf{R}\alpha_*(C)) &\simeq \varinjlim_i H^n_{\text{ét}}(V_i, \mathbf{R}\alpha_*(C)) \\ &\simeq \varinjlim_i H^n_{\mathrm{h}}(V_i, C) \\ &\simeq H^n_{\mathrm{h}}(V, u^*(C)) \\ &\simeq H^n_{\text{ét}}(V, \mathbf{R}\alpha_* u^*(C)). \end{aligned}$$

In other words, the natural map $u^*\mathbf{R}\alpha_* \to \mathbf{R}\alpha_* u^*$ is invertible, and, as we already know that the functor $\mathbf{R}\alpha_*$ commutes with small sums over S_ξ, this achieves the proof of the lemma. $\square$

Proposition 5.3.3. *Let R be a ring of positive characteristic, and S be a noetherian scheme. The functor (5.3.1.c) is fully faithful. In other words, for any complex C of sheaves of R-modules over $S_{\text{ét}}$, and for any morphism of finite type $f : X \to S$, the natural map*

$$H^i_{\text{ét}}(X, f^*C) \to H^i_{\mathrm{h}}(X, \alpha^*C)$$

is invertible for any integer i.

Proof. We must prove that, for any complex of sheaves of R-modules C over $S_{\text{ét}}$, the natural map

$$C \to \mathbf{R}\alpha_*\mathbf{L}\alpha^*(C)$$

is invertible in $\mathrm{D}(\mathrm{Sh}_{\mathrm{h}}(S, R))$. The functor $\mathbf{R}\alpha_*$ preserves small sums (Lemma 5.3.2). Therefore, it is sufficient restrict ourselves to the case of bounded complexes. Then, by virtue of [SGA4, Exposé Vbis, 3.3.3], it is sufficient to prove that any h-cover is a morphism of universal cohomological 1-descent (with respect to the fibered category of étale sheaves of R-modules). The h-topology is the minimal Grothendieck topology generated by open coverings as well as by coverings of shape $\{p : Y \to X\}$ with p proper and surjective; see [Voe96, 1.3.9] in the context of excellent schemes, and [Ryd10, 8.4] in general. We know that the class of morphisms of universal cohomological 1-descent form a pretopology on the category of schemes; see [SGA4, Exposé Vbis, 3.3.2]. To conclude the proof, it is thus sufficient to note that any étale surjective morphism (any proper surjective morphism, respectively) is a morphism of universal cohomological 1-descent; see [SGA4, Exposé Vbis, 4.3.5 and 4.3.2]. $\square$

5.4 Basic change of coefficients

5.4.1. Let R' be an R-algebra and S be a base scheme. We associate with R'/R the classical adjunction

$$\rho^* : \mathrm{Sh}_h(S, R) \leftrightarrows \mathrm{Sh}_h(S, R') : \rho_* \tag{5.4.1.a}$$

such that $\rho^*(F)$ is the h-sheaf associated with the presheaf $X \mapsto F(X) \otimes_R R'$. The functor ρ_* is faithful, exact and commutes with arbitrary direct sums. Note also the formula

$$\rho_*\rho^*(F) = F \otimes_R R' \tag{5.4.1.b}$$

where R' is seen as the constant h-sheaf associated with the R-module R'.

Note that the adjunction (5.4.1.a) is an adjunction of $\mathscr{S}^{\mathrm{ft}}$-premotivic abelian categories. As such, it can be derived and induces a $\mathscr{S}^{\mathrm{ft}}$-premotivic adjunction

$$\mathbf{L}\rho^* : \underline{\mathrm{DM}}_h(-, R) \rightleftarrows \underline{\mathrm{DM}}_h(-, R') : \mathbf{R}\rho_*$$

which restricts, according to Definition 5.1.3, to a premotivic adjunction

$$\mathbf{L}\rho^* : \mathrm{DM}_h(-, R) \rightleftarrows \mathrm{DM}_h(-, R') : \mathbf{R}\rho_*. \tag{5.4.1.c}$$

Recall that the stable category of h-motives over S is a localization of the derived category of symmetric Tate spectra of h-sheaves over S.[10] Here we will simply denote this category by $\mathrm{Spt}_h(S, R)$ and call its objects *spectra*. The adjunction (5.4.1.a) can be extended to an adjunction of $\mathscr{S}^{\mathrm{ft}}$-premotivic abelian categories:

$$\rho^* : \mathrm{Spt}_h(-, R) \rightleftarrows \mathrm{Spt}_h(-, R') : \rho_*. \tag{5.4.1.d}$$

[10] See [CD12], Definition 5.3.16 for symmetric Tate spectra and Definition 5.3.22 for the stable $\mathbf{A}^1$-derived category.

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Again, ρ_* is faithful, exact and commutes with arbitrary sums. Note that the model category structure on $\mathrm{Spt}_h(-,R')$ is a particular instance of a general construction (see [CD12, 7.2.1 and Theorem 7.2.2]), from which we immediately get the following useful result (which is not difficult to prove directly though).

Lemma 5.4.2. *The functor $\rho_* : \mathrm{C}(\mathrm{Spt}_h(S,R')) \to \mathrm{C}(\mathrm{Spt}_h(S,R))$ preserves and detects stable weak $\mathbf{A}^1$-equivalences.*

As a corollary, we get the following proposition.

Proposition 5.4.3. *Consider the notations of Paragraph 5.4.1. The functors $\mathbf{R}\rho_* = \rho_*$ is conservative and admits a right adjoint:*

$$\rho^! : \underline{\mathrm{DM}}_h(S,R) \to \underline{\mathrm{DM}}_h(S,R').$$

For any h-motive M over S, the following computations hold:

$$\rho_*\mathbf{L}\rho^*(M) = M \otimes^{\mathbf{L}}_R R',$$
$$\rho_*\rho^!(M) = \mathbf{R}\underline{\mathrm{Hom}}_R(R', M).$$

5.4.4. We consider the particular case of the discussion above when $R = \mathbf{Z}$ and $R' = \mathbf{Z}/n\mathbf{Z}$ for a positive integer n. For any h-motive M over S, we put

$$M/n := M \otimes^{\mathbf{L}} \mathbf{Z}/n\mathbf{Z}. \tag{5.4.4.a}$$

Then the short exact sequence

$$0 \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \longrightarrow \mathbf{Z}/n\mathbf{Z} \to 0$$

induces a canonical distinguished triangle in $\mathrm{DM}_h(S,\mathbf{Z})$:

$$M \xrightarrow{n} M \longrightarrow M/n \longrightarrow. \tag{5.4.4.b}$$

In the next statement, we will use the fact that $\underline{\mathrm{DM}}_h(S,R)$ is a dg-category (see [CD12, Remark 5.1.19]). We denote the enriched Hom by $\mathbf{R}\mathrm{Hom}$.

Proposition 5.4.5. *Consider the previous notations. Let S be a scheme and $f : X \to S$ be a morphism of* Sch*, M and N be h-motives over X. Then the natural exchange transformations,*

$$\begin{array}{lrcl} (1) & \mathbf{R}f_*(N)/n & \longrightarrow & \mathbf{R}f_*(N/n), \\ (2) & \mathbf{R}\underline{\mathrm{Hom}}(M,N)/n & \longrightarrow & \mathbf{R}\underline{\mathrm{Hom}}_{\mathbf{Z}/n\mathbf{Z}}(M/n,N/n), \\ (3) & \mathbf{R}\mathrm{Hom}(M,N)/n & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\mathbf{Z}/n\mathbf{Z}}(M/n,N/n), \end{array}$$

are isomorphisms.

Proof. In each case, this follows from the distinguished triangle (5.4.4.b); or its analog in the derived category of abelian groups. □

5.4.6. Next we consider the case of $\mathbf{Q}$-localization.

Proposition 5.4.7. *Let S be a noetherian scheme of finite dimension. Then S is of finite cohomological dimension for $\mathbf{Q}$-linear coefficients with respect to the* h*-topology. In particular, for any complex of* h*-sheaves K over S, for any S-scheme of finite type, and for any localization R of $\mathbf{Z}$, we have a canonical isomorphism*

$$H^0_{\mathrm{h}}(X,K) \otimes R \simeq H^0_{\mathrm{h}}(X, K \otimes R).$$

Proof. Any field is of cohomological dimension zero for $\mathbf{Q}$-linear coefficients with respect to the étale topology, and thus any noetherian scheme of finite dimension is of finite cohomological dimension for $\mathbf{Q}$-linear coefficients with respect to the h-topology (see [GL01b]). The last assertion of the proposition is then a direct application of Lemma 1.1.10. □

For the next corollaries, let us write simply $\underline{\mathrm{DM}}_{\mathrm{h}}(S)$ (respectively $\mathrm{DM}_{\mathrm{h}}(S)$) for $\underline{\mathrm{DM}}_{\mathrm{h}}(S, \mathbf{Z})$ (respectively $\mathrm{DM}_{\mathrm{h}}(S, \mathbf{Z})$). As an immediate corollary of the previous theorem, we get the following corollary.

Corollary 5.4.8. *Let R be a localization of $\mathbf{Z}$. For any noetherian scheme S of finite dimension, tensoring by R preserves fibrant symmetric Tate spectra. Furthermore, for any S-scheme of finite type X, and for any object M of $\underline{\mathrm{DM}}_{\mathrm{h}}(S)$, we have*

$$\mathrm{Hom}_{\underline{\mathrm{DM}}_{\mathrm{h}}(S)}(\underline{\mathbf{Z}}^{\mathrm{h}}_S(X), M) \otimes R \simeq \mathrm{Hom}_{\underline{\mathrm{DM}}(S)}(\underline{\mathbf{Z}}_S(X), M \otimes R).$$

Proof. The previous proposition shows that tensoring with R preserves the property of cohomological h-descent, while it obviously preserves the properties of being homotopy invariant and of being an Ω-spectrum. This proves the first assertion. The second one, is a direct translation of the first. □

Corollary 5.4.9. *Consider a noetherian scheme S of finite dimension and any localization R of $\mathbf{Z}$. For any objects M and N of $\mathrm{DM}_{\mathrm{h}}(S)$, if M is constructible, then*

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(S)}(M, N) \otimes R \simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(S)}(M, N \otimes R).$$

Proof. We may assume that $M = \mathbf{Z}(U)(n)$ for some smooth scheme U over S and some integer n. Replacing N by $N(-n)$, we may assume that $n = 0$, and we deduce from the preceding corollary that it is equivalent to show that the functor

$$\nu^* : \underline{\mathrm{DM}}_{\mathrm{h}}(S) \to \mathrm{DM}_{\mathrm{h}}(S)$$

commutes with R-linearization (where, for an object E of $\mathrm{DM}_{\mathrm{h}}(S)$, one defines $E \otimes R = \nu^*(\nu_\sharp(E) \otimes R)$). Let N be any object of $\underline{\mathrm{DM}}_{\mathrm{h}}(S)$, and X be a smooth separated S-scheme of finite type. Then we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(S)}(\mathbf{Z}(X), \nu^*(N) \otimes R) &\simeq \mathrm{Hom}_{\underline{\mathrm{DM}}_{\mathrm{h}}(S)}(\mathbf{Z}(X), \nu_\sharp(\nu^*(N)) \otimes R) \\
&\simeq \mathrm{Hom}_{\underline{\mathrm{DM}}_{\mathrm{h}}(S)}(\mathbf{Z}(X), \nu_\sharp(\nu^*(N))) \otimes R \\
&\simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(S)}(\mathbf{Z}(X), \nu^*(N)) \otimes R \\
&\simeq \mathrm{Hom}_{\underline{\mathrm{DM}}_{\mathrm{h}}(S)}(\mathbf{Z}(X), N) \otimes R \\
&\simeq \mathrm{Hom}_{\underline{\mathrm{DM}}_{\mathrm{h}}(S,R)}(R(X), N \otimes R) \\
&\simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(S,R)}(R(X), \nu^*(N \otimes R)) \\
&\simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(S)}(\mathbf{Z}(X), \nu^*(N \otimes R)).
\end{aligned}$$

As both functors $\nu_\sharp$ and ν^* preserve Tate twists, this implies that the canonical map $\nu^*(N) \otimes R \to \nu^*(N \otimes R)$ is invertible for any N. □

Remark 5.4.10. This corollary says in particular that the category $\mathrm{DM}_{\mathrm{h},c}(S, R \otimes \mathbf{Q})$ of constructible h-motives with $R \otimes \mathbf{Q}$-coefficients is the pseudo-abelian envelope of the *naive* $\mathbf{Q}$-localization of the triangulated category $\mathrm{DM}_{\mathrm{h},c}(S, R)$. This is not an obvious fact as the category $\mathrm{DM}_{\mathrm{h}}(S, R)$ is not compactly generated for general base schemes S and ring of

https://doi.org/10.1112/S0010437X15007459 Published online by Cambridge University Press

coefficients R. To find examples, it is sufficient to know that the unbounded derived category $\mathrm{D}(S_{\text{ét}}, R)$ may not be compactly generated. Indeed, it is easy to see that if $\mathrm{DM}_h(S, \mathbf{Z})$ is compactly generated, then so is $\mathrm{DM}_{\mathrm{h}}(S, R)$ for any ring of coefficients R. For a noetherian scheme S and any prime number ℓ which is invertible in $\mathscr{O}_S$, we will see later that $\mathrm{DM}_{\mathrm{h}}(S, \mathbf{Z}/\ell\mathbf{Z})$ is canonically equivalent to $\mathrm{D}(S_{\text{ét}}, \mathbf{Z}/\ell\mathbf{Z})$ (see Corollary 5.5.4 below). Therefore, if the unbounded derived category $\mathrm{D}(S_{\text{ét}}, \mathbf{Z}/\ell\mathbf{Z})$, of sheaves of $\mathbf{Z}/\ell\mathbf{Z}$-modules on the small étale site of S, is not compactly generated for some ℓ as above, then $\mathrm{DM}_{\mathrm{h}}(S, \mathbf{Z})$ is not compactly generated. This may happen if S is the spectrum of a field with non-discrete absolute Galois group, and with infinite ℓ-cohomological dimension.

Even worse, it may happen that the category $\mathrm{DM}_{\mathrm{h}}(X, R)$ is compactly generated while $\mathrm{DM}_{\mathrm{h},c}(X, R)$ contains objects which are not compact. For instance, this is the case for $X = \operatorname{Spec}(\mathbf{R})$: the constant h-motive $\mathbf{Z}$ is not compact in $\mathrm{DM}_h(\operatorname{Spec}(\mathbf{R}), \mathbf{Z})$. Indeed, if this were the case, then its reduction modulo 2 would be a compact object as well, and, in particular, the constant motive $\mathbf{Z}/2\mathbf{Z}$ would be compact in the category $\mathrm{DM}_h(\operatorname{Spec}(\mathbf{R}), \mathbf{Z}/2\mathbf{Z})$. But the latter is nothing else than $\mathrm{D}(\operatorname{Spec}(\mathbf{R})_{\text{ét}}, \mathbf{Z}/2\mathbf{Z})$, which, in turns is the unbounded derived category of the category of $\mathbf{Z}/2\mathbf{Z}$-linear representations of the group with two elements $G = \mathrm{Gal}(\mathbf{C}/\mathbf{R})$. It is well known that the cohomology of the group G with $\mathbf{Z}/2\mathbf{Z}$-coefficients is non-trivial in infinitely many degrees. On the other hand, for any ring of coefficients R, the unbounded derived category $\mathrm{D}(G, R)$ of the category of R-linear right representations of G is compactly generated: a generating family of compact objects is given by the single representation $R(G)$ (obtained as the free R-module on the underlying set of G, the action being induced by right translations). The functor

$$\mathbf{R}\mathrm{Hom}(R(G), -) : \mathrm{D}(G, R) \to \mathrm{D}(R)$$

is canonically isomorphic to the functor which consists to forget the action of G. Therefore, the complex of R-modules $\mathbf{R}\mathrm{Hom}(M, R)$ is perfect for any compact object M of $\mathrm{D}(G, R)$. But $\mathbf{R}\mathrm{Hom}(R, R)$ is the complex which computes the cohomology of the group G with coefficients in R, so that it cannot be perfect for $R = \mathbf{Z}/2\mathbf{Z}$.

As a corollary, we get the following analog of Proposition 5.4.5.

Corollary 5.4.11. *Let S be a noetherian scheme of finite dimension, and $f : X \to S$ be a morphism of finite type, M and N be* h*-motives with R-coefficients over X, with M constructible. Then the natural exchange transformations below are isomorphisms:*

$$\begin{array}{lrcl}
(1) & \mathbf{R}f_*(N) \otimes \mathbf{Q} & \longrightarrow & \mathbf{R}f_*(N \otimes \mathbf{Q}), \\
(2) & \mathbf{R}\underline{\mathrm{Hom}}_R(M, N) \otimes \mathbf{Q} & \longrightarrow & \mathbf{R}\underline{\mathrm{Hom}}_{R\otimes\mathbf{Q}}(M \otimes \mathbf{Q}, N \otimes \mathbf{Q}), \\
(3) & \mathbf{R}\mathrm{Hom}_R(M, N) \otimes \mathbf{Q} & \longrightarrow & \mathbf{R}\mathrm{Hom}_{R\otimes\mathbf{Q}}(M \otimes \mathbf{Q}, N \otimes \mathbf{Q}).
\end{array}$$

Proof. To prove (1), it is sufficient to check this after applying the functor $\mathbf{R}\mathrm{Hom}_R(P, -)$, when P runs over a generating family of constructible objects. In particular, we may assume that $P = R \otimes^{\mathbf{L}} U$ for some constructible object U of $\mathrm{DM}_{\mathrm{h}}(S, \mathbf{Z})$, in which case we have $\mathbf{R}\mathrm{Hom}_R(P, -) = \mathbf{R}\mathrm{Hom}_{\mathbf{Z}}(U, -)$. for any constructible h-motive P with coefficients in R. Then the result follows from Corollary 5.4.9. Similarly, to prove (3), it is sufficient to consider the case where M is the R-linearization of a constructible object of $\mathrm{DM}_{\mathrm{h}}(X)$, and we conclude again with Corollary 5.4.9. It is easy to see that (3) implies (2). $\square$

As a notable application of the results proved so far, we get the following proposition.

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Proposition 5.4.12. *Let $\mathscr{P}$ be the set of prime integers and S be a noetherian scheme of finite dimension. If R is flat over $\mathbf{Z}$, then the family of change of coefficients functors,*

$$\rho^* : \mathrm{DM}_h(S,R) \to \mathrm{DM}_h(S, R \otimes \mathbf{Q}),$$
$$\rho_p^* : \mathrm{DM}_h(S,R) \to \mathrm{DM}_h(S, R/p), \quad p \in \mathscr{P},$$

defined above is conservative.

Proof. Let K be an h-motive over S with coefficients in R such that $\rho^*(K) = 0$ and $\rho_p^*(K) = 0$ for all $p \in \mathscr{P}$.

It is sufficient to prove that for any constructible h-motive M, $\mathrm{Hom}(M,K) = 0$. Given any prime p, the fact $\rho_p^*(K) = 0$ together with the distinguished triangle (5.4.4.b) implies that the abelian group $\mathrm{Hom}(M,K)$ is uniquely p-divisible. As this is true for any prime p, we get: $\mathrm{Hom}(M,K) = \mathrm{Hom}(M,K) \otimes \mathbf{Q}$. But, as M is constructible, Corollary 5.4.9 implies the later group is isomorphic to $\mathrm{Hom}(\rho^*(M), \rho^*(K))$ which is zero by assumption on K. □

5.5 Comparison with étales motives

5.5.1. Recall Λ is a sub-ring of $\mathbf{Q}$ and R is a Λ-algebra. As it appears already in Paragraph 2.1.1, finite S-correspondences with coefficients in Λ are defined for separated S-schemes of finite type. According to [CD12, Definition 9.1.8], they define a category which we will denote by $\mathscr{S}^{\mathrm{cor}}_{\Lambda,S}$.

Given any S-scheme X, we denote by $\underline{R}^{\mathrm{tr}}_S(X)$ the presheaf of R-modules on $\mathscr{S}^{\mathrm{cor}}_{\Lambda,S}$ represented by X. Moreover the graph functor induces a canonical morphism of presheaves on $\mathscr{S}^{\mathrm{ft}}_S$:

$$\underline{R}_S(X) \to \underline{R}^{\mathrm{tr}}_S(X). \tag{5.5.1.a}$$

Recall the following result of Suslin and Voevodsky (see [VSF00, ch. 2, 4.2.7 and 4.2.12]).

Proposition 5.5.2. *The map (5.5.1.a) induces an isomorphism after* h*-sheafification. Furthermore, if S is a noetherian $\mathbf{Z}[1/n]$-scheme and if any integer prime to n is invertible in R, then, for any S-scheme X of finite type, the presheaf $\underline{R}^{\mathrm{tr}}_S(X)$ is a* qfh*-sheaf, and the morphism (5.5.1.a) induces an isomorphism of* qfh*-sheaves:*

$$\underline{R}^{qfh}_S(X) \to \underline{R}^{\mathrm{tr}}_S(X).$$

This implies in particular that any h-sheaf F over S defines by restriction an étale sheaf with transfers $\psi^*(F)$, on $\mathrm{Sm}^{\mathrm{cor}}_S$ (without any restriction on the characteristic). This gives a canonical functor

$$\psi^* : \mathrm{Sh}_{\mathrm{h}}(S,R) \to \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{\acute{e}t}}(S,R)$$

which preserves small limits as well as small filtering colimits. Using the argument of the proof of [CD12, Theorem 10.5.14], one can show this functor admits a left adjoint $\psi_!$ uniquely defined by the property that $\psi_!(R^{\mathrm{tr}}_S(X)) = \underline{R}^{\mathrm{h}}_S(X)$ for any smooth S-scheme X.

Thus, we have defined an adjunction of abelian premotivic categories over Sch:

$$\psi_! : \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{\acute{e}t}}(-,R) \rightleftarrows \mathrm{Sh}_{\mathrm{h}}(-,R) : \psi^*. \tag{5.5.2.a}$$

According to [CD12, 5.2.19], these functors can be derived and induce an adjunction of premotivic categories over Sch:

$$\mathbf{L}\psi_! : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(-,R) \rightleftarrows \underline{\mathrm{DM}}^{\mathrm{eff}}_{\mathrm{h}}(-,R) : \mathbf{R}\psi^*.$$

As a consequence of the rigidity Theorem 4.5.2 and of the cohomological h-descent property for étale topology Proposition 5.3.3, we get the following theorem.

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Theorem 5.5.3. *Assume that the ring R is of positive characteristic. For any noetherian scheme S, the functor $\mathbf{L}\psi_! : \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(S,R) \to \underline{\mathrm{DM}}^{\mathrm{eff}}_{\mathrm{h}}(S,R)$ is fully faithful and induces an equivalence of triangulated categories*

$$\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(S,R) \xrightarrow{\sim} \mathrm{DM}^{\mathrm{eff}}_{\mathrm{h}}(S,R) \xrightarrow{\sim} \mathrm{DM}_{\mathrm{h}}(S,R).$$

Proof. The equivalence $\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(S,R) \simeq \mathrm{DM}^{\mathrm{eff}}_{\mathrm{h}}(S,R)$ follows from the first assertion: the essential image of $\mathbf{L}\psi_!$ is obviously included in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{h}}(S,R)$ because $\mathbf{L}\psi_!(R^{\mathrm{tr}}_S(X)) = \underline{R}^{\mathrm{h}}_S(X)$ for any smooth S-scheme. Let n be the characteristic of R. As R is a $\mathbf{Z}/n\mathbf{Z}$-algebra, to prove that the functor $\mathbf{L}\psi_!$ is fully faithful, it is sufficient to consider the case where $R = \mathbf{Z}/n\mathbf{Z}$. Decomposing n into its prime factors, we are thus reduced to prove that $\mathbf{L}\psi_!$ is fully faithful in the case where $n = p^a$ with p a prime and $a \geqslant 1$. Furthermore, by virtue of Proposition A.3.4, we may assume that n is invertible in the residue fields of S. In this case, we know that the composite functor

$$\tilde{\rho}_! : \mathrm{D}(S_{\text{ét}},R) \xrightarrow{\rho_!} \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(S,R) \xrightarrow{\mathbf{L}\psi_!} \underline{\mathrm{DM}}^{\mathrm{eff}}_{\mathrm{h}}(S,R)$$

is fully faithful (Proposition 5.3.3) and that the functor $\rho_!$ is an equivalence of categories (by the rigidity Theorem 4.5.2). This obviously implies that the functor $\mathbf{L}\psi_!$ is fully faithful.

For the last equivalence, we simply notice that, for any ring of positive characteristic R, the premotivic triangulated category $\underline{\mathrm{DM}}^{\mathrm{eff}}_{\mathrm{h}}(S,R)$ satisfies the stability property with respect to the Tate object $R(1)$, so that we get a canonical equivalence of categories

$$\underline{\mathrm{DM}}^{\mathrm{eff}}_{\mathrm{h}}(S,R) \simeq \underline{\mathrm{DM}}_{\mathrm{h}}(S,R).$$

This induce an equivalence of categories $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{h}}(S,R) \simeq \mathrm{DM}_{\mathrm{h}}(S,R)$. □

Using the preceding theorem, together with Theorem 4.5.2, we finally get the following theorem.

Corollary 5.5.4. *Assume R is a ring of positive characteristic n. Then for any noetherian scheme X, with n invertible in the residue fields of X, there are canonical equivalences of triangulated monoidal categories*

$$D(X_{\text{ét}},R) \simeq \mathrm{DM}_{\mathrm{h}}(X,R).$$

These equivalences of categories are functorial in the precise sense that they induce an equivalence of premotivic triangulated categories over the category of $\mathbf{Z}[1/n]$-schemes:

$$D((-)_{\text{ét}},R) \simeq \mathrm{DM}_{\mathrm{h}}(-,R).$$

Finally, if R is noetherian, these equivalences induce fully faithful monoidal triangulated functors

$$\mathrm{DM}_{\mathrm{h},c}(X,R) \to \mathrm{D}^b_{\mathrm{ctf}}(X_{\text{ét}},R).$$

Proof. The only thing that remains to be checked is the last assertion (when R is noetherian). To prove that the object C of $\mathrm{D}(X_{\text{ét}},R)$ corresponding to some constructible object M of $\mathrm{DM}_{\mathrm{h}}(X,R)$ belongs to $\mathrm{D}^b_{\mathrm{ctf}}(X_{\text{ét}},R)$, it is sufficient to consider the case of $M = f_*(R)$ with $f : Y \to X$ projective; see [Ayo07, Lemma 2.2.23]. The fact that such an object belongs to $\mathrm{D}^b_{\mathrm{ctf}}(X_{\text{ét}},R)$ is well known; see [SGA4$\frac{1}{2}$, Rapport, Theorem 4.9], for instance. □

Combining Theorem 5.5.3 together with the comparison theorems of [CD12, Theorem 16.1.2, 16.1.4], one gets the following generalization of [VSF00, ch. 5, 4.1.12].

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Corollary 5.5.5 (R is any commutative ring). (1) *Let S be a quasi-excellent geometrically unibranch noetherian scheme of finite dimension.*

Then the adjunction (5.5.2.a) induces an equivalence of triangulated monoidal categories:

$$\mathbf{L}\psi_! : \mathrm{DM}_{\text{ét}}(S,R) \rightleftarrows \mathrm{DM}_{\mathrm{h}}(S,R) : \mathbf{R}\psi^*.$$

(2) *Let k be any field. Then the following composite functor*

$$\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(k,R) \xrightarrow{\Sigma^\infty} \mathrm{DM}_{\text{ét}}(k,R) \xrightarrow{\mathbf{L}\psi_!} \mathrm{DM}_{\mathrm{h}}(k,R)$$

is fully faithful.

Proof. Consider point (1). By definition, $\mathrm{DM}_{\mathrm{h}}(S,R)$ is exactly the image of $\mathbf{L}\psi_!$ in $\underline{\mathrm{DM}}_h(S,R)$. Thus we have only to prove that $\mathbf{L}\psi_!$ is fully faithful.

Taking any étale motive M in $\mathrm{DM}_{\text{ét}}(S,R)$, we prove that the canonical adjunction map

$$M \to \mathbf{R}\psi^*\,\mathbf{L}\psi_!(M)$$

is an isomorphism in $\mathrm{DM}_{\text{ét}}(S,R)$. Applying Proposition 5.4.12, it is sufficient to prove that the image of this map is an isomorphism after applying one of the functor ρ^* or ρ_p^* for a prime p.

Note the functors of the type ρ^* ($\mathbf{Q}$-localization of the coefficients) and ρ_p^* (reduction modulo p of the coefficients) are also defined for the triangulated category $\mathrm{DM}_{\text{ét}}(S,R)$ (see [CD12, 10.5.a]). According to the preceding theorem (respectively to [CD12, Theorems 16.1.2 and 16.1.4]), it is sufficient to prove that the functor ρ_p^* (respectively ρ^*) commutes with $\mathbf{L}\psi_!$ and $\mathbf{R}\psi^*$.

This last assertion, in the case of ρ_p^*, follows easily using the distinguished triangle (5.4.4.b); and its analog version in $\mathrm{DM}_{\text{ét}}(-,R)$. In the case of ρ^*, it follows as in the proof of Corollary 5.4.11 from Corollary 5.4.9 and its analog in $\mathrm{DM}_{\text{ét}}(S,R)$; the proof is the same using in particular Proposition 2.2.3.

Consider point (2). We have to show that for any object K of $\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(k,R)$, the adjunction map

$$\alpha : K \to \Sigma^\infty\Omega^\infty(K)$$

is an isomorphism. Let us denote abusively by ρ_p^* (respectively ρ^*) the change of coefficients functors

$$\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(k,R) \xrightarrow{\rho_p^*} \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(k,R/p), \mathrm{DM}_{\text{ét}}(k,R) \xrightarrow{\rho_p^*} \mathrm{DM}_{\text{ét}}(k,R/p),$$
$$(\text{respectively } \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(k,R) \xrightarrow{\rho^*} \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(k,R_{\mathbf{Q}}), \mathrm{DM}_{\text{ét}}(k,R) \xrightarrow{\rho^*} \mathrm{DM}_{\text{ét}}(k,R_{\mathbf{Q}})).$$

As for point (1), it is sufficient to check that the map α is an isomorphism after applying ρ_p^* of ρ^*; by the obvious analog of Proposition 5.4.12.

The case of the functor ρ_p^* is easily reduced to Corollary 4.1.2.

Next, we consider the case of the functor ρ^*. We can see that the functors Σ^∞ and Ω^∞ commute with tensor product by $\mathbf{Q}$: for the first one, this is obvious, while for Ω^∞, this follows from the fact that tensoring by $\mathbf{Q}$ preserves the properties of being $\mathbf{A}^1$-homotopy invariant, of satisfying étale decent, and of being an Ω-spectrum (which readily follows from the Yoneda lemma and from a repeated use of Proposition 1.1.11). Using the same arguments as in the end of point (1), we deduce that ρ^* commutes with Ω^∞. The case of the functor Σ^∞ is obvious. Thus, we are finally reduced to the case where R is a $\mathbf{Q}$-algebra. Then, for any inseparable

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extension of fields k'/k, the associated pullback functor defines an equivalence of categories $\mathrm{DM}^{\mathrm{eff}}_{\acute{e}t}(k,R) \simeq \mathrm{DM}^{\mathrm{eff}}_{\acute{e}t}(k',R)$. Therefore, it is sufficient to consider the case of a perfect field. Furthermore, as the $\mathbf{Q}$-linear categories of Nisnevich sheaves with transfers and of étale sheaves with transfers are equivalent, we have canonical equivalences of triangulated categories

$$\mathrm{DM}^{\mathrm{eff}}(k,R) \simeq \mathrm{DM}^{\mathrm{eff}}_{\acute{e}t}(k,R) \quad \text{and} \quad \mathrm{DM}(k,R) \simeq \mathrm{DM}_{\acute{e}t}(k,R).$$

We easily conclude with Voevodsky's cancellation theorem. □

5.5.6. Recall from [CD12, 5.3.31] the triangulated category

$$\mathrm{D}_{\mathbf{A}^1,\acute{e}t}(X,R) = \mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_{\acute{e}t}(X,R))$$

obtained as the stabilization of the $\mathbf{A}^1$-derived category of étale sheaves on the smooth-étale site of X. The category $\mathrm{D}_{\mathbf{A}^1,\acute{e}t}(X,R)$ is taken in Ayoub's paper [Ayo14] as a model for étale motives.

Corollary 5.5.7. *Let X be a noetherian scheme of finite dimension. We also assume that, either X is of characteristic zero or that 2 is invertible in R. Then the canonical functor*

$$\mathrm{D}_{\mathbf{A}^1,\acute{e}t}(X,R) \to \mathrm{DM}_{\mathrm{h}}(X,R)$$

is an equivalence of triangulated categories (and is part of an equivalence of premotivic triangulated categories as we let X vary).

Proof. We only sketch the proof. We see that it is sufficient to consider the cases where $R = \mathbf{Q}$ or $R = \mathbf{Z}/p\mathbf{Z}$, with p a prime. The case where $R = \mathbf{Q}$ is already known: this follows right away from Theorem 5.2.2 and from [CD12, Theorem 16.2.18]. The case of torsion coefficients follows from the fact that we may assume that p is prime to the residue characteristics of X (by Proposition A.3.4), and that we have a commutative diagram of the form

$$\begin{tikzcd} & \mathrm{D}(X_{\acute{e}t},\mathbf{Z}/p\mathbf{Z}) \arrow[dl] \arrow[dr] & \\ \mathrm{D}_{\mathbf{A}^1,\acute{e}t}(X,\mathbf{Z}/p\mathbf{Z}) \arrow[rr] & & \mathrm{DM}_{\mathrm{h}}(X,\mathbf{Z}/p\mathbf{Z}) \end{tikzcd}$$

in which the non-horizontal functors are equivalences of categories (see [Ayo14, Theorem 4.1] and the preceding corollary, respectively). □

Remark 5.5.8. If the reader believes [Ayo14, Theorem 4.1], she or he can drop the constraint that 'X is of characteristic zero or that 2 is invertible in R' in the statement of Corollary 5.5.7. The reason why we put this extra assumption is that the proof of Ayoub's result [Ayo14, Theorem 4.1] used above relies on the fact that the 2-functor $\mathrm{D}_{\mathbf{A}^1,\acute{e}t}(-,R)$ is separated (this is [Ayo14, Theorem 3.9]). On the other hand, the proof of [Ayo14, Theorem 3.9] relies on the assumption that a certain property $(\mathbf{SS})_p$ (see [Ayo14, p. 7]) is satisfied by $\mathrm{D}_{\mathbf{A}^1,\acute{e}t}(X,R)$ whenever p is a prime number which is not invertible in R (and invertible in $\mathscr{O}_X$). In the case where 'X is of characteristic zero or that 2 is invertible in R', this property $(\mathbf{SS})_p$ is provided by [Ayo14, Theorem 2.8], whose proof we understand. If X not of characteristic zero and if $p = 2$, the property $(\mathbf{SS})_p$ is discussed in [Ayo14, Appendix C]. The problem (at least for us) is that we think the proof of [Ayo14, Theorem C.1] is incomplete. To be more precise, what is presented as a proof of [Ayo14, Lemma C.9] is far from being convincing: it consists to make the reader believe (without even an heuristic explanation) that a large amount of constructions and computations done by Morel over a perfect field are meaningful for an arbitrary base field (Morel makes this perfectness assumption pervasively for the simple but essential reason that he needs to know that field extensions of finite type have smooth models).

On the other hand, it is very plausible that Ayoub's property $(\mathbf{SS})_p$ is true in full generality. In fact, it can be derived from [Ayo14, Theorem 4.1], and the main difficulty to prove the latter consists to justify that we have a canonical isomorphism $\mathbf{Z}/\ell\mathbf{Z}(1) \simeq \mu_\ell$ in $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(X, \mathbf{Z}/\ell\mathbf{Z})$ for any prime ℓ invertible in $\mathscr{O}_X$ (one may then essentially reproduce the proof of Theorem 4.5.2, or, even more easily, prove that the triangulated categories $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(X, R)$ and $\mathrm{DM}_{\text{ét}}(X, R)$ are canonically equivalent for any ring of positive characteristic R, and then use Theorem 4.5.2). It is easy to see that the case where X is the spectrum of a (perfect, or even prime) field is sufficient, and the establishment of such an isomorphism $\mathbf{Z}/\ell\mathbf{Z}(1) \simeq \mu_\ell$ is then one of the main points in the work of Morel on the Friedlander–Milnor conjecture; see [Mor11, Corollary 4.12]. Therefore, Morel's work should justify that the results of Ayoub's paper are all true with the claimed level of generality, and thus that Corollary 5.5.7 is true without any assumption on the ring of coefficients.

Remark 5.5.9. Once we are able to compare $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(X, R)$ and $\mathrm{DM}_{\mathrm{h}}(X, R)$ as in Corollary 5.5.7, we can use Corollary 5.5.5 to compare $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(X, R)$ and $\mathrm{DM}_{\text{ét}}(X, R)$. The equivalence

$$\mathrm{D}_{\mathbf{A}^1,\text{ét}}(X, R) \simeq \mathrm{DM}_{\text{ét}}(X, R)$$

is also proved by Ayoub in [Ayo14, Theorem B.1] under the assumption that any prime number is invertible in $\mathscr{O}_X$ or in R, and that X is normal and universally Japanese (and requiring that X is of characteristic zero or that 2 is invertible in R, because his proof relies again on the validity of [Ayo14, Theorem 3.9]: see the preceding remark). The main point in the proof of [Ayo14, Theorem B.1] consists to reduce to the case where R is a $\mathbf{Q}$-algebra (in which case this is a variant of [CD12, Theorems 16.1.2 and 16.1.4]) and to the case where R is of positive characteristic n, with n invertible in $\mathscr{O}_X$. In the latter case, Ayoub proves that we have an equivalence for normal schemes (combining [Ayo14, Proposition B.13 and Lemma B.15]), but this is far from being optimal: for torsion coefficients, combining [Ayo14, Theorem 4.1], Theorem 4.5.2 and Proposition A.3.4, we have an equivalence of triangulated categories $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(X, R) \simeq \mathrm{DM}_{\text{ét}}(X, R)$ for any noetherian (and possibly non-normal) scheme X of finite dimension for any ring R of positive characteristic (with the constraint that X is of characteristic zero or that 2 is invertible in R, for the reason explained in Remark 5.5.8).

Proposition 5.5.10. *Let $f : X \to Y$ a morphism between noetherian schemes of finite dimension. Assume that, either f is of finite type, or that X is the projective limit of a projective system of quasi-finite Y-schemes with affine transition maps. Then the functor*

$$\mathbf{R}f_* : \mathrm{DM}_h(X, R) \to \mathrm{DM}_{\mathrm{h}}(Y, R)$$

preserves small sums. In particular, this functor has a right adjoint. In the case where f is proper, we will denote by $f^!$ the right adjoint to $\mathbf{R}f_*$.

Proof. As the forgetful functors $\mathrm{DM}_{\mathrm{h}}(X, R) \to \mathrm{DM}_{\mathrm{h}}(X, \mathbf{Z})$ are conservatives and commute with operations of type $\mathbf{R}f_*$, it is sufficient to prove this for $R = \mathbf{Z}$. Hence, using Proposition 5.4.5 and Corollary 5.4.11, we see that it is sufficient to prove the result in the case where $R = \mathbf{Q}$ or $R = \mathbf{Z}/p\mathbf{Z}$ for some prime p. For $R = \mathbf{Q}$ and any noetherian scheme of finite dimension S, the triangulated category $\mathrm{DM}_{\mathrm{h}}(S, \mathbf{Q})$ is compactly generated and the functor $\mathbf{L}f^*$ preserves compact objects (this follows from Theorem 5.2.4 with $R = \mathbf{Q}$, which makes sense thanks to

Lemma 1.1.4). and this implies the claim. For $R = \mathbf{Z}/p\mathbf{Z}$, if p is invertible in the residue fields of Y, we conclude with Corollary 1.1.15 and Theorem 5.5.3. The general case follows from Proposition A.3.4. The existence of a right adjoint of $\mathbf{R}f_*$ is a direct consequence of the Brown representability theorem. □

Remark 5.5.11. Note that a sufficient condition for a triangulated functor between triangulated categories to preserve compact objects is that it has a right adjoint which preserves small sums. The preceding proposition implies that, for any morphism $f : X \to Y$ between noetherian scheme of finite dimension, the functor $\mathbf{L}f^*$ preserves compact objects in $\mathrm{DM}_{\mathrm{h}}(-, R)$. Therefore, one can interpret the last part of Remark 5.4.10 as follows: for any noetherian scheme X of finite dimension which admits a real point, if 2 is not invertible in R, then the constant motive R_X is not compact in $\mathrm{DM}_{\mathrm{h}}(X, R)$.

COROLLARY 5.5.12. *Let $f : X \to Y$ be a morphism between noetherian schemes of finite dimension. For any object M of $\mathrm{DM}_{\mathrm{h}}(X, R)$ and any R-algebra R', there is a canonical isomorphism*

$$R' \otimes^{\mathbf{L}}_R \mathbf{R}f_*(M) \to \mathbf{R}f_*(R' \otimes^{\mathbf{L}}_R M).$$

Proof. Given a complex of R-modules C, we still denote by C the object of $\mathrm{DM}_{\mathrm{h}}(X, R)$ defined as the free Tate spectrum associated with the constant sheaf of complexes C. This defines a left Quillen functor from the projective model category on the category of complexes of R-modules (with quasi-isomorphisms as weak equivalences, and degree-wise surjective maps as fibrations) to the model category of Tate spectra. Therefore, we have a triangulated functor

$$\mathrm{D}(R\text{-Mod}) \to \mathrm{DM}_{\mathrm{h}}(S, R), \quad C \mapsto C$$

which preserves small sums and is symmetric monoidal. By virtue of the preceding proposition, for any fixed M, we thus have a natural transformation between triangulated functors which preserve small sums:

$$C \otimes^{\mathbf{L}}_R \mathbf{R}f_*(M) \to \mathbf{R}f_*(C \otimes^{\mathbf{L}}_R M).$$

To prove that the map above is an isomorphism for any complex of R-modules C, as the derived category of R is compactly generated by R (seen as a complex concentrated in degree zero), it is sufficient to consider the case where $C = R$, which is trivial. □

COROLLARY 5.5.13. *Let X be a noetherian scheme of finite dimension. Then, for any constructible motive M in $\mathrm{DM}_{\mathrm{h}}(X, R)$, the functor $\underline{\mathrm{Hom}}_R(M, -)$ preserves small sums. Furthermore, for any R-algebra R', we have canonical isomorphisms*

$$\mathbf{R}\underline{\mathrm{Hom}}_R(M, N) \otimes^{\mathbf{L}}_R R' \simeq \mathbf{R}\underline{\mathrm{Hom}}_R(M, N \otimes^{\mathbf{L}}_R R')$$

for any object N in $\mathrm{DM}_{\mathrm{h}}(X, R)$.

Proof. It is sufficient to prove this in the case where M is of the form $M = \mathbf{L}f_\sharp(\mathbb{1}_Y)$ for a separated smooth morphism of finite type $f : Y \to X$. But then, we have

$$\mathbf{R}\underline{\mathrm{Hom}}_R(M, N) \simeq \mathbf{R}f_* f^*(N).$$

This corollary is thus a reformulation of Proposition 5.5.10 and Corollary 5.5.12. □

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Corollary 5.5.14. *For any separated morphism of finite type $f : X \to Y$ between noetherian schemes of finite dimension, the functor*

$$f^! : \mathrm{DM}_{\mathrm{h}}(Y, R) \to \mathrm{DM}_{\mathrm{h}}(X, R)$$

preserves small sums, and, for any R-algebra R', there is a canonical isomorphism

$$f^!(M) \otimes^{\mathbf{L}}_R R' \simeq f^!(M \otimes^{\mathbf{L}}_R R').$$

Proof. For any constructible object C in $\mathrm{DM}_{\mathrm{h}}(X, R)$, we have

$$\mathbf{R}f_*\mathbf{R}\underline{\mathrm{Hom}}_R(C, f^!(M)) \simeq \mathbf{R}\underline{\mathrm{Hom}}_R(f_!(C), M).$$

Using that the functor $f_!$ preserves constructible objects (see [CD12, Corollary 4.2.12]), we deduce from Proposition 5.5.10 and Corollary 5.5.13 the following computation, for any small family of objects M_i in $\mathrm{DM}_{\mathrm{h}}(Y, R)$:

$$\begin{aligned}
\mathrm{Hom}\Big(C, \bigoplus_i f^!(M_i)\Big) &\simeq \mathrm{Hom}\Big(\mathbb{1}_Y, \mathbf{R}\underline{\mathrm{Hom}}_R\Big(C, \bigoplus_i f^!(M_i)\Big)\Big) \\
&\simeq \mathrm{Hom}\Big(\mathbb{1}_Y, \bigoplus_i \mathbf{R}\underline{\mathrm{Hom}}_R\Big(C, \bigoplus_i f^!(M_i)\Big)\Big) \\
&\simeq \mathrm{Hom}\Big(\mathbb{1}_X, \mathbf{R}f_* \bigoplus_i \mathbf{R}\underline{\mathrm{Hom}}_R(C, f^!(M_i))\Big) \\
&\simeq \mathrm{Hom}\Big(\mathbb{1}_Y, \bigoplus_i \mathbf{R}f_*\mathbf{R}\underline{\mathrm{Hom}}_R(C, f^!(M_i))\Big) \\
&\simeq \mathrm{Hom}\Big(\mathbb{1}_Y, \bigoplus_i \mathbf{R}\underline{\mathrm{Hom}}_R(f_!(C), M_i)\Big) \\
&\simeq \mathrm{Hom}\Big(\mathbb{1}_Y, \mathbf{R}\underline{\mathrm{Hom}}_R\Big(f_!(C), \bigoplus_i M_i\Big)\Big) \\
&\simeq \mathrm{Hom}\Big(f^!(C), \bigoplus_i M_i\Big) \\
&\simeq \mathrm{Hom}\Big(C, f^!\Big(\bigoplus_i M_i\Big)\Big).
\end{aligned}$$

The change of coefficients formula is proved similarly (or with the same argument as in the proof of Corollary 5.5.12). □

5.6 h-Motives and Grothendieck's six functors

5.6.1. Let R be any commutative ring. Recall from [Voe96, Theorem 4.2.5] that we get a canonical isomorphism in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{h}}(S, R)$:

$$\mathbb{1}_S(1) \simeq R \otimes^{\mathbf{L}} \mathbf{G}_m[-1]$$

where $\mathbf{G}_m$ is identified with the h-sheaf of abelian groups over S represented by the scheme $\mathbf{G}_m$.

This gives a canonical morphism of groups

$$\begin{aligned}
c_1 : \mathrm{Pic}(S) = H^1_{\mathrm{Zar}}(S, \mathbf{G}_m) &\to \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{h}}(S,R)}(\mathbb{1}_S, \mathbb{1}_S(1)[2]) \\
&\to \mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(S,R)}(\mathbb{1}_S, \mathbb{1}_S(1)[2])
\end{aligned}$$

so that the premotivic triangulated category $\mathrm{DM}_h(S, R)$ is oriented in the sense of Definition A.1.5.

Moreover, as a corollary of the results obtained above, we get the following theorem.

THEOREM 5.6.2. *The triangulated premotivic category* $\mathrm{DM}_h(-,R)$ *satisfies the formalism of the Grothendieck six functors for noetherian schemes of finite dimension (Definition A.1.10) as well as the absolute purity property (Definition A.2.9).*

Proof. Taking into account Corollaries 5.5.12–5.5.14, we see that we may assume $R = \mathbf{Z}$ at will.

Consider the first assertion. Taking into account Theorem A.1.13, we have only to prove the localization property for $\mathrm{DM}_\mathrm{h}(-,R)$. Fix a closed immersion $i : Z \to S$. The analog of Proposition 2.3.4 for the h-topology obviously holds. This means we have to prove that for any smooth S-scheme X, if $R_S(X/X - X_Z)$ denotes the (infinite suspension) of the quotient of representable h-sheaves $R_S(X)/R_S(X - Z)$, then the canonical map

$$R_S(X/X - X_Z) \to i_*R_Z(X_Z)$$

is an isomorphism in $\mathrm{DM}_\mathrm{h}(X,R)$. According to Proposition 5.4.12, together with Proposition 5.4.5 and Corollary 5.4.11, we are reduced to check this when $R = \mathbf{Q}$ or $R = \mathbf{Z}/p\mathbf{Z}$. In the first case, it follows from Theorem 5.2.2 and the localization property for Beilinson motives DM_B. The latter property is part of the statement of [CD12, Corollary 14.2.11]. In the second case, it follows from Theorem 5.5.3 and Theorem 4.3.1.

Concerning the second assertion, the absolute purity for $\mathrm{DM}_\mathrm{h}(-,\mathbf{Z})$, we use the same argument as in the proof of Theorem 4.6.1: using Theorem A.2.8, we can apply Proposition 5.4.12, together with Proposition 5.4.5 and Corollary 5.4.11 to reduced to the case where $R = \mathbf{Q}$ or $R = \mathbf{Z}/p\mathbf{Z}$. The first case follows from Theorem 5.2.2 and [CD12, Theorem 14.4.1]; the second one follows from Theorems 5.5.3 and 4.6.1. $\square$

6. Finiteness theorems

6.1 Transfers and traces

6.1.1 (Transfers). Consider the notations of Paragraph 5.5.1. Let X and Y be proper S-schemes and $\alpha \in c_S(X,Y)_\Lambda$ a finite S-correspondence. According to Proposition 5.5.2, we get a morphism of h-sheaves on $\mathscr{S}_S^{\mathrm{ft}}$

$$\alpha_* : \underline{R}_S(X) \to \underline{R}_S(Y) \tag{6.1.1.a}$$

which induces a morphism in $\underline{\mathrm{DM}}_\mathrm{h}(S,R)$:

$$\alpha_* : \Sigma^\infty \underline{R}_S(X) \to \Sigma^\infty \underline{R}_S(Y).$$

Let p and q be the respective structural morphisms of the S-schemes X and Y. Applying the functor $\underline{\mathrm{Hom}}(-,\mathbb{1}_S)$ to this map, we get a morphism in $\underline{\mathrm{DM}}_\mathrm{h}(S,R)$:

$$\alpha^* : q_*(\mathbb{1}_X) \to p_*(\mathbb{1}_Y).$$

Then we can apply to this functor the right adjoint ν^* of the adjunction (5.1.4.a) and, because it commutes with p_* and q_* and we have the isomorphism $\nu^*\mathbb{1} = \mathbb{1}$, the above morphism can be seen in $\mathrm{DM}_\mathrm{h}(S,R)$.

Given moreover any h-motive E over S, and using the projection formula, cf. Definition A.1.10, (2) and (5), applied to the proper morphisms p and q, we obtain finally a canonical morphism

$$q_*q^*(E) = q_*(\mathbb{1}_X) \otimes E \xrightarrow{\alpha^* \otimes \mathrm{Id}_E} p_*(\mathbb{1}_Y) \otimes E = p_*p^*(E)$$

which is natural in E.

DEFINITION 6.1.2. Consider the notations above. The following natural transformation of endofunctors of $\mathrm{DM_h}(S,R)$

$$\alpha^\star : q_*q^* \to p_*p^* \tag{6.1.2.a}$$

is called the cohomological h-transfer along the finite S-correspondence α.

The following results are easily derived from this definition.

PROPOSITION 6.1.3. *Consider the above definition.*

(1) Normalization. *Consider a commutative diagram of schemes*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

such that p and q are proper. Let α be the finite S-correspondence associated with the graph of f. Then the natural transformation $\alpha^\star$ is equal to the composite,

$$q_*q^* \xrightarrow{ad(f^*,f_*)} q_*f_*f^*q^* \simeq p_*p^*.$$

(2) Composition. *For composable finite S-correspondences $\alpha \in c_S(X,Y)_\Lambda$, $\beta \in c_S(Y,Z)_\Lambda$ with X, Y, Z proper over S, one has $\alpha^\star\beta^\star = (\beta\circ\alpha)^\star$.*

(3) Base change. *Let $f : T \to S$ be a morphism of schemes, $\alpha \in c_S(X,Y)_\Lambda$ a finite S-correspondence between proper S-schemes and put $\alpha_T = f^*(\alpha)$ obtained using the premotivic structure on $\mathscr{S}^{\mathrm{cor}}_\Lambda$. Let p (respectively q, p', q') be the structural morphism of X/S (respectively Y/S, $X\times_S T/T$, $Y\times_S T/T$), $f' = f\times_S T$. Then the following diagram commutes*

$$\begin{tikzcd} f^*q_*q^* \arrow[rrr, "f^*.\alpha^\star"] \arrow[d, "\mathrm{Ex}(f^*,q_*)"', "\sim"] & & & f^*p_*p^* \arrow[d, "\sim"', "\mathrm{Ex}(f^*,p_*)"] \\ q'_*f'^*q^* \arrow[r, equal] & q'_*q'^* \arrow[r, "\alpha_T^\star"] & p'_*p'^* \arrow[r, equal] & p'_*p'^*q^* \end{tikzcd}$$

where the vertical maps are the proper base change isomorphisms: Definition A.1.10(4).

(4) Restriction. *Let $\pi : S \to T$ be a proper morphism of schemes. Consider a finite S-correspondence $\alpha \in c_S(X,Y)_\Lambda$ between proper schemes and put $\alpha|_T = \pi_\sharp(\alpha)$ using the $\mathscr{S}^{\mathrm{ft}}$-premotivic structure on $\mathscr{S}^{\mathrm{cor}}_{\Lambda_S}$. Let p (respectively q) be the structural morphism of X/S (respectively Y/S), and put $p' = \pi\circ p$, $q' = \pi\circ q$. Then the following diagram is commutative.*

$$\begin{tikzcd} \pi_*q_*q^*\pi^* \arrow[r, "\pi_*.\alpha^\star.\pi^*"] \arrow[d, equal] & \pi_*p_*p^*\pi^* \arrow[d, equal] \\ q'_*q'^* \arrow[r, "(\alpha|_T)^\star"] & p'_*p'^* \end{tikzcd}$$

Proof. Properties (1) and (2) are clear as they are obviously true for the morphism α_* of (6.1.1.a).

Similarly, property (3) (respectively (4)) follows from the fact the morphism (5.5.1.a) is compatible with the functor f^* (respectively the functor $\pi_\sharp$). This boils down to the fact that the graph functor[11] $\gamma : \mathscr{S}^{\mathrm{ft}} \to \mathscr{S}^{\mathrm{cor}}_\Lambda$ is a morphism of $\mathscr{S}^{\mathrm{ft}}$-fibered category: see [CD12, 9.4.1]. □

[11] Recall that it is the identity on objects and it associates with a morphism of separated S-schemes of finite type its S-graph seen as a finite S-correspondence.

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6.1.4. Let $f : Y \to X$ be a morphism of schemes. Recall we say that f is Λ-universal if the fundamental cycle associated with Y is Λ-universal over X (Def. [CD12, 8.1.48]).

Let us denote by ${}^t f$ the cycle associated with the graph of f over X seen as a subscheme of $X \times_X Y$. Then, by the very definition, the following conditions are equivalent.

(i) The morphism f is finite Λ-universal.
(ii) The cycle ${}^t f$ is a finite X-correspondence from X to Y.

For matching the existing literature, we introduce, the following definition, redundant with the previous one.

Definition 6.1.5. Let $f : Y \to X$ be a finite Λ-universal morphism of schemes. Using the preceding notations, we define the *trace of* f as the natural transformation of endofunctors of $\mathrm{DM}_{\mathrm{h}}(X, R)$:

$$\mathrm{Tr}_f := ({}^t f)^\star : f_* f^* \to \mathrm{Id}\,.$$

Remark 6.1.6. We will say that a morphism of schemes is pseudo-dominant if it sends any generic point to a generic point. Recall that a finite Λ-universal $f : Y \to X$ is in particular pseudo-dominant.

Let us recall the following example of finite Λ-universal morphisms of schemes:

(1) finite flat;
(2) finite pseudo-dominant morphisms whose aim is regular;
(3) finite pseudo-dominant morphisms whose aim is geometrically unibranch and has residue fields whose exponential characteristic is invertible in Λ.

6.1.7. One readily obtain from Proposition 6.1.3 that our trace maps are compatible with composition.

Recall that given a finite Λ-universal morphism $f : Y \to X$ and a generic point x of X, we can define an integer $\deg_x(f)$, the degree of f at x, by choosing any generic point y of Y such that $f(y) = x$ and putting

$$\deg_x(f) := [\kappa(y) : \kappa(x)]$$

(see [CD12, 9.1.13]). We will say that f has *constant degree* d if for any generic point $x \in X$, $\deg_x(f) = d$.

Applying Proposition 6.1.3 to the particular case of traces, one gets the following formulas.

Proposition 6.1.8. *Consider the above definition.*

(1) Normalization. *Let $f : Y \to X$ be a finite étale morphism. Then the following diagram commutes*

$$\begin{tikzcd} f_* f^* \arrow[r, "\mathrm{Tr}_f"] \arrow[d, "\alpha_f \cdot \mathfrak{p}'_f"', "\sim"] & \mathrm{Id} \\ f_! f^! \arrow[ru, "ad(f_!, f^!)"'] & \end{tikzcd}$$

where α_f and $\mathfrak{p}'_f$ are the isomorphisms from Definition A.1.10(2),(3).

(2) Composition. *Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be finite Λ-universal morphisms. Then the following diagram commutes.*

$$\begin{tikzcd} f_* g_* g^* f^* \arrow[r, "f_* \mathrm{Tr}_g . f^*"] \arrow[d, equal] & f_* f^* \arrow[r, "\mathrm{Tr}_f"] & \mathrm{Id} \arrow[d, equal] \\ (fg)_*(fg)^* \arrow[rr, "\mathrm{Tr}_{fg}"] & & \mathrm{Id} \end{tikzcd}$$

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(3) Base change. *Consider a pullback square of schemes*

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "\pi'"'] & X' \arrow[d, "\pi"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

such that f is a finite flat morphism. Then, the following diagram is commutative

$$\begin{tikzcd} \pi^* f_* f^* \arrow[rr, "\pi^*.\,\mathrm{Tr}_f"] \arrow[d, "\mathrm{Ex}(\pi^*,p_*)"', "\sim"] & & \pi^* \arrow[d, equal] \\ f'_*\pi'^* f^* \arrow[r, equal] & f'_* f'^* \pi^* \arrow[r, "\mathrm{Tr}_{f'}.\pi^*"] & \pi^* \end{tikzcd}$$

where the left vertical map is the proper base change isomorphism.

(4) Degree formula. *Let $f : Y \to X$ be a finite Λ-universal morphism of constant degree d, the following composite*

$$f_* f^* \xrightarrow{\mathrm{Tr}_f} \mathrm{Id} \xrightarrow{ad(f^*,f_*)} f_* f^*$$

is equal to $d \cdot \mathrm{Id}$.

Proof. Point (1) follows from the fact that, in the category $\mathrm{Sh}_{\mathrm{h}}(S,R)$, the representable sheaf $\underline{R}_X(Y)$ is strongly dualizable with itself as a dual and with duality pairings:

$$\underline{R}_X(Y) \otimes \underline{R}_X(Y) = \underline{R}_X(Y \times_X Y) \xrightarrow{({}^t\delta)_*} \underline{R}_X(Y) \xrightarrow{f_*} \underline{R}_X(X)$$
$$\underline{R}_X(X) \xrightarrow{({}^tf)_*} \underline{R}_X(Y) \xrightarrow{\delta_*} \underline{R}_X(Y \times_X Y) = \underline{R}_X(Y) \otimes \underline{R}_X(Y).$$

where δ is diagonal embedding (which is open and closed).

Point (2) is obtained from Proposition 6.1.3, properties (2) and (4). Point (3) is a special case of Proposition 6.1.3(3), given the fact that $\pi^*({}^tf) = {}^tf'$ as f is flat; see property (P3) of the tensor product of relative cycles in [CD12, Paragraph 8.1.34]. Point (4) follows from Proposition 6.1.3(1), (2) and the formula of [CD12, Proposition 9.1.13]. □

Remark 6.1.9. According to Corollary 5.5.4, this notion of trace generalizes the one introduced in [SGA4, XVII, § 6.2] in the case of finite morphisms, taking into account Remark 6.1.6.

Let us consider the more general case of a quasi-finite separated morphism $f : Y \to X$. According to the theorem of Nagata [Con07], there exists a factorization, $f = \bar{f} \circ j$, such that $\bar{f}$ is proper, thus finite according to Zariski's main theorem, and j is an open immersion.

We will say that f is *strongly* Λ-*universal* if there exists such a factorization such that in addition $\bar{f}$ is Λ-universal.[12]

In this condition, one checks easily using Proposition 6.1.8, properties (1) and (2), that the following composite is independent of the chosen factorization of f:

$$\mathrm{Tr}_f : f_! f^* = \bar{f}_! j_! j^* \bar{f}^* \xrightarrow{\bar{f}_!.ad(j_!,j^*).\bar{f}^*} f_! f^* = f_* f^* \xrightarrow{\mathrm{Tr}_{\bar{f}}} \mathrm{Id}\,. \tag{6.1.9.a}$$

This composition is called the trace of f

[12] This implies in particular that f is Λ-universal according to [CD12, Corollary 8.2.6]. The converse is not true.

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Properties (1), (2), (3) of the preceding proposition immediately extend to this notion of trace.

However, this construction is not optimal as it is not clear that a flat quasi-finite separated morphism is strongly Λ-universal.

In particular, it only partially generalizes the construction of [SGA4, Theorem 6.2.3] when $R = \mathbf{Z}/n\mathbf{Z}$ and X has residual characteristics prime to n. However, in the case where X is geometrically unibranch, and has residual characteristics prime to n, any quasi-finite separated pseudo-dominant morphism is strongly Λ-universal (cf. Remark 6.1.6). Thus, in this case, our notion does generalize the finer notion of trace introduced in [SGA4, 6.2.5, 6.2.6].

6.2 Constructible h-motives

In this subsection, devoted to the study of constructible h-motives (5.1.3), we will simplify the notations by dropping the symbols $\mathbf{L}$ and $\mathbf{R}$; in other words, by default, all the functors will be the derived ones. We will prove the main theorems about constructible h-motives: their stability by the six operations (Theorem 6.2.13 and its corollary) and the duality theorem (Theorem 6.2.17).

6.2.1. Let S be a noetherian scheme. For any prime ideal $\mathfrak{p}$ of $\mathbf{Z}$, we have a fully faithful functor

$$(\mathrm{DM}_{\mathrm{h},c}(S,\mathbf{Z})_{\mathfrak{p}})^{\sharp} \to (\mathrm{DM}_{\mathrm{h}}(S,\mathbf{Z})_{\mathfrak{p}})^{\sharp}, \tag{6.2.1.a}$$

where, for a triangulated category T, $T^{\sharp}$ denotes its idempotent completion and $T_{\mathfrak{p}}$ its $\mathbf{Z}_{\mathfrak{p}}$-linearization; see Appendix B.

Definition 6.2.2. An object M of $\mathrm{DM}_{\mathrm{h}}(S,\mathbf{Z})$ will be called $\mathfrak{p}$-*constructible* if its image in $(\mathrm{DM}_{\mathrm{h}}(S,\mathbf{Z})_{\mathfrak{p}})^{\sharp}$ lies in the essential image of the functor (6.2.1.a).

Let us state explicitly the proposition that we will use below.

Proposition 6.2.3. *Let S be a noetherian scheme and M be an object of $\mathrm{DM}_{\mathrm{h}}(S,\mathbf{Z})$. Then the following conditions are equivalent.*

(i) *The* h*-motive M is constructible.*

(ii) *For any maximal ideal $\mathfrak{p} \in \mathrm{Spec}(\mathbf{Z})$, M is $\mathfrak{p}$-constructible.*

Proof. We just apply the abstract Proposition B.1.7 (from the Appendix A) to the $\mathbf{Z}$-linear category $T = \mathrm{DM}_{\mathrm{h}}(S,\mathbf{Z})$ and its thick subcategory $U = \mathrm{DM}_{\mathrm{h},c}(S,\mathbf{Z})$. □

Proposition 6.2.4. *Let p be a prime number and X a noetherian scheme of characteristic p. An object M of $\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Z})$ is (p)-constructible if and only if it is (0)-constructible.*

Proof. The Artin–Schreier short exact sequence (see the proof of Proposition A.3.1) implies that the category $\mathrm{DM}_{\mathrm{h}}(S,\mathbf{Z})$ is $\mathbf{Z}[1/p]$-linear, so that we have

$$\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Z})_{(p)} = \mathrm{DM}_{\mathrm{h}}(X,\mathbf{Z}) \otimes \mathbf{Q},$$

and similarly for $\mathrm{DM}_{\mathrm{h},c}(X,\mathbf{Z})$. □

Remark 6.2.5. When $\mathfrak{p} = (0)$, the functor $\rho_{\mathfrak{p}}^*$ which appears in this corollary coincide on constructible objects with the functor ρ^* of Paragraph 5.4.1 in the case $R = \mathbf{Z}$ and $R' = \mathbf{Q}$ (this is the meaning of Corollary 5.4.9).

The proof of the stability of constructible h-motives by direct image (Theorem 6.2.13), which is based on an argument of Gabber, is intricate. We divide it with the help of the following two results. The first one is due to Ayoub.

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Proposition 6.2.6 (Ayoub). *Let X be a noetherian scheme. The category $\mathrm{DM}_{\mathrm{h},c}(X,R)$ is the smallest thick triangulated subcategory of the triangulated category $\mathrm{DM}_{\mathrm{h}}(X,R)$ which contains the objects of the form $f_*(R_{X'}(n))$ where $f : X' \to X$ is a projective morphism and $n \in \mathbf{Z}$.*

In fact, if X is a noetherian schemes having an ample family of line bundles, this is [Ayo07, Lemma 2.2.23] but it is easy to check that this assumption is not used in the proof of [Ayo07, Lemma 2.2.23].

The second result used in the proof of the forthcoming Theorem 6.2.13 is a variation on an argument of Gabber, used in the étale torsion case (see [ILO14, XIII, § 3]).

Lemma 6.2.7 (Gabber's lemma). *Let X be a quasi-excellent noetherian scheme, and $\mathfrak{p}$ a prime ideal of $\mathbf{Z}$. Assume that, for any point x of X, the exponent characteristic of the residue field $\kappa(x)$ is not in $\mathfrak{p}$. Then, for any dense open immersion $j : U \to X$, the* h*-motive $j_*(\mathbb{1}_U)$ is $\mathfrak{p}$-constructible.*

Proof. We will use the following geometrical consequence of the *local uniformization theorem prime to* $\mathfrak{p}$ of Gabber (see [ILO14, VII, 1.1 and IX, 1.1]).

Lemma 6.2.8. *Let $j : U \to X$ be a dense open immersion such that X is reduced and quasi-excellent, and $\mathfrak{p}$ a prime ideal of $\mathbf{Z}$. Assume that, for any point x of X, the exponent characteristic of the residue field $\kappa(x)$ is not in $\mathfrak{p}$. Then, there exists the following data:*

(i) *a finite* h*-cover $\{f_i : Y_i \to X\}_{i\in I}$ such that for all i in I, f_i is a morphism of finite type, the scheme Y_i is regular, and $f_i^{-1}(U)$ is either Y_i itself or the complement of a strict normal crossing divisor in Y_i (we shall write*

$$f : Y = \coprod_{i\in I} Y_i \to X$$

for the induced global h-cover);

(ii) *a commutative diagram*

$$\begin{tikzcd} X''' \arrow[rr, "g"] \arrow[d, "q"'] & & Y \arrow[d, "f"] \\ X'' \arrow[r, "u"] & X' \arrow[r, "p"] & X \end{tikzcd} \tag{6.2.8.a}$$

in which p is a proper birational morphism, u is a Nisnevich cover, and q is a flat finite surjective morphism of degree not in $\mathfrak{p}$.

Let T (respectively T') be a closed subscheme of X (respectively X') and assume that for any irreducible component T_0 of T, the following inequality is satisfied:

$$\mathrm{codim}_{X'}(T') \geqslant \mathrm{codim}_X(T_0).$$

Then, possibly after shrinking X in an open neighborhood of the generic points of T in X, one can replace X'' by an open cover and X''' by its pullback along this cover, in such a way that we have in addition the following properties.

(iii) *One has the inclusion $p(T') \subset T$ and the induced map $T' \to T$ is finite and sends any generic point to a generic point.*

(iv) *If we write $T'' = u^{-1}(T')$, the induced map $T'' \to T'$ is an isomorphism.*

Points (i) and (ii) are proved in [ILO14, Exp. XIII, part 3.2.1]. Then points (iii) and (iv) are proved in [CD12, proof of Lemma 4.2.14].

6.2.9. We introduce the following notations: for any scheme Y, we let $\mathscr{T}_0(Y)$ be the subcategory of $\mathrm{DM}_{\mathrm{h}}(Y,\mathbf{Z})$ made of $\mathfrak{p}$-constructible objects K. Then $\mathscr{T}_0$ becomes a fibered subcategory of $\mathrm{DM}_{\mathrm{h}}(-,\mathbf{Z})$ and we can moreover check the following properties.

(a) For any scheme Y in Sch, $\mathscr{T}_0(Y)$ is a triangulated thick subcategory of the triangulated category $\mathrm{DM}_{\mathrm{h}}(Y,\mathbf{Z})$ which contains the objects of the form $\mathbb{1}_Y(n)$, $n\in\mathbf{Z}$.

(b) For any separated morphism of finite type $f: Y'\to Y$ in Sch, $\mathscr{T}_0$ is stable under $f_!$.

(c) For any dense open immersion $j: V\to Y$, with Y regular, which is the complement of a strict normal crossing divisor, $j_*(\mathbb{1}_V)$ is in $\mathscr{T}_0(V)$.

Indeed, (a) is obvious, (b) follows from the fact the functor $f_!$ preserves constructible motives, while (c) comes from the absolute purity property for $\mathrm{DM}_{\mathrm{h}}(-,\mathbf{Z})$; see Theorem 5.6.2. With this notation, we have to prove that $j_*(\mathbb{1}_U)$ is in $\mathscr{T}_0$.

We now return to the proof of Lemma 6.2.7. Following the argument of [ILO14, XIII, 3.1.3], we may assume that X is reduced, and it is sufficient to prove by induction on $c\geqslant 0$ that here exists a closed subscheme $T\subset X$ of codimension greater than c such that the restriction of $j_*(\mathbb{1}_U)$ to $(X-T)$ is in $\mathscr{T}_0$.

Indeed, if this is the case, let us chose a closed subset T_c of X satisfying the condition above with respect to an arbitrary integer $c\geqslant 0$. As X is noetherian, we get that X is covered by the family of open subschemes $(X-T_c)$ indexed by $c\geqslant 0$. Moreover, X is quasi-compact so that only a finite number of these open subschemes are sufficient to cover X. Thus we can conclude that $j_*(\mathbb{1}_U)$ is in $\mathscr{T}_0$ iteratively using the Mayer–Vietoris exact triangle and property (a) of Paragraph 6.2.9.

The case where $c=0$ is clear: we can choose T such that $(X-T)=U$. If $c>0$, we choose a closed subscheme T of X, of codimension greater than $c-1$, such that the restriction of $j_*(\mathbb{1}_U)$ to $(X-T)$ is in $\mathscr{T}_0$. It is then sufficient to find a dense open subscheme V of X, which contains all the generic points of T, and such that the restriction of $j_*(\mathbb{1}_U)$ to V is in $\mathscr{T}_0$: for such a V, we shall obtain that the restriction of $j_*(\mathbb{1}_U)$ to $V\cup(X-T)$ is in $\mathscr{T}_0$, the complement of $V\cup(X-T)$ being the support of a closed subscheme of codimension greater than c in X. In particular, using the smooth base change isomorphism (for open immersions), we can always replace X by a generic neighborhood of T. It is sufficient to prove that, possibly after shrinking X as above, the pullback of $j_*(\mathbb{1}_U)$ along $T\to X$ is in $\mathscr{T}_0$ (as we already know that its restriction to $(X-T)$ is in $\mathscr{T}_0$).

We may assume that T is purely of codimension c. We may assume that we have data as in points (i) and (ii) of Lemma 6.2.8. We let $j': U'\to X'$ denote the pullback of j along $p: X'\to X$. Then, we can find, by induction on c, a closed subscheme T' in X', of codimension greater than $c-1$, such that the restriction of $j'_*(\mathbb{1}_{U'})$ to $(X'-T')$ is in $\mathscr{T}_0$. By shrinking X, we may assume that conditions (iii) and (iv) of Lemma 6.2.8 are fulfilled as well.

Given any morphism $i: Z\to W$ of X-schemes, we consider the following commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "i"] \arrow[dr, "\pi"'] & W \arrow[d] & W_U \arrow[l, "j_W"'] \arrow[d] \\
& X & U \arrow[l, "j"']
\end{tikzcd}$$

where the right-hand square is cartesian, and we define the following h-motive of $\mathrm{DM}_{\mathrm{h}}(X,R)$:

$$\varphi(W,Z) := \pi_*\, i^*\, j_{W,*}(\mathbb{1}_{W_U}).$$

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This notation is slightly abusive but it will most of the time be used when i is the immersion of a closed subscheme. This construction is contravariantly functorial: given any commutative diagram of X-schemes

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, "i'"'] & Z \arrow[d, "i"] \\ W' \arrow[r] & W \end{tikzcd}$$

we get a natural map $\varphi(W,Z) \to \varphi(W',Z')$. Remember that we want to prove that $\varphi(X,T)$ is in $\mathscr{T}_0$. This will be done via the following lemmas (which hold assuming all the conditions stated in Lemma 6.2.8 as well as our inductive assumptions).

Lemma 6.2.10. *The cone of the map $\varphi(X,T) \to \varphi(X',T')$ is in $\mathscr{T}_0$.*

The map $\varphi(X,T) \to \varphi(X',T')$ factors as

$$\varphi(X,T) \to \varphi(X',p^{-1}(T)) \to \varphi(X',T').$$

By the octahedral axiom, it is sufficient to prove that each of these two maps has a cone in $\mathscr{T}_0$.

We shall prove first that the cone of the map $\varphi(X',p^{-1}(T)) \to \varphi(X',T')$ is in $\mathscr{T}_0$. Given an immersion $a : S \to X'$, we shall write

$$M_S = a_!\, a^*(M).$$

We then have distinguished triangles

$$M_{p^{-1}(T)-T'} \to M_{p^{-1}(T)} \to M_{T'} \to M_{p^{-1}(T)-T'}[1].$$

For $M = j'_*(\mathbb{1}_{U'})$ (recall j' is the pullback of j along p) the image of this triangle by p_* gives a distinguished triangle

$$p_*(M_{p^{-1}(T)-T'}) \to \varphi(X',p^{-1}(T)) \to \varphi(X',T') \to p_*(M_{p^{-1}(T)-T'})[1].$$

As the restriction of $M = j'_*(\mathbb{1}_{U'})$ to $X' - T'$ is in $\mathscr{T}_0$ by assumption on T', the object $M_{p^{-1}(T)-T'}$ is in $\mathscr{T}_0$ as well (by property (b) of Paragraph 6.2.9), from which we deduce that $p_*(M_{p^{-1}(T)-T'})$ is in $\mathscr{T}_0$ (using the condition (iii) of Lemma 6.2.8 and property (b) of Paragraph 6.2.9).

Let V be a dense open subscheme of X such that $p^{-1}(V) \to V$ is an isomorphism. We may assume that $V \subset U$, and write $i : Z \to U$ for the complement closed immersion. Let $p_U : U' = p^{-1}(U) \to U$ be the pullback of p along j, and let $\bar{Z}$ be the reduced closure of Z in X. We thus get the commutative squares of immersions below

$$\begin{tikzcd} Z \arrow[r, "k"] \arrow[d, "i"'] & \bar{Z} \arrow[d, "l"] \\ U \arrow[r, "j"'] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Z' \arrow[r, "k'"] \arrow[d, "i'"'] & \bar{Z}' \arrow[d, "l'"] \\ U' \arrow[r, "j'"'] & X' \end{tikzcd}$$

where the square on the right is obtained from the one on the left by pulling back along $p : X' \to X$. Recall that the triangulated motivic category $\mathrm{DM}_{\mathrm{h}}(-,\mathbf{Z})$ satisfies cdh-descent (see [CD12, Proposition 3.3.10]). Thus, as p is an isomorphism over V, we get the homotopy cartesian square below.

$$\begin{tikzcd} \mathbb{1}_U \arrow[r] \arrow[d] & p_{U,*}(\mathbb{1}_{U'}) \arrow[d] \\ i_*\, i^*(\mathbb{1}_Z) \arrow[r] & i_*\, i^*\, p_{U,*}(\mathbb{1}_{U'}) \end{tikzcd}$$

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If $a : T \to X$ denotes the inclusion, applying the functor $a_* \, a^* \, j_*$ to the commutative square above, we see from the proper base change formula and from the identification $j_* \, i_* \simeq l_* \, k_*$ that we get a commutative square isomorphic to the following one

$$\begin{tikzcd} \varphi(X,T) \arrow[r] \arrow[d] & \varphi(X',p^{-1}(T)) \arrow[d] \\ \varphi(\bar{Z},\bar{Z}\cap T) \arrow[r] & \varphi(\bar{Z}',p^{-1}(\bar{Z}\cap T)) \end{tikzcd}$$

which is thus homotopy cartesian as well. It is sufficient to prove that the two objects $\varphi(\bar{Z},\bar{Z}\cap T)$ and $\varphi(\bar{Z}',p^{-1}(\bar{Z}\cap T))$ are in $\mathscr{T}_0$. It follows from the proper base change formula that the object $\varphi(\bar{Z},\bar{Z}\cap T)$ is canonically isomorphic to the restriction to T of $l_* \, k_*(\mathbb{1}_Z)$. As $\dim \bar{Z} < \dim X$, we know that the object $k_*(\mathbb{1}_Z)$ is in $\mathscr{T}_0$. By property (b) of Paragraph 6.2.9, we obtain that $\varphi(\bar{Z},\bar{Z}\cap T)$ is in $\mathscr{T}_0$. Similarly, the object $\varphi(\bar{Z}',p^{-1}(\bar{Z}\cap T))$ is canonically isomorphic to the restriction of $p_* \, l'_* \, k'_*(\mathbb{1}_{Z'})$ to T, and, as $\dim \bar{Z}' < \dim X'$ (because, p being an isomorphism over the dense open subscheme V of X, $\bar{Z}'$ does not contain any generic point of X'), $k'_*(\mathbb{1}_{Z'})$ is in $\mathscr{T}_0$. We deduce again from property (b) of Paragraph 6.2.9 that $\varphi(\bar{Z}',p^{-1}(\bar{Z}\cap T))$ is in $\mathscr{T}_0$ as well, which achieves the proof of the lemma.

Lemma 6.2.11. *The map $\varphi(X',T') \to \varphi(X'',T'')$ is an isomorphism in $\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Z})$.*

Condition (iv) of Lemma 6.2.8 can be reformulated by saying that we have the Nisnevich distinguished square below.

$$\begin{tikzcd} X''-T'' \arrow[r] \arrow[d] & X'' \arrow[d, "v"] \\ X'-T' \arrow[r] & X' \end{tikzcd}$$

This lemma follows then by Nisnevich excision [CD12, 3.3.4] and smooth base change (for étale maps).

In the next lemma, we call $\mathfrak{p}$-*quasi-section* of a morphism $f : K \to L$ in $\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Z})$ any morphism $s : L \to K$ such that there exists an integer n, not in $\mathfrak{p}$, and such that $f \circ s = n \cdot \mathrm{Id}$.

Lemma 6.2.12. *Let T''' be the pullback of T'' along the finite surjective morphism $X''' \to X''$. The map $\varphi(X'',T'') \to \varphi(X''',T''')$ admits a $\mathfrak{p}$-quasi-section.*

We have the following pullback squares

$$\begin{tikzcd} T''' \arrow[r, "t"] \arrow[d, "r"'] & X''' \arrow[d, "q"] & U''' \arrow[l, "j'''"'] \arrow[d, "q_U"] \\ T'' \arrow[r, "s"] & X'' & U' \arrow[l, "j''"'] \end{tikzcd}$$

in which j'' and j''' denote the pullback of j along pu and puq respectively, while s and t are the inclusions. By the proper base change formula applied to the left-hand square, we see that the map $\varphi(X'',T'') \to \varphi(X''',T''')$ is isomorphic to the image of the map

$$j''_*(\mathbb{1}_{U''}) \to q_* \, q^* \, j''_*(\mathbb{1}_{U''}) \to q_* \, j'''_*(\mathbb{1}_{U'''})$$

by $f_* \, s^*$, where $f : T'' \to T$ is the map induced by p (note that f is proper as $T'' \simeq T'$ by assumption). As $q_* \, j'''_* \simeq j''_* \, q_{U,*}$, we are thus reduced to prove that the unit map

$$\mathbb{1}_{U''} \to q_{U,*}(\mathbb{1}_{U'''})$$

admits a p-quasi-section. By property (iii) of Lemma 6.2.8, q_U is a flat finite surjective morphism of degree n not in $\mathfrak{p}$. Thus the $\mathfrak{p}$-quasi-section is given by the trace map (Definition 6.1.5) associated with q_U, taking into account the *degree formula* of Proposition 6.1.8.

Now, we can finish the proof of Lemma 6.2.7. Let us apply the functoriality of the construction φ with respect to the following commutative squares

$$\begin{tikzcd} T''' \arrow[r, equal] \arrow[d, "t"'] & T''' \arrow[r] \arrow[d, "a"'] & T \arrow[d] \\ X''' \arrow[r, "g"] & Y \arrow[r, "f"] & X \end{tikzcd}$$

where $T''' = q^{-1}u^{-1}(T')$, t is the natural map and $a = g \circ t$, we get the following commutative diagram of $\mathrm{DM}_{\mathrm{h}}(X, \mathbf{Z})$.

$$\begin{tikzcd} \varphi(X,T) \arrow[rr, "(1)"] \arrow[dr] & & \varphi(X''',T''') \\ & \varphi(Y,T''') \arrow[ur] & \end{tikzcd}$$

We consider the image of that diagram through the functor

$$\bar{\rho} : \mathrm{DM}_{\mathrm{h}}(X, \mathbf{Z}) \to (\mathrm{DM}_{\mathrm{h}}(X, \mathbf{Z}) / \mathrm{DM}_{\mathrm{h},c}(X, \mathbf{Z})) \to (\mathrm{DM}_{\mathrm{h}}(X, \mathbf{Z}) / \mathrm{DM}_{\mathrm{h},c}(X, \mathbf{Z}))_{\mathfrak{p}}.$$

By virtue of Proposition B.1.7, we have to show that the image of $\varphi(X,T)$ under $\bar{\rho}$ is 0. According to Lemmas 6.2.10, 6.2.12, and 6.2.11, the image of (1) under $\bar{\rho}$ is a split monomorphism. Thus it is sufficient to prove that this image is the zero map, and according to the commutativity of the above diagram, this will follow if we prove that $\bar{\rho}(\varphi(Y,T''')) = 0$, which amounts to prove that $\varphi(Y,T''')$ is $\mathfrak{p}$-constructible.

We come back to the definition of $\varphi(Y,T''')$: considering the following commutative diagram

$$\begin{tikzcd} T''' \arrow[r, "a"] \arrow[dr, "\pi"'] & Y \arrow[d, "f"] & Y_U \arrow[l, "j_Y"'] \arrow[d] \\ & X & U \arrow[l, "j"'] \end{tikzcd}$$

we have $\varphi(Y,T''') = \pi_* \, a^* \, j_{Y,*}(\mathbb{1}_{Y_U})$. By assumption, the morphism π is finite; this follows more precisely from the following conditions of Lemma 6.2.8: (ii) saying that q is finite, (iii) and (iv). Thus by assumption on j_Y (see point (i) of Lemma 6.2.8), we obtain that $\varphi(Y,T''')$ is $\mathfrak{p}$-constructible, according to properties (b) and (c) stated in Paragraph 6.2.9. This achieves the proof of Gabber's Lemma 6.2.7. □

Theorem 6.2.13. *Let $f : Y \to X$ be a morphism of finite type such that X is a quasi-excellent noetherian scheme of finite dimension. Then for any constructible* h*-motive K of* $\mathrm{DM}_{\mathrm{h}}(Y,R)$, *$f_*(K)$ is constructible in* $\mathrm{DM}_{\mathrm{h}}(X,R)$.

Proof. The case where f is proper is already known from [CD12, Proposition 4.2.11]. Then, a well-known argument allows to reduce to prove that for any dense open immersion $j : U \to X$, the h-motive $j_*(R_U)$ is constructible. Indeed, assume this is known. We want to prove that $f_*(K)$ is constructible whenever K is constructible. According to Proposition 6.2.6, and because f_* commutes with Tate twists, it is sufficient to consider the case $K = \mathbb{1}_Y$. Moreover, we easily conclude from Corollary 5.5.12 that we may assume that $R = \mathbf{Z}$. Then, as this property is assumed to be known for dense open immersions, by an easy Mayer–Vietoris argument, we see

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that the condition that $f_*(\mathbb{1}_Y)$ is constructible is local on Y and X with respect to the Zariski topology. Therefore, we may assume that X and Y are affine, thus f is affine [EGA2, (1.6.2)] and in particular quasi-projective [EGA2, (5.3.4)]: it can be factored as $f = \bar{f} \circ j$ where f is projective and j is a dense open immersion. The case of $\bar{f}$ being already known from [CD12, Proposition 4.2.11], we may assume $f = j$.

Thus, as j_* commutes with Tate twist, it is sufficient to prove that for any dense open immersion $j : U \to X$, with X a quasi-excellent, the h-motive $j_*(\mathbb{1}_U)$ is constructible. Applying Proposition 6.2.3, it is sufficient to prove that, given any prime ideal $\mathfrak{p} \in \operatorname{Spec}(\mathbf{Z})$, the h-motive $j_*(\mathbb{1}_U)$ is $\mathfrak{p}$-constructible.

The case where $\mathfrak{p} = (0)$ directly follows from Gabber's Lemma 6.2.7. Assume now that $\mathfrak{p} = (p)$ for a prime number $p > 0$. Let us consider the following cartesian square of schemes, in which $X_p = X \times \operatorname{Spec}(\mathbf{Z}[1/p])$.

$$\begin{tikzcd}
U_p \arrow[r, "i_U"] \arrow[d, "j_p"'] & U \arrow[d, "j"] & U' \arrow[l, "j_U"'] \arrow[d, "j'"] \\
X_p \arrow[r, "i_X"] & X & X' \arrow[l, "j_X"']
\end{tikzcd}$$

Then we can consider the following localization distinguished triangle

$$j_{X!}j_X^*j_*(\mathbb{1}_U) \to j_*(\mathbb{1}_U) \to i_{X*}i_X^*j_*(\mathbb{1}_U) \to j_{X!}j_X^*j_*(\mathbb{1}_U)[1]$$

so that it is sufficient to prove that the first and third motives in the above triangle are $\mathfrak{p}$-constructible. Note that the functors $j_{X!}$ and i_{X*} preserve, $\mathfrak{p}$-constructible objects, so that it is sufficient to prove that $i_X^*j_*(\mathbb{1}_U)$ and $j_X^*j_*(\mathbb{1}_U)$ are $\mathfrak{p}$-constructible.

The object $i_X^*j_*(\mathbb{1}_U)$ being (0)-constructible, it is $\mathfrak{p}$-constructible, by virtue of Proposition 6.2.4. It remains to prove that the following h-motive is $\mathfrak{p}$-constructible:

$$j_X^*j_*(\mathbb{1}_U) = j'_*(\mathbb{1}_{U'})$$

(for the isomorphism, we have used the smooth base change theorem, which is trivially true in $\mathrm{DM_h}$, by construction). Thus, we are finally reduced to Gabber's Lemma 6.2.7, and this completes the proof. □

Corollary 6.2.14. *The six operations preserve constructibility in* $\mathrm{DM_h}(-, R)$ *over quasi-excellent noetherian schemes of finite dimension. In other words, we have the following stability properties.*

(a) *For any quasi-excellent noetherian scheme of finite dimension* X*, any constructible objects* M *and* N *in* $\mathrm{DM_h}(X, R)$*, both* $M \otimes_R N$ *and* $\underline{\mathrm{Hom}}_R(M, N)$ *are constructible.*

(b) *For any separated morphism of finite type between quasi-excellent noetherian schemes of finite dimension* $f : X \to Y$*, and for any constructible object* M *of* $\mathrm{DM_h}(X, R)$*, the objects* $f_*(M)$ *and* $f_!(M)$ *are constructible, and for any constructible object* N *of* $\mathrm{DM_h}(Y, R)$*, the objects* $f^*(N)$ *and* $f^!(N)$ *are constructible.*

Proof. The fact that f^* preserves constructibility is obvious. The case of f_* follows from the preceding theorem. The tensor product also preserves constructibility on the nose. To prove that $\underline{\mathrm{Hom}}_R(M, N)$ is constructible for any constructible objects M and N in $\mathrm{DM_h}(X, R)$, we may assume that $M = f_\sharp(\mathbb{1}_Y)$ for a separated smooth morphism of finite type $f : Y \to X$. In this case, we have the isomorphism

$$\underline{\mathrm{Hom}}_R(M, N) \simeq f_*f^*(N),$$

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from which we get the expected property. The fact that the functors of the form $f_!$ preserve constructibility is well known (see for instance [CD12, Corollary 4.2.12]). Let $f : X \to Y$ be a separated morphism of finite type between quasi-excellent noetherian schemes of finite dimension. The property that $f^!$ preserves constructibility is local on X and on Y with respect to the Zariski topology (see [CD12, Lemma 4.2.27]), so that we may assume that f is affine. From there, we see that we may assume that f is an open immersion, or that f is the projection of the projective space $\mathbf{P}^n_Y$ to the base, or that f is a closed immersion. The case of an open immersion is trivial. In the case where f is a projective space of dimension n, the purity isomorphism $f^! \simeq f^*(n)[2n]$ allows to conclude. Finally, if $f = i$ is a closed immersion with open complement $j : U \to Y$, then we have distinguished triangles

$$i_*i^!(M) \to M \to j_*j^*(M) \to i_*i^!(M)[1]$$

from which deduce that $i_*i^!(M)$ is constructible, and thus that $i^!(M) \simeq i^*i_*i^!(M)$ is constructible, whenever M has this property. □

6.2.15. An object U of $\mathrm{DM}_{\mathrm{h}}(X, R)$ will be said to be *dualizing* if it has the following two properties.

(i) The h-motive U is constructible.

(ii) For any constructible object M in $\mathrm{DM}_{\mathrm{h}}(X, R)$, the canonical morphism

$$M \to \underline{\mathrm{Hom}}_R(\underline{\mathrm{Hom}}_R(M, U), U)$$

is an isomorphism.

Lemma 6.2.16. *Let X be a quasi-excellent noetherian scheme of finite dimension.*

(i) *If an object U of $\mathrm{DM}_{\mathrm{h}}(X, \mathbf{Z})$ is dualizing, then, for any commutative ring R, the (derived) tensor product $R \otimes U$ is dualizing in $\mathrm{DM}_{\mathrm{h}}(X, R)$.*

(ii) *A constructible object U of $\mathrm{DM}_{\mathrm{h}}(X, R)$ is dualizing if an only if $\mathbf{Q} \otimes U$ is dualizing in $\mathrm{DM}_{\mathrm{h}}(X, \mathbf{Q})$ and, for any prime p, U/p is dualizing in $\mathrm{DM}_{\mathrm{h}}(X, \mathbf{Z}/p\mathbf{Z})$.*

Proof. Assume that the object U of $\mathrm{DM}_{\mathrm{h}}(X, \mathbf{Z})$ is dualizing. To prove that the canonical map

$$M \to \underline{\mathrm{Hom}}_R(\underline{\mathrm{Hom}}_R(M, R \otimes U), R \otimes U)$$

is invertible for any constructible object M in $\mathrm{DM}_{\mathrm{h}}(X, R)$, we may assume that

$$M = f_\sharp(R_Y) \simeq R \otimes f_\sharp(\mathbf{Z}_Y)$$

for a separated smooth morphism of finite type $f : Y \to X$. In particular, we may assume that $M = R \otimes C$ for a constructible object C in $\mathrm{DM}_{\mathrm{h}}(X, \mathbf{Z})$. But then, by virtue of Corollary 5.5.13, we have a canonical isomorphism

$$\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(C, U), U) \otimes R \simeq \underline{\mathrm{Hom}}_R(\underline{\mathrm{Hom}}_R(M, R \otimes U), R \otimes U),$$

from which we conclude that $R \otimes U$ is dualizing. The proof of the second assertion is similar. Indeed, for any constructible object C of $\mathrm{DM}_{\mathrm{h}}(X, \mathbf{Z})$, by virtue of Corollary 5.4.11, we have canonical isomorphisms

$$\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(C, U), U) \otimes \mathbf{Q} \simeq \underline{\mathrm{Hom}}_{\mathbf{Q}}(\underline{\mathrm{Hom}}_{\mathbf{Q}}(\mathbf{Q} \otimes C, \mathbf{Q} \otimes U), \mathbf{Q} \otimes U),$$

and, by Proposition 5.4.5, for any positive integer n, canonical isomorphisms

$$\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(C, U), U)/n \simeq \underline{\mathrm{Hom}}_{\mathbf{Z}/n\mathbf{Z}}(\underline{\mathrm{Hom}}_{\mathbf{Z}/n\mathbf{Z}}(C/n, U/n), U/n).$$

By virtue of Proposition 5.4.12, this readily implies assertion (ii). □

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Theorem 6.2.17. *Let B be an excellent noetherian scheme of dimension less than or equal to 2 (or, more generally, which admits wide resolution of singularities up to quotient singularities in the sense of [CD12, Definition 4.1.9]).*

(a) *For any regular B-scheme of finite type S, an object U of $\mathrm{DM}_{\mathrm{h}}(S,R)$ is dualizing if and only if it is constructible and $\otimes$-invertible.*

(b) *For any separated morphism of B-schemes of finite type $f : X \to S$, with S regular, and for any dualizing object U in $\mathrm{DM}_{\mathrm{h}}(S,R)$, the object $f^!(U)$ is a dualizing object in $\mathrm{DM}_{\mathrm{h}}(X,R)$.*

Proof. Consider separated morphism of B-schemes of finite type $f : X \to S$, with S regular. Then we claim that the object $f^!(R_S)$ is dualizing in $\mathrm{DM}_{\mathrm{h}}(X,R)$. Indeed, by virtue of Corollary 5.5.14 and Lemma 6.2.16, we may assume that $R = \mathbf{Q}$ or $R = \mathbf{Z}/p\mathbf{Z}$ for some prime p. In the first case, this is already known (see [CD12, Theorems 15.2.4 and 16.1.2]). If $R = \mathbf{Z}/p\mathbf{Z}$, as, for any open immersion j, the functor j^* is symmetric monoidal and preserves internal Hom's, by virtue of Corollaries 4.5.3 and 5.5.4, we may assume that p is invertible in the residue fields of S and that we have equivalence of triangulated categories

$$\mathrm{D}(Y_{\text{ét}}, \mathbf{Z}/p\mathbf{Z}) \simeq \mathrm{DM}_{\mathrm{h}}(Y, \mathbf{Z}/p\mathbf{Z})$$

for any S-scheme of finite type Y, in a functorial way with respect to the six operations. As, by virtue of the last assertion of Corollary 5.5.4, this equivalence restricts to a monoidal full embedding

$$\mathrm{DM}_{\mathrm{h},c}(X, \mathbf{Z}/p\mathbf{Z}) \subset \mathrm{D}^b_{\mathrm{ctf}}(X_{\text{ét}}, \mathbf{Z}/p\mathbf{Z}),$$

this property boils down to the analogous result in classical étale cohomology (which, at this level of generality, has been proved by Gabber; see [ILO14, XVII, Theorem 0.2]).[13] This implies the theorem through classical and formal arguments; see [CD12, Proposition 4.4.22]. □

6.3 Continuity and locally constructible h-motives

Definition 6.3.1. An object M of $\mathrm{DM}_{\mathrm{h}}(X,R)$ is *locally constructible* (with respect to the étale topology) if there exists an étale covering $\{u_i : X_i \to X\}_{i\in I}$ such that, for any $i \in I$, the object $u_i^*(M)$ is constructible (of geometric origin) in the sense of Definition 5.1.3. We denote by $\mathrm{DM}_{\mathrm{h,lc}}(X,R)$ the full subcategory of $\mathrm{DM}(X,R)$ which consists of locally constructible objects. We have embeddings

$$\mathrm{DM}_{\mathrm{h},c}(X,R) \subset \mathrm{DM}_{\mathrm{h,lc}}(X,R) \subset \mathrm{DM}_{\mathrm{h}}(X,R).$$

Remark 6.3.2. The heuristic reason why the notion of locally constructible object is a natural one is the following. In a setting in which one has the six operations (e.g. a motivic triangulated category in the sense of [CD12]), it is natural to look at the smallest subsystem generated by the constant coefficient (i.e. the unit object of the monoidal structure) and closed under the six operations. Finiteness theorems such as Corollary 6.2.14 mean that the notion of constructible motive, as in Definition 5.1.3, gives such a thing. But, in practice (e.g. in this article), we have more than a system of triangulated categories: we have a system of stable Quillen model categories (or, in a more intrinsic language, of stable $(\infty,1)$-categories in the sense of Lurie), and this extra structure is rich enough to speak of descent: we can speak of stacks (in an adequate homotopical sense) for appropriate topologies (in the language of Lurie: sheaves of $(\infty,1)$-categories). In fact the formalism of the six operations always ensures that we have

[13] In Gabber's theorem, the existence of a dualizing object is subject a dimension function, which, in our situation, readily follows from [ILO14, XIV, Corollaries 2.4.4 and 2.5.2].

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descent for the Nisnevich topology. Therefore, whenever constructible objects are closed under the six operations, they form a Nisnevich stack. But, in the case of $\mathrm{DM}_{\mathrm{h}}(-,R)$, we have a stack with respect to the étale topology, and it is thus natural to ask for a notion of constructible h-motives which also form a stack for the étale topology. Essentially by definition, the system of locally constructible h-motives (expressed in the language of stable $(\infty,1)$-categories) is the étale stack associated with the fibered $(\infty,1)$-category of constructible h-motives. Even though we will not go very deep into such considerations about descent and higher categories, we can say that much of the results of this section are devoted to the understanding of the étale stack of locally constructible h-motives by understanding its stalks. This will be expressed by continuity phenomena, and will have as consequences that we still have the formalism of the six operations in this context. Note finally that, even though we will not develop this very far here, locally constructible h-motives do form a stack for the h-topology. This is suggested by Propositions 6.3.16 and 6.3.18 below, together with the proper base change formula.

Proposition 6.3.3. *Let X be a noetherian scheme of finite dimension. For any $\mathbf{Q}$-algebra R, one has $\mathrm{DM}_{\mathrm{h},c}(X,R)=\mathrm{DM}_{\mathrm{h,lc}}(X,R)$.*

Proof. This follows right away from Lemma 1.1.4 and from Theorem 5.2.4. □

Proposition 6.3.4. *Let X be a noetherian scheme of finite dimension. Consider a localization A of $\mathbf{Z}$, and a ring of coefficients R. For any objects M and N of $\mathrm{DM}_{\mathrm{h}}(X,R)$, if M is locally constructible, then the natural map*

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(X,R)}(M,N)\otimes A\to \mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(X,R\otimes A)}(M\otimes A,N\otimes A)$$

is bijective.

Proof. We must prove that the natural map

$$\mathbf{R}\mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(X,R)}(M,N)\otimes A\to \mathbf{R}\mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(X,R\otimes A)}(M\otimes A,N\otimes A)$$

is an isomorphism in the derived category of the category of A-modules. Let us consider the case where M is constructible. We easily reduce the problem to the case where $M=R(Y)$ for some smooth X-scheme Y. In particular, we may assume that $M=R\otimes^{\mathbf{L}}M'$ for some constructible object M' of $\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Z})$. In other words, in the case where M is constructible, we may assume that $R=\mathbf{Z}$, in which case we already know this property to hold; see Corollary 5.4.9. To prove the general case, note that, for any ring of coefficients R, and any objects E and F of $\mathrm{DM}_{\mathrm{h}}(X,R)$, one can associate a presheaf of complexes $C(E,F;R)$ on the small étale site of X such that, for any étale map $u:U\to X$, we have canonical isomorphisms

$$H^i(C(E,F;R)(U))\simeq H^i_{\mathrm{\acute{e}t}}(U,C(E,F;R))\simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(U,R)}(u^*(E),u^*(F)[i])$$

(see [CD12, Paragraph 3.2.11 and Corollary 3.2.18] for a rigorous definition and construction of such a C). Therefore, the complex $C(M,N;R\otimes A)$ satisfies étale descent, and Proposition 1.1.11 implies that the complex $C(M,N;R)\otimes A$ has the same property. Since, locally for the étale topology over X, the canonical map

$$C(M,N;R)\otimes A\to C(M,N;R\otimes A)$$

is a quasi-isomorphism, its evaluation at X is a quasi-isomorphism, which is precisely what we wanted to prove. □

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Proposition 6.3.5. *Let X be a noetherian scheme, and $\{X_i\}_{i\in I}$ a projective system of noetherian schemes of finite dimension with affine transition maps. Let us consider a noetherian ring of coefficients R. Then the canonical functors*

$$2\text{-}\varinjlim_i \mathrm{D}^b_c(X_i, R) \to \mathrm{D}^b_c(X, R) \tag{6.3.5.a}$$

and

$$2\text{-}\varinjlim_i \mathrm{D}^b_{\mathrm{ctf}}(X_i, R) \to \mathrm{D}^b_{\mathrm{ctf}}(X, R) \tag{6.3.5.b}$$

are equivalences of triangulated categories.

Proof. The fact that (6.3.5.a) is an equivalence easily follows from [SGA4, Exp. IX, Corollaries 2.7.3 and 2.7.4]. This readily implies that (6.3.5.b) is fully faithful. To prove the essential surjectivity of the latter, we easily deduce from [SGA4$\frac{1}{2}$, Rapport, 4.6] that it is sufficient to prove the following property: given some constructible sheaf of R-modules F_i on some X_i whose pullback F along the projection $X \to X_i$ is flat, there exists an index $j \geqslant i$ such that the pullback F_j of F_i along the transition map $X_j \to X_i$ is flat. Choosing an adequate stratification of X_i, we may assume that F_i is locally constant and that X_i is integral. By virtue of [SGA4, Exp. IX, Proposition 2.11], is thus sufficient to prove that there exists a geometric point x_i of X_i such that the fiber of F_i at x_i is a flat R-module. But, for any (geometric) point x of X over x_i, it is isomorphic to the fiber $x^*(F) = F_x$, which is flat. □

Definition 6.3.6. A commutative ring R will be said to be *good enough* if it is noetherian, and if, for any prime number p, the localized ring $R_{(p)} = \mathbf{Z}_{(p)} \otimes R$ has the property that p is either nilpotent or is not a zero divisor.[14] For instance, any noetherian ring which is flat over $\mathbf{Z}$, or any noetherian ring of positive characteristic is good enough.

Proposition 6.3.7. *Assume that R is good enough. Let X be a noetherian scheme of finite dimension, and $\{X_i\}_{i\in I}$ a projective system of noetherian schemes of finite dimension with affine transition maps.*

Consider an index $i_0 \in I$ and two locally constructible R-linear h*-motives M_{i_0}, N_{i_0} over X_{i_0}. We denote by M, N (respectively M_i, N_i) for the respective pullbacks of M_{i_0}, N_{i_0} along the projection $X \to X_{i_0}$ (respectively transition map $X_i \to X_{i_0}$ for a map $i \to i_0$ in I).*

Then we have a canonical isomorphism of R-modules

$$\varinjlim_i \mathrm{Hom}_{\mathrm{DM}_{\mathrm{h,lc}}(X_i,R)}(M_i, N_i) \simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{h,lc}}(X,R)}(M, N). \tag{6.3.7.a}$$

Proof. We want to prove that the morphism

$$\mathbf{L}\varinjlim_i \mathbf{R}\mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(X_i,R)}(M_i, N_i) \simeq \mathbf{R}\mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(X,R)}(M, N) \tag{6.3.7.b}$$

is an isomorphism in the derived category of R-modules. By virtue of Proposition 6.3.4, we may assume that R is a $\mathbf{Z}_{(p)}$-algebra for some prime number p. Under these assumptions,

[14] This notion is introduced as a possible constraint on the rings of coefficients. However, it is only a simplifying hypothesis for the proof of Proposition 6.3.7 and, in an even less trivial way, of Theorem 6.3.11: in fact, this proposition (as well as the theorem, but the latter is not used to prove anything else), and therefore, all the results of this section, remain valid for arbitrary rings of coefficients (although one has to take the appropriate definition of $\mathrm{D}^b_{\mathrm{ctf}}(X, R)$ for a non-noetherian ring R), but such level of generality demands either enough abnegation to do ingrate computations or to present the theory into the more advanced language of higher categories.

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if R is furthermore a $\mathbf{Q}$-algebra, the invertibility of the map (6.3.7.b) is a particular case of Proposition 5.2.5. If R is a $\mathbf{Z}/n\mathbf{Z}$-algebra with $n = p^a$ a power of some prime number p, then, by virtue of Proposition A.3.4, we may assume that all the schemes are of characteristic prime to p, and thus, by virtue of the last assertion of Corollary 5.5.4, we can replace $\mathrm{DM}_{\mathrm{h,lc}}(X, R)$ by $\mathrm{D}^b_{\mathrm{ctf}}(X_{\text{ét}}, R)$ and use Proposition 6.3.5. For the general case when R is a $\mathbf{Z}_{(p)}$-algebra, we may assume that R is of mixed characteristic and flat over $\mathbf{Z}$. Using Proposition 5.4.5 and Proposition 6.3.4, we deduce that the map (6.3.7.b) is an isomorphism after we tensor (in the derived sense) by $\mathbf{Z}/p\mathbf{Z}$, or by $\mathbf{Q}$. This implies that it is an isomorphism in the derived category of R-modules. □

Remark 6.3.8. In the previous proposition, if separated étale X-schemes of finite type are of finite étale cohomological dimension (e.g. if X is of finite type over a strictly henselian scheme (Theorem 1.1.5)), and if the transition maps of the projective system $\{X_i\}_{i\in I}$ are étale, then we still have the isomorphism (6.3.7.a) without the assumption that the objects N_i are locally constructible. The proof remains exactly the same, except that we use Lemma 1.1.12 (applied to the adequate family of small étale topoi) instead of Proposition 6.3.5.

Theorem 6.3.9. *Under the assumptions of Proposition 6.3.7, the canonical functor*

$$2\text{-}\varinjlim_i \mathrm{DM}_{\mathrm{h},c}(X_i, R) \to \mathrm{DM}_{\mathrm{h},c}(X, R) \tag{6.3.9.a}$$

is an equivalence of triangulated categories. If, moreover, the étale cohomological dimension of the residue fields of the scheme X is uniformly bounded (e.g. if X is of finite type over a noetherian strictly henselian scheme), then the functor

$$2\text{-}\varinjlim_i \mathrm{DM}_{\mathrm{h,lc}}(X_i, R) \to \mathrm{DM}_{\mathrm{h,lc}}(X, R) = \mathrm{DM}_{\mathrm{h},c}(X, R) \tag{6.3.9.b}$$

is an equivalence of triangulated categories as well.

Proof. The isomorphism (6.3.7.a) implies that the functor (6.3.9.a) is fully faithful. Let us prove that it is essentially surjective. As we already know that it is fully faithful, it identifies the idempotent complete triangulated category $2\text{-}\varinjlim_i \mathrm{DM}_{\mathrm{h},c}(X_i, R)$ with a thick subcategory of the triangulated category $\mathrm{DM}_{\mathrm{h},c}(X, R)$. But, by definition of the latter, the smallest thick subcategory of $\mathrm{DM}_{\mathrm{h},c}(X, R)$ containing the objects of the form $R(U)(n)$, with U a separated smooth scheme of finite type over X and $n \in \mathbf{Z}$, is the whole category $\mathrm{DM}_{\mathrm{h},c}(X, R)$ itself. Moreover, for any such U and any Zariski covering $U = V \cup W$, we have a Mayer–Vietoris distinguished triangle of the form

$$R(V \cap W) \to R(V) \oplus R(W) \to R(U) \to R(V \cap W)[1].$$

Hence, to prove that $R(U)(n)$ belongs to the essential image of (6.3.9.a), it is sufficient to prove that $R(V)$, $R(W)$ and $R(V \cap W)$ have this property. In particular, it is sufficient to consider the case where U is affine over X. Therefore, the fact that the functor (6.3.9.a) is essentially surjective comes from the fact that any affine smooth scheme of finite type over X is the pullback of an affine smooth scheme of finite type over X_i for some index $i \in I$; see [EGA4, Theorem 8.10.5, Proposition 17.7.8].

Under our additional assumption, the proof that the functor (6.3.9.b) is an equivalence of categories readily follows from there: it is fully faithful by Proposition 6.3.7, and it is essentially surjective because the functor (6.3.9.a) is essentially surjective and because, by virtue of Theorem 5.2.4, we have the equality $\mathrm{DM}_{\mathrm{h,lc}}(X, R) = \mathrm{DM}_{\mathrm{h},c}(X, R)$. □

Proposition 6.3.10. *Let X be a noetherian scheme, and R a noetherian ring. Assume as well that any separated quasi-finite X-scheme is of finite étale cohomological dimension with R-linear coefficients. Then the triangulated category $\mathrm{D}^b_{\mathrm{ctf}}(X_{\mathrm{\acute{e}t}}, R)$ is the full subcategory of compact objects in the unbounded derived category $\mathrm{D}(X_{\mathrm{\acute{e}t}}, R)$. If, moreover, X is of finite dimension, R is of characteristic invertible in $\mathscr{O}_X$, and if the étale cohomological dimension with R-linear coefficients of the residue fields of X is uniformly bounded, then the equivalence of triangulated categories $\mathrm{D}(X_{\mathrm{\acute{e}t}}, R) \simeq \mathrm{DM_h}(X, R)$ provided by Corollary 5.5.4 induces an equivalence of categories*

$$\mathrm{D}^b_{\mathrm{ctf}}(X_{\mathrm{\acute{e}t}}, R) \simeq \mathrm{DM}_{\mathrm{h},c}(X, R).$$

Proof. It follows from Proposition 1.1.9 that the family of representable sheaves $R(U)$, where U runs over the (separated) étale X-schemes of finite type, form a generating family of compact objects of the triangulated category $\mathrm{D}(X_{\mathrm{\acute{e}t}}, R)$. Therefore, the category $\mathrm{D}(X_{\mathrm{\acute{e}t}}, R)_c$ of compact objects of $\mathrm{D}(X_{\mathrm{\acute{e}t}}, R)$ can be described as the smallest thick subcategory of $\mathrm{D}(X_{\mathrm{\acute{e}t}}, R)$ which contains the sheaves $R(U)$ as above. As these sheaves obviously belong to $\mathrm{D}^b_{\mathrm{ctf}}(X_{\mathrm{\acute{e}t}}, R)$, this proves that any compact object of $\mathrm{D}(X_{\mathrm{\acute{e}t}}, R)$ belongs to $\mathrm{D}^b_{\mathrm{ctf}}(X_{\mathrm{\acute{e}t}}, R)$. It remains to prove the reverse inclusion. Note that, for any closed immersion $i : Z \to X$ with open complement $j : U \to X$, we have short exact sequences

$$0 \to j_! j^*(F) \to F \to i_* i^*(F) \to 0$$

from which we deduce that $\mathrm{D}(X_{\mathrm{\acute{e}t}}, R)_c$ is stable by the operations $j_!$, j^*, i^* and i_*. Proceeding as in the proof of the equivalence $(c) \Leftrightarrow (d)$ of Theorem 5.2.4, we see that the property of being compact in $\mathrm{D}(X_{\mathrm{\acute{e}t}}, R)$ is local with respect to the étale topology: if there exists an étale surjective map $u : X' \to X$ such that $u^*(C)$ is compact in $\mathrm{D}(X'_{\mathrm{\acute{e}t}}, R)$, then C is compact.

Let C be an object of $\mathrm{D}^b_{\mathrm{ctf}}(X_{\mathrm{\acute{e}t}}, R)$. To prove that C is compact, it is sufficient to prove that there exists a stratification of X by locally closed subsets X_i such that the restriction $C_i = C|_{X_i}$ is compact for any i. Moreover, it is sufficient to check that each C_i is compact after we pull it back along an étale surjective map $X'_i \to X_i$. By virtue of [SGA4$\frac{1}{2}$, Rapport, Lemma 4.5.1 and Proposition 4.6], we thus may assume that there exists a perfect complex of R-modules M such that C is isomorphic in $\mathrm{D}(X_{\mathrm{\acute{e}t}}, R)$ to the constant sheaf M_X associated with M. On the other hand, the functor $M \mapsto M_X$ being exact, the complexes of R-modules M such that M_X is compact form a thick subcategory of the derived category $\mathrm{D}(R)$ of the category of R-modules. But the category of perfect complexes of R-modules is the smallest thick subcategory of $\mathrm{D}(R)$ which contains R (seen as a complex of R-modules concentrated in degree zero). Therefore, we may assume that $C = R_X$, which is compact. This proves the equality $\mathrm{D}(X_{\mathrm{\acute{e}t}}, R)_c = \mathrm{D}^b_{\mathrm{ctf}}(X_{\mathrm{\acute{e}t}}, R)$.

As equivalences of categories preserve compact objects, the last assertion readily follows from Theorem 5.2.4. □

Theorem 6.3.11. *Let X be a noetherian scheme of finite dimension, and consider a noetherian ring of coefficients R, of positive characteristic prime to the residue characteristics of X. Then the canonical equivalence of triangulated categories $\mathrm{D}(X_{\mathrm{\acute{e}t}}, R) \simeq \mathrm{DM_h}(X, R)$ restricts to an equivalence of triangulated categories*

$$\mathrm{D}^b_{\mathrm{ctf}}(X_{\mathrm{\acute{e}t}}, R) \simeq \mathrm{DM}_{\mathrm{h,lc}}(X, R).$$

Proof. The equivalence of categories $\mathrm{D}(X_{\mathrm{\acute{e}t}}, R) \simeq \mathrm{DM_h}(X, R)$ are compatible with the six operations and thus induce fully faithful functors

$$\mathrm{DM}_{\mathrm{h,lc}}(X, R) \to \mathrm{D}^b_{\mathrm{ctf}}(X_{\mathrm{\acute{e}t}}, R)$$

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which are compatible with pullback functors; see Corollary 5.5.4. It is sufficient to prove the essential surjectivity in the étale neighborhood of each geometric point x of X. On the other hand, by virtue of Theorems 1.1.5 and 6.3.9 and of Propositions 6.3.10 and 6.3.5, all the functors in the obvious commutative diagram below, in which V runs over the étale neighborhoods of x,

$$\begin{tikzcd} 2\text{-}\varinjlim_V \mathrm{DM}_{\mathrm{h},c}(V,R) \arrow[rr] \arrow[dr] & & 2\text{-}\varinjlim_V \mathrm{DM}_{\mathrm{h,lc}}(V,R) \arrow[dl] \\ & 2\text{-}\varinjlim_V \mathrm{D}^b_{\mathrm{ctf}}(V,R) & \end{tikzcd}$$

are equivalences of categories, which implies our assertion. □

We can now complete Theorem 6.3.9 as follows.

Theorem 6.3.12. *Let R be a good enough ring of coefficients. All the schemes below are assumed to be noetherian and of finite dimension. Assume that the scheme X is the limit of a projective system of schemes $\{X_i\}_{i\in I}$ with affine transition maps. Then the canonical functor*

$$2\text{-}\varinjlim_i \mathrm{DM}_{\mathrm{h,lc}}(X_i,R) \to \mathrm{DM}_{\mathrm{h,lc}}(X,R) \qquad (6.3.12.a)$$

is an equivalence of categories.

Proof. We already know that this functor is fully faithful. Therefore, the left-hand side of (6.3.12.a) can be seen as a thick subcategory of the right-hand side. As all the categories involved here are idempotent complete, using Proposition 6.3.4 together with Proposition B.1.7 from the Appendix A, we see that we may assume R to be a $\mathbf{Z}_{(p)}$-algebra. This also means that, to prove that an object of $\mathrm{DM}_{\mathrm{h,lc}}(X,R)$ is in the essential image of this functor, it is sufficient to prove that it is a direct factor of an object in the essential image.

Henceforth, all integers prime to p are supposed to be invertible in R. If R is a $\mathbf{Q}$-algebra, Proposition 6.3.3, together with Theorem 6.3.9, show that the functor (6.3.12.a) is an equivalence of categories. If R is of positive characteristic, we easily deduce from Theorem 6.3.11 and Proposition A.3.4 that, for any scheme V, we have canonical equivalences of triangulated categories

$$\mathrm{D}^b_{\mathrm{ctf}}(W_{\text{ét}},R) \simeq \mathrm{DM}_{\mathrm{h,lc}}(W,R) \simeq \mathrm{DM}_{\mathrm{h,lc}}(V,R)$$

where $W = V \times \mathrm{Spec}\,(\mathbf{Z}[1/p])$. The fact that the functor (6.3.12.a) is an equivalence of categories whenever R is of positive characteristic is now a reformulation of Proposition 6.3.5 and of Theorem 6.3.11.

It remains to consider the case where R is a good enough $\mathbf{Z}_{(p)}$-algebra R of characteristic zero. Using Proposition 6.3.4 (with $M = N$), what precedes implies that any object M of $\mathrm{DM}_{\mathrm{h,lc}}(X,R)$ such that $M \otimes \mathbf{Q} = 0$ in $\mathrm{DM}_{\mathrm{h}}(X,R)$ belongs to the essential image of the functor (6.3.12.a): indeed, this implies that, for $\nu \geqslant 0$ big enough, M is a direct factor of $M \otimes^{\mathbf{L}} \mathbf{Z}/p^\nu\mathbf{Z}$, which belongs to the essential image, as it comes from $\mathrm{DM}_{\mathrm{h,lc}}(X, R\otimes \mathbf{Z}/p^\nu\mathbf{Z})$. On the other hand, one can interpret the conjunction of Propositions 6.3.3 and 6.3.4 as follows: the triangulated category $\mathrm{DM}_{\mathrm{h,lc}}(X,R\otimes\mathbf{Q})$ is the idempotent completion of the triangulated category $\mathrm{DM}_{\mathrm{h},c}(X,R)\otimes\mathbf{Q}$. This means that, for any object M of $\mathrm{DM}_{\mathrm{h,lc}}(X,R)$, there exists M_0' in $\mathrm{DM}_{\mathrm{h,lc}}(X,R\otimes\mathbf{Q})$ and N in $\mathrm{DM}_{\mathrm{h},c}(X,R)$ as well as an isomorphism

$$M\otimes\mathbf{Q}\oplus M_0' \simeq N\otimes\mathbf{Q}$$

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in $\mathrm{DM}_{\mathrm{h}}(X, R \otimes \mathbf{Q})$. But Proposition 6.3.3 also tells us that the corresponding embedding $M \otimes \mathbf{Q} \to N \otimes \mathbf{Q}$ is defined over R: there is a map $\lambda : M \to N$ in $\mathrm{DM}_{\mathrm{h}}(X, R)$ which identifies $M \otimes \mathbf{Q}$ with a direct factor of $N \otimes \mathbf{Q}$. If M' denotes a cone of this map λ, there exists an isomorphism $M' \otimes \mathbf{Q} \simeq M'_0$. Hence, again by Proposition 6.3.4, there is a morphism

$$\varphi : M \oplus M' \to N$$

in $\mathrm{DM}_{\mathrm{h,lc}}(X, R)$ such that $\varphi \otimes \mathbf{Q}$ is invertible. Let C be a cone of φ. Then $C \otimes \mathbf{Q} = 0$. Therefore, the locally constructible h-motive C is in the essential image of the functor (6.3.12.a). But the h-motive N has the same property (because it is constructible, using the first part of Theorem 6.3.9). Hence $M \oplus M'$ is in the essential image of the functor (6.3.12.a) and this completes the proof as explained in the beginning of this proof. □

Proposition 6.3.13. *Let $p : X \to S$ be a morphism of finite type between noetherian schemes of finite dimension. Consider a good enough ring of coefficients R. Then, for R-linear* h-*motives over X, the property of local constructibility is local over S with respect to the étale topology. In other words, for any object M of $\mathrm{DM}_{\mathrm{h,lc}}(X, R)$, there exists a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ S' \arrow[r, "v"] & S \end{tikzcd}$$

with v étale surjective and such that $u^*(M)$ belongs to $\mathrm{DM}_{\mathrm{h},c}(X', R)$.

Proof. For each geometric point s of S, we must find an étale neighborhood $w : W \to S$ of s such that the pullback of M along the first projection of $W \times_S X$ on X is constructible. But Theorems 6.3.9 and 1.1.5 imply that we have a canonical equivalence of categories

$$2\text{-}\varinjlim_{W} \mathrm{DM}_{\mathrm{h},c}(W \times_S X, R) \simeq 2\text{-}\varinjlim_{W} \mathrm{DM}_{\mathrm{h,lc}}(W \times_S X, R),$$

where W runs over the étale neighborhoods of s. The essential surjectivity of this functor precisely expresses what we seek. □

Corollary 6.3.14. *Let $f : X \to S$ be a separated morphism of finite type with S noetherian of finite dimension, and assume that the ring R is good enough. Then the functor $f_! : \mathrm{DM}_{\mathrm{h}}(X, R) \to \mathrm{DM}_{\mathrm{h}}(S, R)$ preserves locally constructible objects.*

Proof. Let M be a locally constructible object of $\mathrm{DM}_{\mathrm{h,lc}}(X, R)$. Then, by virtue of the preceding proposition, one can form a cartesian square of schemes

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ S' \arrow[r, "v"] & S \end{tikzcd}$$

in which v is a surjective separated étale morphism of finite type, such that $u^*(M)$ is constructible. The base change isomorphism $v^* f_!(M) \simeq g_! u^*(M)$ thus shows that it is sufficient to know that the functor $g_!$ preserves constructible objects. This is then a well-known consequence of the formalism of the six operations (which makes sense here by Theorem 5.6.2); see [CD12, Corollary 4.3.12]. □

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COROLLARY 6.3.15. *Let R be a good enough ring. The subcategories $\mathrm{DM}_{\mathrm{h,lc}}(X,R)$ are closed under the six operations in $\mathrm{DM}_{\mathrm{h}}(X,R)$ for quasi-excellent noetherian schemes of finite dimension.*

Furthermore, consider an excellent scheme B of dimension less than or equal to 2 as well as a regular separated B-scheme of finite type S, endowed with a locally constructible and $\otimes$-invertible object U in $\mathrm{DM}_{\mathrm{h}}(S,R)$. For any separated morphism of finite type $f : X \to S$, define the duality functor D_X by the formula $D_X(M) = \mathbf{R}\underline{\mathrm{Hom}}_R(M, f^!(U))$. Then, for any locally constructible object M in $\mathrm{DM}_{\mathrm{h}}(X,R)$, the canonical map

$$M \to D_X(D_X(M))$$

is an isomorphism.

Proof. Consider the first assertion. We already know it is true in the case for the subcategories $\mathrm{DM}_{\mathrm{h},c}(X,R)$ (Corollary 6.2.14). This will imply our claim as follows. The stability by operations f^* for any morphism f is obvious. If $u : X' \to X$ is a surjective separated étale morphism of finite type, the functor u^* is conservative. As it is monoidal, this implies the stability of $\mathrm{DM}_{\mathrm{h,lc}}(X,R)$ by the derived tensor product $\otimes^{\mathbf{L}}_R$. As u^* commutes with the formation of the derived internal Hom

$$u^*\underline{\mathrm{Hom}}_R(A,B) \simeq \underline{\mathrm{Hom}}_R(u^*A, u^*B),$$

we easily get the stability by the bifunctor $\underline{\mathrm{Hom}}_R$. The stability by the operation $f_!$ for f separated and of finite type has already been considered in, Corollary 6.3.14, and the stability by the operation f_* for any morphism of finite type f is proved similarly.

The last assertion about duality follows from Theorem 6.2.17 and Proposition 6.3.13, using again the stability of local constructibility by pullbacks and derived internal Hom. $\square$

PROPOSITION 6.3.16. *Let $p : X \to S$ be a surjective, integral and radicial morphism between noetherian schemes of finite dimension. The pullback functor*

$$p^* : \mathrm{DM}_{\mathrm{h}}(S,R) \to \mathrm{DM}_{\mathrm{h}}(X,R)$$

is an equivalence of triangulated categories, and it restricts to an equivalence of categories

$$\mathrm{DM}_{\mathrm{h},c}(S,R) \simeq \mathrm{DM}_{\mathrm{h},c}(X,R).$$

In particular, its right adjoint p_* preserves constructible objects.

Proof. It is sufficient to prove this proposition when R is good enough. Indeed, if p^* is an equivalence with integral coefficients and restricts to an equivalence on constructible objects, then to prove that the unit and counit

$$M \to p_*p^*(M) \quad \text{and} \quad p^*p_*(N) \to N$$

are invertible for any M and N, as both functors p^* and p_* preserve small sums (see Proposition 5.5.10 for the second one), it is sufficient to prove it when M and N run over a generating family of $\mathrm{DM}_{\mathrm{h}}(S,R)$ and of $\mathrm{DM}_{\mathrm{h}}(X,R)$, respectively. This means that we may assume that both M and N are R-linearization of integral h-motives, and we finish with Corollary 5.5.12. The same kind of arguments show that p_* preserves constructible objects.

Henceforth, we will thus assume that R is good enough. Let us first consider the particular case where p is of finite type (and thus finite). For any finite surjective and radicial morphism of noetherian schemes $g : Y' \to Y$, the functor

$$g^* : \underline{\mathrm{DM}}_{\mathrm{h}}(Y,R) \to \underline{\mathrm{DM}}_{\mathrm{h}}(Y',R)$$

is conservative (by h-descent, because g is a covering for the h topology; see [Voe96, Proposition 3.2.5]). This implies that the functor

$$p^* : \mathrm{DM}_{\mathrm{h}}(S, R) \to \mathrm{DM}_{\mathrm{h}}(X, R)$$

is an equivalence of categories (see [CD12, Proposition 2.1.9]). Its restriction

$$p^* : \mathrm{DM}_{\mathrm{h},c}(S, R) \to \mathrm{DM}_{\mathrm{h},c}(X, R)$$

is an equivalence of categories as well, for its right adjoint $p_! = p_*$ preserves constructible objects.

If p is not of finite type, it is still affine and thus one can describe X as a limit of a projective system of affine Y-schemes X_i such that the structural maps $X_i \to S$ are finite, surjective and radicial. By continuity (Theorem 6.3.9), we see that the functor

$$p^* : \mathrm{DM}_{\mathrm{h},c}(S, R) \to \mathrm{DM}_{\mathrm{h},c}(X, R)$$

is an equivalence of categories as a filtered 2-colimit of such things. As both functors p^* and p_* commute with small sums, this implies that p^* is fully faithful on the whole category $\mathrm{DM}_{\mathrm{h}}(S)$. This ends the proof, as what precedes exhibits the essential image of $\mathrm{DM}_{\mathrm{h}}(S)$ in $\mathrm{DM}_{\mathrm{h}}(X)$ as a localizing subcategory containing a generating family of $\mathrm{DM}_{\mathrm{h}}(X)$. □

Corollary 6.3.17. *Under the assumptions of the preceding proposition, the functor p_* preserves locally constructible objects, and the functor p^* defines an equivalence of triangulated categories*

$$\mathrm{DM}_{\mathrm{h,lc}}(S, R) \simeq \mathrm{DM}_{\mathrm{h,lc}}(X, R).$$

Proof. Let M be an object of $\mathrm{DM}_{\mathrm{h,lc}}(X, R)$. We want to prove that $N = p_*(M)$ is locally constructible. By virtue of [SGA1, Exp. IX, Corollary 4.11], any surjective étale map $u : X' \to X$ is isomorphic to the pullback of a surjective étale map $v : S' \to S$ along p. Therefore, there exists an étale surjective morphism of finite type $v : S' \to S$ such that the pullback of M along the second projection $u : X' = S' \times_S X \to X$ is constructible. If $q : X' \to S'$ denotes the first projection, the base change map $v^*(N) = v^* p_*(M) \to q_* u^*(M)$ is invertible. Finally, the morphism q is also surjective, integral and radicial, so that the functor q_* preserves constructible objects (Proposition 6.3.16); this proves that N is locally constructible. □

Proposition 6.3.18. *Let R be a good enough ring of coefficients, and consider a surjective morphism of finite type between noetherian schemes of finite dimension $f : X \to S$. Then pulling back along f detects locally constructible motives: if an object M of $\mathrm{DM}_{\mathrm{h}}(S, R)$ has the property that $f^*(M)$ is locally constructible, then it is locally constructible. If, furthermore, the scheme S is quasi-excellent and if the morphism f is separated, then one can replace the functor f^* by $f^!$: the local constructibility of $f^!(M)$ implies the same property for M.*

Proof. Assume that $f^*(M)$ ($f^!(M)$, respectively) is locally constructible (with S quasi-excellent in the respective case). It is harmless to assume that $f^*(M)$ (respectively $f^!(M)$) is constructible (in the respective case, we use that $u^* = u^!$ for any separated étale morphism of finite type).

As both constructible and locally constructible objects are stable under the operations $f_!$ and f^* (respectively f_* and $f^!$) for any separated morphism of finite type, using the localization triangles

$$j_!j^*(M) \to M \to i_!i^*(M) \to j_!j^*(M)[1]$$
$$(i_*i^!(M) \to M \to j_*j^!(M) \to i_*i^!(M)[1], \text{ respectively}),$$

for any closed immersion i with open complement j, we see that it is sufficient to prove that there exists a stratification $\{S_i\}$ of S such that, if we denote by $j_i : S_i \to S$ the embedding of each strata, each restriction $j_i^*(M)$ (respectively $j_i^!(M)$) is constructible. By virtue of [EGA4, 17.16.4], we may thus assume that $f = hg$ is the composition of a finite, faithfully flat and radicial morphism g with a finite surjective étale map h. But the functor g^* is an equivalence of categories with right adjoint $g_! \simeq g_*$ (Proposition 6.3.16), so that we get an isomorphism of functors $g^* \simeq g^!$. This means that the h-motive $g^*(M) \simeq g^!(M)$ is locally constructible, and we conclude with Corollary 6.3.17. □

6.3.19. Recall that an object M of a closed symmetric monoidal category $\mathscr{C}$ is *rigid* if there exists an object $M^\vee$ of $\mathscr{C}$ such that tensoring by $M^\vee$ is a right adjoint of the functor $A \mapsto A \otimes M$. One checks easily that an object M of $\mathscr{C}$ is rigid if and only if, for any other object N, the canonical map

$$\underline{\mathrm{Hom}}(M, \mathbb{1}) \otimes N \to \underline{\mathrm{Hom}}(M, N)$$

is an isomorphism, in which case we have a canonical isomorphism

$$M^\vee \simeq \underline{\mathrm{Hom}}(M, \mathbb{1}).$$

The latter characterization implies that, whenever $\mathscr{C}$ is a triangulated category, its rigid objects form a thick subcategory. Moreover, if ever the unit object of $\mathscr{C}$ is compact, then all the rigid objects are compact in $\mathscr{C}$. For instance, given any ring R, the rigid objects of the unbounded derived category of R-modules are precisely the perfect complexes of R-modules (up to isomorphism in $\mathrm{D}(R)$).

Lemma 6.3.20. *The property of being rigid in $\mathrm{DM_h}(X, R)$ is local for the étale topology: for an object M of $\mathrm{DM_h}(X, R)$, if there exists a surjective étale morphism $u : X' \to X$ such that $u^*(M)$ is a rigid object of $\mathrm{DM_h}(X', R)$, then M is rigid.*

Proof. As the formation of the internal Hom in $\mathrm{DM_h}$ commutes with the functor u^*, this follows right away from the fact that the functor u^* is conservative. □

A source of rigid objects is provided by the following proposition.

Proposition 6.3.21. *Let $f : X \to S$ be a morphism between noetherian schemes of finite dimension. Assume that f is the composition of a surjective finite radicial morphism $g : T \to S$ with a smooth and proper morphism $p : X \to T$. Then, for any integer $n \in \mathbf{Z}$, the* h*-motive $f_*(R_X)(n)$ is a rigid object in $\mathrm{DM_h}(S, R)$.*

Proof. By virtue of Proposition 6.3.16, the symmetric monoidal functor g^* is an equivalence of categories, with quasi-inverse g_*. It is thus sufficient to prove that $p_*(R_X)$ is a rigid object of $\mathrm{DM_h}(T, R)$, which follows from the general formalism of the six operations: see [CD12, Proposition 2.4.31]. □

DEFINITION 6.3.22. Let S be a noetherian scheme. An object of $\mathrm{DM}_{\mathrm{h}}(S,R)$ is said to be *strictly smooth* if it belongs to the smallest thick subcategory generated by objects of the form $f_*(R_X)(n)$ for f as in Proposition 6.3.21 and $n \in \mathbf{Z}$. An object M of $\mathrm{DM}_{\mathrm{h}}(S,R)$ is *smooth* if there exists a surjective étale morphism $u : T \to S$ such that $u^*(M)$ is strictly smooth.

LEMMA 6.3.23. *Let S be the spectrum of a field k with finite étale cohomological dimension. Then the category of locally constructible object of $\mathrm{DM}_{\mathrm{h}}(S,R)$ is the thick subcategory generated by objects of the form $f_*(R_X)(n)$ with X smooth and projective over a purely inseparable finite extension of k, with structural map $f : X \to S$, and $n \in \mathbf{Z}$.*

Proof. Since the (locally) constructible h-motives over S precisely are the compact objects of $\mathrm{DM}_{\mathrm{h}}(S,R)$ (see Theorem 5.2.4), it is sufficient to prove that the family of compact objects of the form $f_*(R_X)(n)$, for $f : X \to S$ projective, X regular, and $n \in \mathbf{Z}$, form a generating family of $\mathrm{DM}_{\mathrm{h}}(S,R)$. Corollary 5.5.12 implies that it is sufficient to consider the case of $R = \mathbf{Z}$. Let M be an object of $\mathrm{DM}_{\mathrm{h}}(S,\mathbf{Z})$ such that, for any f and n as above, we have

$$\mathbf{R}\mathrm{Hom}(f_*(\mathbf{Z}_X)(n), M) = 0.$$

We want to prove that $M = 0$. But then, we also have

$$\mathbf{R}\mathrm{Hom}(f_*(\mathbf{Z}_X)(n), M \otimes \mathbf{Q}) = \mathbf{R}\mathrm{Hom}(f_*(\mathbf{Z}_X)(n), M) \otimes \mathbf{Q} = 0.$$

Since the property we seek is known for $\mathbf{Q}$-linear coefficients (see [CD12, Corollary 4.4.3]), we see that $M \otimes \mathbf{Q} = 0$. It is thus sufficient to prove that $M/p = M \otimes^{\mathbf{L}} \mathbf{Z}/p\mathbf{Z}$ vanishes in $\mathrm{DM}_{\mathrm{h}}(X,\mathbf{Z}/p\mathbf{Z})$ for any prime number p. If p is the characteristic of K, we conclude with Corollary A.3.3. Otherwise, Corollary 5.5.4 implies that $M = 0$, because the objects of the form $\mathbf{Z}(X)/p \simeq f_*(\mathbf{Z}_X)/p$, for $f : X \to S$ any Galois covering, do form a generating family of $\mathrm{D}(S_{\text{ét}},\mathbf{Z}/p\mathbf{Z})$. □

DEFINITION 6.3.24. A property $\mathbf{P}$ of R-linear h-motives is said to be *generic* if it satisfies the following conditions.

(g1) Given any noetherian scheme of finite dimension X, the objects of $\mathrm{DM}_{\mathrm{h}}(X,R)$ which have property $\mathbf{P}$ form a thick subcategory, which we will denote by $\mathbf{P}(X)$.

(g2) For any morphism between noetherian schemes of finite dimension $f : X \to Y$, the pullback functor sends $\mathbf{P}(Y)$ in $\mathbf{P}(X)$.

(g3) If S is the spectrum of a separably closed field, then any object of $\mathbf{P}(S)$ is locally constructible.

(g4) For any integral noetherian scheme of finite dimension X with generic point η, if M and N are two objects of $\mathbf{P}(X)$, then the canonical map

$$\varinjlim_{v:V\to X} \mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(V,R)}(v^*(M), v^*(N)) \to \mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(\bar{\eta},R)}(u^*(M), u^*(N)) \tag{6.3.24.a}$$

is an isomorphism of R-modules, where $v : V \to X$ runs over the étale neighborhoods of η, while $u : \bar{\eta} \to X$ denotes a geometric point associated with η.

(g5) Any strictly smooth object has property $\mathbf{P}$ (over noetherian schemes of finite dimension).

LEMMA 6.3.25. *Let X be a noetherian scheme of finite dimension and R a good enough ring of coefficients. Assume that a generic property $\mathbf{P}$ is defined. For any object M of $\mathrm{DM}_{\mathrm{h}}(X,R)$ which has property $\mathbf{P}$, there exists a dense open immersion $j : U \to X$ such that the restriction $M_{|U} = j^*(M)$ is smooth.*

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Proof. We may always assume that X is reduced, and replace X by any dense open subscheme at will. It is thus sufficient to consider the case where X is integral. For a noetherian scheme of finite dimension Y, let us write $\mathrm{DM}_{\mathrm{h,ss}}(Y,R)$ for the thick subcategory of strictly smooth objects in $\mathrm{DM}_{\mathrm{h}}(Y,R)$. Conditions (g1), (g2), (g3) and (g5) of Definition 6.3.24 then imply that we have the commutative triangle of triangulated functors below, in which $v : V \to X$ runs over the étale neighborhoods of η, while $u : \bar{\eta} \to X$ denotes a generic geometric point (i.e. a separable closure of the field of functions on X).

$$\begin{tikzcd}
2\text{-}\varinjlim_V \mathrm{DM}_{\mathrm{h,ss}}(V,R) \arrow[rr, "(1)"] \arrow[dr, "(3)"'] & & 2\text{-}\varinjlim_V \mathbf{P}(V) \arrow[dl, "(2)"] \\
& \mathrm{DM}_{\mathrm{h,lc}}(\bar{\eta},R) &
\end{tikzcd}$$

Condition (g4) ensures that (2) is fully faithful. As (1) obviously has the same property, (3)=(2)$\circ$(1) must be fully faithful as well. Using standard limit arguments [EGA4, Theorem 8.10.5, Proposition 17.7.8] together with Lemma 6.3.23, we see that the thick subcategory generated by the essential image of (3) is the whole category $\mathrm{DM}_{\mathrm{h,lc}}(\bar{\eta},R)$. Hence the functor (3) is an equivalence of categories. Therefore, all the functors in the commutative triangle above are equivalences of categories. In particular, the essential surjectivity of (1) tells us that, for any object M of $\mathbf{P}(X)$, there exists a dense open subscheme $U \subset X$ and a surjective étale morphism $v : V \to U$ such that $v^*(M_{|U})$ is strictly smooth. □

Theorem 6.3.26. *Let X be a noetherian scheme of finite dimension, and R a good enough ring of coefficients. For an object M of $\mathrm{DM}_{\mathrm{h}}(X,R)$, the following conditions are equivalent.*

(i) *The h-motive M is locally constructible in $\mathrm{DM}_{\mathrm{h}}(X,R)$.*

(ii) *There exists a stratification $\{X_i\}_{i\in I}$ by locally closed subschemes of X, such that the restriction $M_{|X_i}$ is smooth in $\mathrm{DM}_{\mathrm{h}}(X_i,R)$ for all $i \in I$.*

(iii) *There exists a stratification $\{X_i\}_{i\in I}$ by locally closed subschemes of X, such that the restriction $M_{|X_i}$ is rigid in $\mathrm{DM}_{\mathrm{h}}(X_i,R)$ for all $i \in I$.*

Proof. The property of being locally constructible is generic (conditions (g1), (g2) and (g3) of Definition 6.3.24 are obvious, while conditions (g4) and (g5) follow right away from Proposition 6.3.7, and Corollary 6.3.14, respectively). Therefore, a suitable noetherian induction, together with Lemma 6.3.25, shows that (i)$\Rightarrow$(ii). The implication (ii)$\Rightarrow$(i) follows from Proposition 6.3.18. After Lemma 6.3.20 and Proposition 6.3.21, it is obvious that (ii)$\Rightarrow$(iii).

It remains to prove that (iii)$\Rightarrow$(i). By virtue of Proposition 6.3.18, this amounts to prove that any rigid R-linear h-motive is locally constructible. Note that rigid objects are stable by inverse image functors of the form f^*, because symmetric monoidal functors always preserve rigid objects. Hence, using noetherian induction together with Lemma 6.3.25, we see that it is sufficient to prove that the property of being rigid is generic. We already know that condition (g1) of Definition 6.3.24 holds, and we have just seen why condition (g2) holds. To prove condition (g3), we remark that, if S is the spectrum of a separably closed field, then the locally constructible objects of $\mathrm{DM}_{\mathrm{h}}(S,R)$ are precisely the compact objects (by Theorems 1.1.5 and 5.2.4). Therefore, it is sufficient to prove that any rigid object is compact in $\mathrm{DM}_{\mathrm{h}}(S,R)$, which readily follows from the fact that the unit object R_S is compact. Since condition (g5) is already known (Lemma 6.3.20 and Proposition 6.3.21), it remains to prove condition (g4). We will prove a slightly better property. Let M and N be two rigid objects of $\mathrm{DM}_{\mathrm{h}}(X,R)$, and pick a point x in X. If we let

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$v : V \to X$ run over the family of étale neighborhoods of x, and if we let $u : S = \mathrm{Spec}(\mathscr{O}^{\mathrm{sh}}_{X,x}) \to X$ denote the strict henselization at x, then the canonical map

$$\varinjlim_{v:V\to X} \mathrm{Hom}(v^*(M), v^*(N)) \to \mathrm{Hom}(u^*(M), u^*(N)) \tag{6.3.26.a}$$

is an isomorphism. Indeed, we have the canonical isomorphisms below:

$$\begin{aligned}
\varinjlim_{v:V\to X} \mathrm{Hom}(v^*(M), v^*(N)) &\simeq \varinjlim_{v:V\to X} \mathrm{Hom}(v^*(R_X), v^*(M)^\vee \otimes^{\mathbf{L}}_R v^*(N))\\
&\simeq \varinjlim_{v:V\to X} \mathrm{Hom}(v^*(R_X), v^*(M^\vee \otimes^{\mathbf{L}}_R N))\\
&\simeq \mathrm{Hom}(u^*(R_X), u^*(M^\vee \otimes^{\mathbf{L}}_R N)) \quad \text{(see Remark 6.3.8)}\\
&\simeq \mathrm{Hom}(R_S, u^*(M)^\vee \otimes^{\mathbf{L}}_R u^*(N))\\
&\simeq \mathrm{Hom}(u^*(M), u^*(N)).
\end{aligned}$$

This shows the invertibility of the map (6.3.24.a) in the case where X is integral and x is its generic point. □

Remark 6.3.27. Assume finally that R is of positive characteristic invertible in $\mathscr{O}_X$. Then $\mathrm{D}(X_{\text{ét}}, R) \simeq \mathrm{DM}_{\mathrm{h}}(X, R)$ (Corollary 5.5.4), and this implies that any R-linear rigid h-motive is smooth. This is because any rigid object of $\mathrm{D}(X_{\text{ét}}, R)$ is locally isomorphic to a constant sheaf of complexes associated with a perfect complex of R-modules. Although this certainly is a folkloric result, we include a proof here. If S is a strictly henselian scheme with closed point s, then taking the fiber of sheaves of R-modules at s is the same thing as taking the global sections. For two rigid objects M and N of $\mathrm{D}(S_{\text{ét}}, R)$, we thus have

$$\mathbf{R}\mathrm{Hom}_{\mathrm{D}(S_{\text{ét}},R)}(M, N) \simeq (M^\vee \otimes^{\mathbf{L}}_R N)_s \simeq \mathbf{R}\mathrm{Hom}_{\mathrm{D}(R)}(M_s, N_s),$$

from which we get

$$\mathrm{Hom}_{\mathrm{D}(S_{\text{ét}},R)}(M, N) \simeq \mathrm{Hom}_{\mathrm{D}(R)}(M_s, N_s).$$

For a geometric point x of X, the constant sheaf of complexes associated with a perfect complex of R-modules is obviously rigid, while taking the fiber at x defines a symmetric monoidal functor and thus sends rigid objects to perfect complexes of R-modules (because the latter are the rigid objects of $\mathrm{D}(R)$). Hence we deduce from what precedes and from the isomorphism (6.3.26.a) that taking the fiber at x defines an equivalence of triangulated categories

$$2\text{-}\varinjlim_{V} \mathrm{D}_{\mathrm{rig}}(V_{\text{ét}}, R) \simeq \mathrm{D}_{\mathrm{perf}}(R),$$

where V runs over the family of étale neighborhoods of x, $\mathrm{D}_{\mathrm{rig}}(V_{\text{ét}}, R)$ denotes the thick subcategory of rigid objects in $\mathrm{D}(V_{\text{ét}}, R)$, and $\mathrm{D}_{\mathrm{perf}}(R)$ is the triangulated category of perfect complexes of R-modules. In particular, if two rigid objects M and N in $\mathrm{D}(X_{\text{ét}}, R)$ have isomorphic fibers at x in $\mathrm{D}(R)$, then there exists an étale neighborhood $v : V \to X$ of x such that $v^*(M)$ and $v^*(N)$ are isomorphic in $\mathrm{D}(V_{\text{ét}}, R)$. This applies to any rigid object M, with N the constant sheaf associated with the fiber of M at x.

We do not know if, for a general ring of coefficients R, any rigid h-motive is smooth or not (except in the very particular situation of Lemma 6.3.23).

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7. Applications

7.1 Algebraic cycles in étale motivic cohomology

7.1.1. Let us fix an integer $n \geqslant 0$.

Consider a smooth k-scheme X of finite type. We let z^n_X be the presheaf on $X_{\text{ét}}$ which with an étale X-scheme U associates Bloch cycle complex $z^n(U,*)[-2n]$ (as in [GL01a, § 2.2]). On the other hand, let $\mathbf{Z}_{SV}(n)$ be Suslin–Voevodsky's motivic complex of Nisnevich sheaves on Sm_k. According to [Voe02, Theorem 1], there is a canonical quasi-isomorphism of complexes of Nisnevich sheaves on the site of étale X-schemes:

$$z^n_X \xrightarrow{\sim} (\mathbf{Z}_{SV}(n))|_{X_{\text{ét}}}.$$

Recall also that, by definition,

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(k,\mathbf{Z})}(M(X),\mathbf{Z}(n)[i]) \simeq H^i_{\text{ét}}(X, L_{\mathbf{A}^1}(\mathbf{Z}_{SV}(n)_{\text{ét}}))$$

where $L_{\mathbf{A}^1}$ is the $\mathbf{A}^1$-localization functor of effective étale motivic complexes. Thus, we deduce from the previous corollary a canonical map

$$\rho^{i,n}_X : H^i_{\text{ét}}(X, z^n_X) \to \mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(k,\mathbf{Z})}(\mathbf{Z}(X),\mathbf{Z}(n)[i])$$

which, up to the isomorphisms described previously, is induced by the canonical map:

$$\mathbf{Z}_{SV}(n)_{\text{ét}} \to L_{\mathbf{A}^1}(\mathbf{Z}_{SV}(n)_{\text{ét}}).$$

We recall the following theorem.

Theorem 7.1.2. *Consider the above notations and let p be the characteristic exponent of k. then $\rho^{n,i}_X$ induce an isomorphism after tensorization by $\mathbf{Z}[1/p]$.*

Proof. We want to show that the map

$$\mathbf{R}\Gamma(X_{\text{ét}}, z^n_X)[1/p] \simeq \mathbf{R}\Gamma(X, \mathbf{Z}_{SV}(n)_{\text{ét}})[1/p] \to \mathbf{R}\mathrm{Hom}_{\mathrm{DM}_{\mathrm{h}}(k,\mathbf{Z})}(\mathbf{Z}(X),\mathbf{Z}(n))$$

is an isomorphism in the derived category of abelian groups. It is sufficient to check that it induces an isomorphism after we apply the functor $C \mapsto C \otimes^{\mathbf{L}} R$ for $R = \mathbf{Q}$ or $R = \mathbf{Z}/\ell\mathbf{Z}$ for prime numbers $\ell \neq p$. For $R = \mathbf{Q}$, this readily follows from Voevodsky's comparison theorem [Voe02] (using Corollary 5.5.5(3), as well as the equivalence $\mathrm{DM}^{\mathrm{eff}}(k,\mathbf{Q}) \simeq \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(k,\mathbf{Q})$). For $R = \mathbf{Z}/\ell\mathbf{Z}$, it is sufficient to check that the map

$$\mathbf{Z}_{SV}(n)_{\text{ét}} \otimes^{\mathbf{L}} \mathbf{Z}/\ell\mathbf{Z} \to L_{\mathbf{A}^1}\big(\mathbf{Z}^{SV}(n)_{\text{ét}}\big) \otimes^{\mathbf{L}} \mathbf{Z}/\ell\mathbf{Z}$$

is a quasi-isomorphism. But this map is an $\mathbf{A}^1$-equivalence with $\mathbf{A}^1$-local codomain. It is thus sufficient to check that the left-hand side is $\mathbf{A}^1$-local as well. By virtue of Corollary 4.5.4, it is sufficient to prove that the cohomology sheaves of the tensor product $\mathbf{Z}_{SV}(n)_{\text{ét}} \otimes^{\mathbf{L}} \mathbf{Z}/\ell\mathbf{Z}$ are locally constant. But this readily follows from the rigidity theorem of Suslin and Voevodsky [SV96, Theorem 4.4] (see [MVW06, Theorem 7.20]). □

Remark 7.1.3. The preceding theorem and its proof are well known. For instance, using Voevodsky's comparison theorem [Voe02], one can find them in [MVW06, 10.2 and 14.27] under the assumption that the field k is of finite cohomological dimension (the later assumption being used to prove Corollary 4.5.4 in the case of $X = \mathrm{Spec}(k)$).

Remark 7.1.4. The source of $\rho_X^{n,i}$ is an important invariant. Let us mention in particular the result [GL00, Theorem 8.3]: if $p > 0$, z_X^n/p^r is isomorphic to the logarithmic De Rham Witt sheaf ν_r^n placed in degree n. This fact alone explains the failure of homotopy invariance of the cohomology $H^*_{\text{ét}}(X, z_X^n)$, equivalent to the failure injectivity for $\rho_X^{n,i}$.[15] This can also be explained by saying that the étale sheafification functor, which goes from Nisnevich complexes to étale complexes of sheaves on Sm_k, does not preserve $\mathbf{A}^1$-local objects. In fact, in characteristic $p > 0$, this functor does not even preserves $\mathbf{A}^1$-invariant sheaves because of the Artin–Schreier étale covers of the affine line.

We will now explain the strong relationship of classical Chow groups with étale motivic cohomology in weight n and degree $2n$ for regular schemes, by combining the absolute purity theorem for étale motives and the fact that the Bloch–Kato conjecture is true; see Theorem 7.1.11 below.

7.1.5. The coniveau filtration and its associated spectral sequence is very well documented in the literature, under an axiomatic treatment. However, the authors usually require a base field in their axioms.[16] It is clearly not necessary so let us quickly recall the construction of this spectral sequence in the case of étale motivic cohomology, and more precisely its version with support:

$$\mathrm{H}^{r,n}_{\text{ét}}(X, Z) = \mathrm{Hom}_{\mathrm{DM}_h(X)}(i_*(\mathbb{1}_Z), \mathbb{1}_X(n)[r]).$$

where $i: Z \to X$ is closed immersion.

First, one defines a *flag* on X has a decreasing sequence $(Z^p)_{p\in\mathbf{Z}}$ of closed subschemes of X such that the following hold.

- For all integer $p \geqslant 0$, Z^p is of codimension greater or equal to p in X.
- For $p < 0$, we have $Z^p = X$.

We let $\mathscr{D}(X)$ be the set of flags of X, ordered by term-wise inclusion. It is an easy fact it is right filtering.

Given such a flag Z_*, and a fixed integer $n \in \mathbf{Z}$, we define an exact couple, denoted by $(D(Z^*, n), E_1(Z^*, n))$ (with cohomological conventions, see [McC01, Theorem 2.8]), as follows

$$\begin{tikzcd}
D^{p-1,q}(Z^*, n) \arrow[r] \arrow[d, equal] & E_1^{p,q}(Z^*, n) \arrow[r] \arrow[d, equal] & D^{p,q}(Z^*, n) \arrow[r] \arrow[d, equal] & D^{p-1,q+1}(Z^*, n) \arrow[d, equal] \\
\mathrm{H}^{p+q-1,n}_{\text{ét}}(X - Z^p) & \mathrm{H}^{p+q,n}_{\text{ét}}(X - Z^{p+1}, Z^p - Z^{p+1}) & \mathrm{H}^{p+q,n}_{\text{ét}}(X - Z^{p+1}) & \mathrm{H}^{p+q,n}_{\text{ét}}(X - Z^p)
\end{tikzcd}$$

where the morphisms are given by localization long exact sequence of cohomology with support associated with the closed immersion: $(Z^p - Z^{p+1}) \to (X - Z^{p+1})$.[17] This exact couple is obviously contravariantly functorial in Z^* as follows from the six functors formalism (more precisely, we need the proper base change theorem with respect to functor i_*, i a closed immersion).

The coniveau exact couple associated with X is obtained by taking the colimit of these exact couples as Z^* runs in the set of flags of X:

$$(D(X, n), E_1(X, n)) = \varinjlim_{Z^* \in \mathscr{D}(X)} (D(Z^*, n), E_1(Z^*, n)).$$

[15] Compare this with the general fact Corollary A.3.3.

[16] The reason for doing so is that at this moment we do not know if Gersten conjecture holds for all regular schemes of unequal characteristics, either for K-theory or torsion étale cohomology.

[17] This sequence is induced by the corresponding localization triangle in DM_h, which exists according to Theorem 5.6.2.

Before stating the main result, we need a final notation. Let $x \in X$ be any point, and Z be its reduced closure in X. Then we will consider the following cohomology groups:

$$\hat{\mathrm{H}}^{r,n}_{\text{ét}}(X_{(x)}, x) = \varinjlim_{U} \mathrm{H}^{r,n}_{\text{ét}}(U, Z \cap U),$$
$$\hat{\mathrm{H}}^{r,n}_{\text{ét}}(\kappa(x)) = \varinjlim_{U} \mathrm{H}^{r,n}_{\text{ét}}(Z \cap U),$$

where U runs over the open neighborhood of x in X.

Proposition 7.1.6. *Consider the notations above and assume that X is excellent and regular. Then for any integers $p, q \in \mathbf{Z}$, there exists canonical isomorphisms:*

$$E_1^{p,q}(X,n) \underset{(1)}{\simeq} \bigoplus_{x \in X^{(p)}} \hat{\mathrm{H}}^{p+q,n}_{\text{ét}}(X_{(x)}, x) \underset{(2)}{\simeq} \bigoplus_{x \in X^{(p)}} \hat{\mathrm{H}}^{q-p,n-p}_{\text{ét}}(\kappa(x)) \underset{(3)}{\simeq} \bigoplus_{x \in X^{(p)}} \mathrm{H}^{q-p,n-p}_{\text{ét}}(\kappa(x)).$$

In particular, we get the usual form of the *coniveau spectral sequence*, associated with the above exact couple:

$$E_1^{p,q}(X,n) = \bigoplus_{x \in X^{(p)}} \mathrm{H}^{q-p,n-p}_{\text{ét}}(\kappa(x)) \Rightarrow \mathrm{H}^{p+q,n}_{\text{ét}}(X). \tag{7.1.6.a}$$

Proof. The isomorphism (1) only uses the additivity in Z of cohomology with support, $\mathrm{H}^{**}_{\text{ét}}(X, Z \sqcup Z') \simeq \mathrm{H}^{**}_{\text{ét}}(X, Z) \oplus \mathrm{H}^{**}_{\text{ét}}(X, Z')$, which is obvious according to our definition.

The isomorphism (2) uses the absolute purity property for DM_h (Theorem 5.6.2) together with the fact that any integral closed subscheme $Z \subset X$ has a dense regular locus (cf. [SGA4, 7.8.6]).

Finally, the isomorphism (3) uses the continuity property of $\mathrm{DM}_{\mathrm{h,c}}$ (see Theorem 6.3.9). □

7.1.7. Let x be any point of X. Let us denote by p_x the exponential characteristic of $\kappa(x)$. Then we get the following canonical isomorphisms. Thus, according to Proposition A.3.1, Corollary 5.5.5, and the Bloch–Kato conjecture (more precisely [Voe11, Theorem 6.17]), we get the following isomorphisms for any integers r, n such that $r \leqslant n+1$:

$$\mathrm{H}^{r,n}_{\text{ét}}(\kappa(x)) \simeq \mathrm{H}^{r,n}_{\text{ét}}(\kappa(x))[p_x^{-1}] \simeq H^{r,n}(\kappa(x))[p_x^{-1}]$$

where the right-hand side denotes the motivic cohomology groups of the field $\kappa(x)$ with $\mathbf{Z}[p_x^{-1}]$-coefficients. Recall the later groups are zero if in addition $n < 0$ or $r > n$.

Corollary 7.1.8. *Under the assumptions of the previous proposition and with the above notations, one gets, for any integers p, q,*

$$E_1^{p,q}(X,n) = \begin{cases} 0 & \text{if } q = n+1, (q < n+1, p > n), (q < n, p = n), \\ \bigoplus_{x \in X^{(p)}} K^M_{n-p}(\kappa(x))[p_x^{-1}] & \text{if } q = n, \end{cases}$$

where K^M_* denotes Milnor K-theory.

7.1.9. Thus, from the coniveau spectral sequence, one deduces the following maps:

$$E_2^{n,n}(X,n) \xrightarrow{b} E_\infty^{n,n}(X,n) \xrightarrow{a} \mathrm{H}^{2n,n}_{\text{ét}}(X,n). \tag{7.1.9.a}$$

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where b is an epimorphism, and an isomorphism if $n \leqslant 2$, while a is always a monomorphism. Moreover, we get a short exact sequence:

$$\bigoplus_{y \in X^{(n-1)}} \kappa(y)^\times[1/p_y] \xrightarrow{d_1} \bigoplus_{x \in X^{(n)}} \mathbf{Z}[1/p_x] \xrightarrow{c} E_2^{n,n}(X,n) \to 0, \tag{7.1.9.b}$$

where d_1 is the differential of the E_1-page of the coniveau spectral sequence with source the $(n-1,n)$-term.

Let N be the set of made of the exponential characteristics of the residue fields of X. Then, if we tensor the above short exact sequence with $\mathbf{Z}[N^{-1}]$, the middle term becomes the group of n-codimensional cycles with $\mathbf{Z}[N^{-1}]$-coefficients. To finish our study of the coniveau spectral sequence, we notice the following critical point (analog of [Qui73, Proposition 5.14]).

Proposition 7.1.10. *Consider the notations above. Then the differential of the coniveau spectral sequence*

$$d_1 : \bigoplus_{y \in X^{(n-1)}} \kappa(y)^\times[p_y^{-1}] = E_1^{n-1,n}(X,n) \to E_1^{n,n}(X,n) = \bigoplus_{x \in X^{(n)}} \mathbf{Z}[1/p_x]$$

is the usual divisor class map: given $(y,x) \in X^{(n-1)} \times X^{(n)}$ such that $y \in Z^{(1)}$ where Z is the reduced closure of x in X, the component $(d_1)_x^y$ is the order function of the local one-dimensional excellent ring $\mathscr{O}_{Z,y}$ up to the denominators indicated.

Proof. The first step is to reduce to the case where X is local regular of dimension 1, y being its closed point.

This reduction works as in [Dég12, 1.16]. Though this proof is written for k-schemes, it works equally fine if one uses the fact that étale motivic cohomology admits Gysin maps between regular schemes for finite morphisms (see [Dég14, § 6]) and the fact these Gysin maps commute with residue morphisms: more precisely, given any cartesian square

$$\begin{array}{ccc} Z' & \rightarrow & T' \\ \downarrow & & \downarrow f \\ Z & \xrightarrow{i} & T \end{array}$$

of regular schemes, such that f is finite and i is a (codimension 1) closed immersion, the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}^{**}_{\text{ét}}(T'-Z') & \xrightarrow{\partial_{T',Z'}} & \mathrm{H}^{**}_{\text{ét}}(Z') \\ h_* \downarrow & & \downarrow f_* \\ \mathrm{H}^{**}_{\text{ét}}(T-Z) & \xrightarrow{\partial_{T,Z}} & \mathrm{H}^{**}_{\text{ét}}(Z) \end{array}$$

where f_* (respectively h_*) is the Gysin morphism mentioned above and $\partial_{T,Z}$ is obtained from the canonical (boundary) map

$$\mathrm{H}^{**}_{\text{ét}}(T-Z) \to \mathrm{H}^{**}_{\text{ét}}(T,Z)$$

using the purity isomorphism: $\mathrm{H}^{**}_{\text{ét}}(T,Z) \simeq \mathrm{H}^{**}_{\text{ét}}(Z)$. Over a field, this commutativity has been proved in [Dég08a, 5.15]. The absolute case considered here is treated likewise using the absolute purity property.

To treat the remaining case, $X = \operatorname{Spec}(A)$ with A a discrete valuation ring, we thus have to prove that d_1 is the valuation map of A. In this case d_1, is the residue map $\mathrm{H}^{1,1}_{\text{ét}}(X-Z) \to \mathrm{H}^{0,0}_{\text{ét}}(Z)$, Z being the closed point of X. Thus d_1 obviously sends units to 0, and because it is additive, we have only to prove that $d_1(\pi) = 1$ where π is a uniformizing parameter of A. This last property follows from the definition of the absolute purity isomorphism (cf. Appendix A and especially Theorem A.2.8) and a careful computation with the deformation space (see the proof of [Dég08b, 2.6.5]). $\square$

One can summarize the information obtained from the above proposition and its preceding paragraph by the following commutative diagram

$$\begin{tikzcd}
\bigoplus_{y\in X^{(n-1)}} \kappa(y)^\times \arrow[r, "\mathrm{div}"] \arrow[d, hook] & Z^n(X) \arrow[d, hook] & \\
\bigoplus_{y\in X^{(n-1)}} \kappa(y)^\times[1/p_y] \arrow[r, "d_1"] & \bigoplus_{x\in X^{(n)}} \mathbf{Z}[1/p_x] \arrow[r, "abc"] & \mathrm{H}^{2n,n}_{\text{ét}}(X)
\end{tikzcd}$$

where the maps a, b and c are those of (7.1.9.a) and (7.1.9.b) and the map div is the usual divisor class map with values the n-codimensional algebraic cycles of X. Thus taking care of the previous study, together with Theorem 5.2.2, one gets the following result.

Theorem 7.1.11. *Let X be a regular excellent scheme and N be the set of integers made by the exponential characteristics of all the residue fields of X.*

Then for any integer $n \geqslant 0$, the above diagram induces a canonical morphism of abelian groups

$$\sigma^n : CH^n(X) \to \mathrm{H}^{2n,n}_{\text{ét}}(X)$$

which satisfies moreover the following properties.

(1) *The morphism $\sigma^n \otimes \mathbf{Q}$ is an isomorphism.*
(2) *The morphism $\sigma^1 \otimes \mathbf{Z}[N^{-1}]$ is an isomorphism.*
(3) *There exists a short exact sequence*

$$0 \to CH^2(X)[N^{-1}] \xrightarrow{\sigma^2} \mathrm{H}^{4,2}_{\text{ét}}(X)[N^{-1}] \to \mathrm{H}^{4,2}_{\text{ét},nr}(X)[N^{-1}] \to 0$$

where $\mathrm{H}^{4,2}_{\text{ét},nr}(X)$ is the kernel of the differential

$$d_1^{4,2} : \mathrm{H}^{4,2}_{\text{ét}}(\kappa(X)) \to \bigoplus_{x\in X^{(1)}} \mathrm{H}^{3,1}_{\text{ét}}(\kappa(x)),$$

in the spectral sequence (7.1.6.a).

Remark 7.1.12. (1) The map σ^n is the *étale cycle class* map. The new information here is that it exists with integral coefficients and, if one inverts the exponential characteristics of X, is an isomorphism for $n = 1$ and a monomorphism for $n = 2$.

Note that the method gives the following explicit way to determine the étale class of a cycle in X: take a reduced closed subscheme $Z \subset X$; there exists an open subscheme $U \subset X$ such that $Z \cap U$ is regular and dense in Z; then the closed immersion $i : Z \cap U \to U$ induces a Gysin map

$$i_* : \mathrm{H}^{**}_{\text{ét}}(Z \cap U) \to \mathrm{H}^{**}_{\text{ét}}(U)$$

and the restriction to U of $\sigma^*(\langle Z\rangle)$ equals $i_*(1)$. The latter is usually called the fundamental class of $Z \cap U$ in U.

(2) The previous method gives back the construction of the cycle class in torsion étale cohomology (cf. [SGA4$\frac{1}{2}$]). The construction used here is more direct but it uses the absolute purity property.

(3) Along the lines of the equal characteristics case, one can show that σ^* is compatible with push-forwards with respect to projective maps between regular schemes, where on the left-hand side one considers the usual functoriality of Chow groups and on the right-hand side the Gysin morphisms of [Dég14]; this is a Riemann–Roch formula where, because the oriented theories CH^* and $\mathrm{H}^{**}_{\text{ét}}$ have an additive formal group law, the Todd class is equal to 1.

(4) It is possible to extend the previous result to the case of a singular scheme X which is separated of finite type over a regular scheme S. Given $f : X \to S$ the corresponding structural morphism, one defines the Borel–Moore motivic étale cohomology of X/S as

$$\mathrm{H}^{BM,\text{ét}}_{r,n}(X/S) = \mathrm{Hom}_{\mathrm{DM}_h(X)}(\mathbb{1}_X(n)[r], f^!(\mathbb{1}_S)).$$

The niveau spectral sequence for this Borel–Moore homology is defined as in the case of coniveau but replacing the indexing by codimension with the one by dimension. One then gets, using similar arguments, a cycle map:

$$\sigma_* : CH_*(X) \to \mathrm{H}^{BM,\text{ét}}_{**}(X/S).$$

The only remark to be done is that one has to take care of the dimension of S which will appear in the computation of the E_1-term of the niveau spectral sequence through absolute purity.

7.2 Completion and ℓ-adic realization

In this section, we fix a discrete valuation ring R with local parameter ℓ. We will write R/ℓ^r for the quotient ring $R/(\ell^r)$, $r \geqslant 0$. Until Paragraph 7.2.18, there is not any constraint on the characteristic of the field $R/(\ell)$; only at this point, the characteristic will be positive.

Definition 7.2.1. Let X be a noetherian scheme.

We denote by $\mathrm{DM}_h(X, \hat{R}_\ell)$ the localizing subcategory of $\mathrm{DM}_h(X, R)$ generated by the objects of the form $M/\ell = R/\ell \otimes^{\mathbf{L}}_R M$, for any constructible object M of $\mathrm{DM}_\mathrm{h}(X, R)$.

7.2.2. Recall from § 5.4 the following adjunctions of triangulated categories, expressing various change of coefficients:

$$\mathbf{L}\rho_\ell^* : \mathrm{DM}_h(X, R) \rightleftarrows \mathrm{DM}_h(X, R/\ell) : \rho_{\ell *},$$
$$\mathbf{L}\rho^* : \mathrm{DM}_h(X, R) \rightleftarrows \mathrm{DM}_h(X, R[\ell^{-1}]) : \rho_*,$$

where $\rho_\ell^*(M) = M/\ell$ and $\rho^*(M) = R[\ell^{-1}] \otimes M$. Note that, for any h-motive M in $\mathrm{DM}_h(X, R)$, the h-motive $R[\ell^{-1}] \otimes M$ is the homotopy colimit of the tower

$$M \xrightarrow{\ell.1_M} M \xrightarrow{\ell.1_M} M \to \cdots \to M \xrightarrow{\ell.1_M} M \to \cdots .$$

Moreover, the functor ρ_* is fully faithful, and identifies $\mathrm{DM}_h(X, R[\ell^{-1}])$ with the full subcategory of $\mathrm{DM}_h(X, R)$ whose objects are those on which the multiplication by ℓ is invertible. Such an object will be said *uniquely ℓ-divisible*.

Lemma 7.2.3. *For an object M of $\mathrm{DM}_\mathrm{h}(X, R)$, the following conditions are equivalent.*

(i) *The* h*-motive M is uniquely ℓ-divisible.*

(ii) *The* h*-motive $M/\ell \simeq 0$.*

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(iii) *For any constructible object C of $\mathrm{DM_h}(X,R)$, any map $C/\ell \to M$ is zero.*

(iv) *For any object C of $\mathrm{DM}_h(X,\hat{R}_\ell)$, any map from C to M is zero.*

Proof. The equivalence between conditions (i) and (ii) is trivial (in view of the distinguished triangle (5.4.4.b)), and the equivalence between conditions (iii) and (iv) is true by definition of $\mathrm{DM}_h(X,\hat{R}_\ell)$. The equivalence between conditions (ii) and (iii) comes from the fact that the objects of the form C/ℓ, with C constructible in $\mathrm{DM_h}(X,R)$, form a generating family of the triangulated category $\mathrm{DM_h}(X,\mathbf{Z}/\ell\mathbf{Z})$. □

7.2.4. We are thus in the *situation of the six gluing functors* as defined in [Nee01, 9.2.1]. This means that we have six functors

$$\mathrm{DM_h}(X,\hat{R}_\ell) \underset{\hat{\rho}_{\ell *}}{\overset{\hat{\rho}_{\ell !}}{\underset{\xleftarrow{\hat{\rho}_\ell^*}}{\rightrightarrows}}} \mathrm{DM_h}(X,R) \underset{\rho^!}{\overset{\mathbf{L}\rho^*}{\underset{\xleftarrow{\rho_*}}{\rightrightarrows}}} \mathrm{DM_h}(X,R[\ell^{-1}]) \tag{7.2.4.a}$$

where $\hat{\rho}_{\ell !}$ denotes the inclusion functor, and that, for any h-motive in $\mathrm{DM}_h(X,R)$ we have functorial distinguished triangles

$$\hat{\rho}_{\ell !}\hat{\rho}_\ell^*(M) \xrightarrow{ad(\hat{\rho}_{\ell !},\hat{\rho}_\ell^*)} M \xrightarrow{ad'(\mathbf{L}\rho^*,\rho_*)} \rho_*\mathbf{L}\rho^*(M) \to M[1], \tag{7.2.4.b}$$

$$\rho_*\rho^!(M) \xrightarrow{ad(\rho_*,\rho^!)} M \xrightarrow{ad'(\hat{\rho}_\ell^*,\hat{\rho}_{l*})} \hat{\rho}_{\ell *}\hat{\rho}_\ell^*(M) \to M[1]. \tag{7.2.4.c}$$

Consider the obvious exact sequence of R-modules:

$$0 \to R \to R[\ell^{-1}] \to R[\ell^{-1}]/R \to 0.$$

It induces the following distinguished triangle in $\mathrm{DM}_h(X,R)$

$$M \otimes^{\mathbf{L}} (R[\ell^{-1}]/R)[-1] \longrightarrow M \longrightarrow M \otimes^{\mathbf{L}} R[\ell^{-1}] \longrightarrow M \otimes^{\mathbf{L}} (R[\ell^{-1}]/R)$$

which is isomorphic to the triangle (7.2.4.b). In other words, we have the formulas

$$\hat{\rho}_{\ell !}\hat{\rho}_\ell^*(M) = M \otimes^{\mathbf{L}} (R[\ell^{-1}]/R)[-1] \quad \text{and} \quad \rho_*\mathbf{L}\rho^*(M) = M[\ell^{-1}] = M \otimes \mathbf{Z}[\ell^{-1}].$$

7.2.5. Let M be a cofibrant object in the model category underlying $\mathrm{DM_h}(X,R)$. The h-motive M/ℓ^r is then represented by the complex of Tate spectra:

$$\mathrm{Coker}(M \xrightarrow{\ell^r.1_M} M).$$

Thus, we get a tower

$$\begin{tikzcd} M \arrow[r, "\ell"] \arrow[d, "\ell"'] & M \arrow[r] \arrow[d, "\ell^2"] & \cdots \arrow[r] & M \arrow[r, "\ell"] \arrow[d, "\ell^r"'] & M \arrow[r] \arrow[d, "\ell^{r+1}"] & \cdots \\ M \arrow[r, equal] & M \arrow[r, equal] & \cdots \arrow[r, equal] & M \arrow[r, equal] & M \arrow[r, equal] & \cdots \end{tikzcd} \tag{7.2.5.a}$$

which defines a projective system $(M/\ell^r)_{r\in\mathbf{N}}$, and it makes sense to take its derived limit. This construction defines a triangulated functor

$$\mathrm{DM_h}(X,R) \to \mathrm{DM_h}(X,R), \quad M \mapsto \mathbf{R}\varprojlim_r M/\ell^r.$$

Furthermore, the towers (7.2.5.a) define a natural transformation

$$\epsilon_\ell^M : M \to \mathbf{R}\varprojlim_{r\in\mathbf{N}} M/\ell^r. \tag{7.2.5.b}$$

Lemma 7.2.6. *For any* h*-motive* M *in* $\mathrm{DM}_h(X,R)$, *we have a canonical isomorphism:*

$$\mathbf{R}\underline{\mathrm{Hom}}_R(R[\ell^{-1}]/R, M)[1] \simeq \mathbf{R}\varprojlim_{r\in\mathbf{N}} M/\ell^r.$$

Proof. We have $R[\ell^{-1}]/R = \varinjlim_r R/\ell^r$. As this colimit is filtering, this is in fact an homotopy colimit, and we conclude from the isomorphisms $\mathbf{R}\underline{\mathrm{Hom}}(R/\ell^r, M)[1] \simeq M/\ell^r$. □

Definition 7.2.7. For any h-motive M in $\mathrm{DM}_h(X,R)$, we define the ℓ*-completion* of M as the h-motive

$$\hat{M}_\ell = \mathbf{R}\varprojlim_{n\in\mathbf{N}} M/\ell^r.$$

We say that M is ℓ*-complete* if the map $\epsilon_l^M : M \to \hat{M}_\ell$ defined above is an isomorphism.

According to Lemma 7.2.6 and Paragraph 7.2.4, the triangle (7.2.4.c) can be identified to the triangle

$$\mathbf{R}\underline{\mathrm{Hom}}(R[\ell^{-1}], M) \longrightarrow M \xrightarrow{\epsilon_\ell^M} \hat{M}_\ell \xrightarrow{+1} .$$

Note in particular the following well-known fact (see for instance [DG02]).

Proposition 7.2.8. *Let* M *be an* h*-motive in* $\mathrm{DM_h}(X,R)$. *Then the following conditions are equivalent.*

(i) *The* h*-motive* M *belongs to the essential image of* $\hat{\rho}_{\ell*} : \mathrm{DM_h}(X,\hat{R}_\ell) \to \mathrm{DM_h}(X,R)$.

(ii) *The* h*-motive* M *is* ℓ*-complete.*

(iii) *The* h*-motive* M *is left orthogonal to uniquely* ℓ*-divisible objects in* $\mathrm{DM_h}(X,R)$.

Lemma 7.2.6 readily implies the following computation, which means (at least when $R/(\ell)$ is of characteristic prime to the residue characteristics of X), in view of the equivalences $\mathrm{DM_h}(X, R/\ell^r) \simeq \mathrm{D}(X_{\text{ét}}, R/\ell^r)$, that the category $\mathrm{DM_h}(X,\hat{R}_\ell)$ is a categorical incarnation of continuous étale cohomology in the sense of Jannsen [Jan88].

Proposition 7.2.9. *For any objects* M *and* N *in* $\mathrm{DM_h}(X,\hat{R}_\ell)$, *we have*

$$\mathbf{R}\mathrm{Hom}_{\mathrm{DM_h}(X,\hat{R}_\ell)}(M,N) \simeq \mathbf{R}\varprojlim_r \mathbf{R}\mathrm{Hom}_{\mathrm{DM_h}(X,R/\ell^r)}(M/\ell^r, N/\ell^r).$$

7.2.10. The right adjoints $\mathbf{R}f_*$, $\mathbf{R}\underline{\mathrm{Hom}}$ commute with homotopy limits in $\mathrm{DM}_h(-,R)$. Moreover, Proposition 5.4.5 shows they preserve ℓ-complete objects.

On the other hand, for any morphism of scheme $f : Y \to X$, and smooth morphism $p : X \to S$ and any ℓ-complete h-motives M, N, we put

$$\hat{f}^*(M) = \mathbf{L}\widehat{f^*(M)}_\ell, \quad \hat{p}_\sharp(M) = \mathbf{L}\widehat{p_\sharp(M)}_\ell, \quad M\hat{\otimes}N = (\widehat{M\otimes^{\mathbf{L}} N})_\ell.$$

This defines a structure of a premotivic triangulated category on $\mathrm{DM_h}(-,\hat{R}_\ell)$, the right adjoints being induced by their counterparts in $\mathrm{DM_h}(-,R)$.

According to these definitions, we get a premotivic adjunction:

$$\hat{\rho}_\ell^* : \mathrm{DM}_h(-,R) \rightleftarrows \mathrm{DM}_h(-,\hat{R}_\ell) : \hat{\rho}_{\ell*}. \tag{7.2.10.a}$$

The functor $\hat{\rho}_\ell^*$ will be called the ℓ*-adic realization functor*. Moreover, $\hat{\rho}_\ell^*$ obviously commutes with f_* and $\underline{\mathrm{Hom}}$.

Taking into account Theorem 5.6.2, Corollary 5.4.11, Proposition 6.2.14, as well as Lemma 7.2.6, we thus obtain the following theorem.

Theorem 7.2.11. *The triangulated premotivic category* $\mathrm{DM}_h(-,\hat{R}_\ell)$ *satisfies the Grothendieck six functors formalism (Definition A.1.10) and the absolute purity property (Definition A.2.9) over noetherian schemes of finite dimension. The premotivic morphism* $\hat{\rho}_\ell^*$ *defined above commutes with the six operations (Definition A.1.17).*

Remark 7.2.12. Note that, if $R/(\ell)$ is of positive characteristic, by virtue of Theorem 5.5.3, if we perform this ℓ-completion procedure to $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(X,R)$ or $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{h}}(X,R)$, this leads to the same category $\mathrm{DM}_{\mathrm{h}}(-,\hat{R}_\ell)$.

Definition 7.2.13. Let X be any scheme. One defines the category $\mathrm{DM}_{\mathrm{h,gm}}(X,\hat{R}_\ell)$ of *geometric ℓ-adic* h-*motives* as the thick triangulated subcategory of $\mathrm{DM}_{\mathrm{h}}(X,\hat{R}_\ell)$ generated by h-motives of the form $\widehat{R(X)}_\ell(n)$ for X/S smooth and $n \in \mathbf{Z}$. An object M of $\mathrm{DM}_{\mathrm{h}}(X,\hat{R}_\ell)$ is said to be *constructible* if, M/ℓ is locally constructible in $\mathrm{DM}_{\mathrm{h}}(X,R/\ell)$ (see Definition 6.3.1). We write $\mathrm{DM}_{\mathrm{h},c}(X,\hat{R}_\ell)$ for the thick subcategory of the triangulated category $\mathrm{DM}_{\mathrm{h}}(X,\hat{R}_\ell)$ generated by constructible ℓ-adic motives. We thus have a natural inclusion

$$\mathrm{DM}_{\mathrm{h,gm}}(X,\hat{R}_\ell) \subset \mathrm{DM}_{\mathrm{h},c}(X,\hat{R}_\ell).$$

Remark 7.2.14. The notion of constructible ℓ-adic motive corresponds to what is usually called (bounded complex of) constructible ℓ-adic sheaves, while geometric ℓ-adic h-motives correspond to (bounded complex of) constructible ℓ-adic sheaves *of geometric origin*.

Remark 7.2.15. It is clear that $\mathrm{DM}_{\mathrm{h},c}(X,\hat{R}_\ell)$ is closed under the six operations in $\mathrm{DM}_{\mathrm{h}}(X,\hat{R}_\ell)$. This readily follows from Corollary 6.3.15 in the case of R/ℓ-linear coefficients. Indeed, the functor

$$\mathrm{DM}_{\mathrm{h}}(X,\hat{R}_\ell) \to \mathrm{DM}_{\mathrm{h}}(X,R/\ell), \quad M \mapsto M/\ell$$

is conservative and preserves the six operations as well as constructible objects (by definition). Note also that an object M of $\mathrm{DM}_{\mathrm{h}}(X,\hat{R}_\ell)$ is constructible if and only if M/ℓ^r is constructible in $\mathrm{DM}_{\mathrm{h}}(X,R/\ell^r)$ for any $r \geqslant 1$.

Theorem 7.2.16. *The ℓ-adic realization functor of Theorem 7.2.11 sends constructible objects to geometric ones (locally constructible objects to constructible ones, respectively). Moreover, the six operations preserve geometric objects (constructible objects, respectively) in* $\mathrm{DM}_{\mathrm{h}}(X,\hat{R}_\ell)$ *for quasi-excellent noetherian schemes of finite dimension.*

Proof. The first assertion is obvious. To prove that the subcategory $\mathrm{DM}_{\mathrm{h,gm}}(X,\hat{R}_\ell)$ is closed under the six operations in $\mathrm{DM}_{\mathrm{h}}(X,\hat{R}_\ell)$, it is sufficient check what happens on objects of the form $\hat{M}_\ell$ with M constructible in $\mathrm{DM}_{\mathrm{h}}(X,R)$. But then, the fact that the ℓ-adic realization functor preserves the six operations on the nose means that they preserve the class of these objects in $\mathrm{DM}_{\mathrm{h}}(X,\hat{R}_\ell)$. The stability of constructible objects under the six operations readily follows from the stability of locally constructible objects for torsion coefficients (Corollary 6.3.15). $\square$

Remark 7.2.17. The triangulated categories $\mathrm{DM}_{\mathrm{h},c}(X,\hat{R}_\ell)$ make sense for any scheme, whether or not the characteristic of R/ℓ is invertible in $\mathscr{O}_X$. Moreover, as we will see now, in the case where R/ℓ is of positive characteristic invertible in $\mathscr{O}_X$, they are equivalent to their classical analogues, whenever that makes sense: the construction of Beilinson *et al.* [BBD82], or the one of Ekedahl [Eke90]; see Propositions 7.2.19 and 7.2.21, respectively.

7.2.18. Let us assume that R/ℓ is of positive characteristic. Consider a noetherian scheme S with residue characteristics prime to the characteristic of R/ℓ, and assume that, for any constructible

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sheaf of R/ℓ-modules F on $S_{\text{ét}}$, the cohomology groups $H^i_{\text{ét}}(S,F)$ are finite (e.g. R/ℓ is finite and S is strictly local or the spectrum of a finite field). Then, for any S-scheme of finite type X, one can define, following Beilinson *et al.* [BBD82, Par. 2.2.14 and Propositon 2.2.15], the *triangulated category of constructible ℓ-adic sheaves* as the following 2-limit of derived categories of constructible sheaves:

$$\mathrm{D}^b_c(X,R_\ell) = 2\text{-}\varprojlim_r \mathrm{D}^b_{\mathrm{ctf}}(X_{\text{ét}},R/\ell^r).$$

On the other hand, we have an obvious family of triangulated functors

$$\mathrm{DM}_{\mathrm{h},c}(X,\hat{R}_\ell) \to \mathrm{DM}_{\mathrm{h,lc}}(X,R/\ell^r), \quad M \mapsto M/\ell^r$$

which, together with the equivalences of categories given by Theorem 6.3.11,

$$\mathrm{D}^b_{\mathrm{ctf}}(X,R/\ell^r) \simeq \mathrm{DM}_{\mathrm{h,lc}}(X,R/\ell^r),$$

induce a triangulated functor

$$\mathrm{DM}_{\mathrm{h},c}(X,\hat{R}_\ell) \to \mathrm{D}^b_c(X,R_\ell). \tag{7.2.18.a}$$

Proposition 7.2.19. *Under the assumptions of Paragraph 7.2.18, the functor (7.2.18.a) is an equivalence of categories.*

Proof. Let M and N be two objects of $\mathrm{DM}_{\mathrm{h},c}(X,\hat{R}_\ell)$. By virtue of Proposition 7.2.8, we have

$$N = \mathbf{R}\varprojlim_r N/\ell^r.$$

Moreover, by assumption, for any $r \geqslant 1$, the groups $\mathrm{Hom}(M/\ell^r, N/\ell^r)$ are finite, and thus, for any integer i, we have

$$\mathrm{Hom}(M,N[i]) = H^i\Big(\mathbf{R}\varprojlim_r \mathbf{R}\mathrm{Hom}(M,N/\ell^r)\Big) \simeq \varprojlim_r \mathrm{Hom}(M,N/\ell^r[i]).$$

The fully faithfulness of the functor (7.2.18.a) readily follows from this computation. Let F be an object of $\mathrm{D}^b_c(X,R_\ell)$, that is a collection of objects F_r in $\mathrm{D}^b_{\mathrm{ctf}}(X,R/\ell^r)$, together with isomorphisms

$$u_r : R/\ell^r \otimes^{\mathbf{L}}_{R/\ell^{r+1}} F_{r+1} \simeq F_r$$

for each $r \geqslant 1$. Such data can be lifted into a collection (E_r, v_r), where E_r is a complex of sheaves of R/ℓ^r-modules on $X_{\text{ét}}$, and

$$v_r : R/\ell^r \otimes_{R/\ell^{r+1}} E_{r+1} \to E_r$$

is a $R/(\ell^r)$-linear morphism of complexes of sheaves for each $r \geqslant 1$, such that $E_r \simeq F_r$ in $\mathrm{D}^b_{\mathrm{ctf}}(X,R/\ell^r)$, and such that the canonical map

$$R/\ell^r \otimes^{\mathbf{L}}_{R/\ell^{r+1}} E_{r+1} \to R/\ell^r \otimes_{R/\ell^{r+1}} E_{r+1} \to E_r$$

coincides with the given isomorphism u_r under these identifications. Applying the functor α^* (5.3.1.a), this defines similar data $(\alpha^*(E_r), \alpha^*(v_r))$ in the category of complexes of sheaves over the h-site of X. We may assume that each sheaf E_r if flat over R/ℓ^r (by choosing them cofibrant for the projective model structure, for instance), in which case the maps v_r already are

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quasi-isomorphisms. Applying the infinite suspension functor Σ^∞ finally leads to a diagram of Tate spectra, and we can define

$$E = \mathbf{R}\varprojlim_r \Sigma^\infty(\alpha^*(E_r)).$$

Note that, for any integer $r \geqslant 1$, we have $E/\ell^r \simeq \Sigma^\infty(\alpha^*(E_r))$ in $\mathrm{DM}_{\mathrm{h},c}(X, R/\ell^r)$. We thus see through the equivalences

$$\mathrm{D}^b_{\mathrm{ctf}}(X, R/\ell^r) \simeq \mathrm{DM}_{\mathrm{h,lc}}(X, R/\ell^r) \quad \text{and} \quad \mathrm{DM}^{\mathrm{eff}}_{\mathrm{h}}(X, R/\ell^r) \simeq \mathrm{DM}_{\mathrm{h}}(X, R/\ell^r)$$

that the functor (7.2.18.a) sends E to an object isomorphic to F. $\square$

7.2.20. More generally, assume now that R is noetherian and that the characteristic of the field R/ℓ is invertible in $\mathscr{O}_X$. Recall that Ekedahl has constructed a triangulated monoidal category $\mathrm{D}(X, R_\ell)$ of ℓ-adic systems; see [Eke90, Definition 2.5].[18] We denote by $\mathrm{D}^b_c(X, R_\ell)$ the full subcategory of $\mathrm{D}(X, R_\ell)$ spanned ℓ-adic constructible systems. By virtue of (the proof of) [Eke90, Theorem 6.3], $\mathrm{D}^b_c(X, R_\ell)$ is stable under the six operations (whenever this property holds for the categories $\mathrm{D}^b_{\mathrm{ctf}}(X, R/\ell)$, which is the case whenever X is noetherian and quasi-excellent by Gabber's theorem [ILO14, XIII, Theorem 1.1.1]).

Proposition 7.2.21. *Under the assumptions of Paragraph 7.2.20, there is a canonical equivalence of categories*

$$\mathrm{D}(X, R_\ell) \simeq \mathrm{DM}_{\mathrm{h}}(X, \hat{R}_\ell)$$

which is compatible with the six operations. This equivalence restricts to an equivalence of triangulated categories

$$\mathrm{D}^b_c(X, R_\ell) \simeq \mathrm{DM}_{\mathrm{h},c}(X, \hat{R}_\ell).$$

Proof. Note that the second equivalence of categories readily follows from the first, using Theorem 6.3.11. We will thus ignore finiteness hypotheses. We may assume that R is a complete discrete valuation ring. Before going further, we should emphasize that, in Ekedahl's article, there are restrictions about boundedness of complexes or about finite tor-dimension: we will ignore them completely because the reason for these is that, at that time, it was not known how to derive the tensor product for unbounded complexes. In particular, [Eke90, Proposition 2.2, Lemma 2.3] are true for unbounded complexes (and the proof does not change). We will try to remain close to the notations of Ekedahl's article. The obvious morphism of ringed topoi $\pi : X^{\mathbf{N}}_{\text{ét}} \to X_{\text{ét}}$ induces an adjunction

$$\mathbf{L}\pi^* : \mathrm{D}(X_{\text{ét}}, R) \rightleftarrows \mathrm{D}(X^{\mathbf{N}}_{\text{ét}}, R_\bullet) : \mathbf{R}\pi_*$$

where $\mathrm{D}(X^{\mathbf{N}}_{\text{ét}}, R_\bullet)$ is the derived category of the category of $R_\bullet$-modules on the topos $X^{\mathbf{N}}_{\text{ét}}$ of inverse systems of sheaves on the small étale site of X (with $R_\bullet$ the sheaf of rings on $X^{\mathbf{N}}_{\text{ét}}$ defined by the sequence $R/\ell^{n+1} \to R/\ell^n$), while $\mathrm{D}(X_{\text{ét}}, R)$ is the derived category of sheaves of R-modules on the small étale site of X. An object C of $\mathrm{D}(X_{\text{ét}}, R)$ will be said *ℓ-complete* is the canonical map

$$C \to \mathbf{R}\varprojlim_n C/\ell^n$$

[18] Ekedahl's notation for this category is $\mathrm{D}(X_{\text{ét}} - R)$, where $X_{\text{ét}}$ denotes the topos of sheaves on the small étale site of X.

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is an isomorphism (remark that the analog of Proposition 7.2.8 holds, with the same proofs). We denote by $\mathrm{D}(X_{\text{ét}}, R)_\ell$ the full subcategory of $\mathrm{D}(X_{\text{ét}}, R)$ which consists of ℓ-complete objects. We notice first that there are natural isomorphisms

$$\mathbf{R}\pi_*(C) \simeq \mathbf{R}\varprojlim_n C_n.$$

Therefore, we have isomorphisms

$$\mathbf{R}\pi_*\mathbf{L}\pi^*(C) \simeq \mathbf{R}\varprojlim_n C/\ell^n$$

and we obtain an adjunction

$$\mathbf{L}\pi^* : \mathrm{D}(X_{\text{ét}}, R)_\ell \rightleftarrows \mathrm{D}(X^{\mathbf{N}}_{\text{ét}}, R_\bullet) : \mathbf{R}\pi_*.$$

By definition of $\mathrm{D}(X_{\text{ét}}, R)_\ell$, the functor $\mathbf{L}\pi^*$ is now fully faithful, so that the functor $\mathbf{R}\pi_*$ identifies $\mathrm{D}(X_{\text{ét}}, R)_\ell$ as a Verdier quotient of $\mathrm{D}(X^{\mathbf{N}}_{\text{ét}}, R_\bullet)$. But we have the identifications $\mathbf{R}\pi_*(C)/\ell^n \simeq \mathbf{R}\pi_*(C/\ell^n)$, so that (the unbounded version of) [Eke90, Proposition 2.2 and Lemma 2.3], together with Corollary 5.5.4, express precisely that this Verdier quotient is Ekedahl's category $\mathrm{D}(X, R_\ell)$. In other words, we have proved that there is a canonical equivalence of triangulated categories

$$\mathrm{D}(X, R_\ell) \simeq \mathrm{D}(X_{\text{ét}}, R)_\ell.$$

We are thus reduced to prove that we have an equivalence

$$\mathrm{D}(X_{\text{ét}}, R)_\ell \simeq \mathrm{DM_h}(X, \hat{R}_\ell).$$

Considering the canonical adjunction

$$\Sigma^\infty\alpha^* : \mathrm{D}(X_{\text{ét}}, R) \rightleftarrows \mathrm{DM_h}(X, R) : \mathbf{R}\alpha_*\mathbf{R}\Omega^\infty,$$

we obtain an adjunction

$$\Sigma^\infty\alpha^*(-)_\ell : \mathrm{D}(X_{\text{ét}}, R)_\ell \rightleftarrows \mathrm{DM_h}(X, \hat{R}_\ell) : \mathbf{R}\alpha_*\mathbf{R}\Omega^\infty,$$

where $\Sigma^\infty\alpha^*(C)_\ell$ denotes the ℓ-completion of $\Sigma^\infty\alpha^*(C)$. As these two adjoint functors commute with the operation $C \mapsto C/\ell$, it is sufficient to check that the counit and unit of this adjunction are invertible modulo ℓ (i.e. are invertible when applied to objects of the form C/ℓ), which is a reformulation of Corollary 5.5.4. □

Corollary 7.2.22. *Under the assumptions of Paragraph 7.2.20, the category $\mathrm{DM}_{h,c}(X, \hat{R}_\ell)$ has a canonical bounded t-structure whose heart is equivalent to the abelian category of constructible ℓ-adic sheaves in the sense of [SGA5, Exp. V, 3.1.1].*

Proof. This follows from Proposition 7.2.21 and, since the ring R is noetherian and regular, from [Eke90, Theorem 6.3(i)]. □

7.2.23. Let Q be the field of fractions of R, and assume furthermore that R is of mixed characteristic. For a noetherian scheme X, we define the category of (constructible) Q_ℓ-sheaves over X

$$\mathrm{D}^b_c(X, Q_\ell) = \mathrm{DM}_{\mathrm{h},c}(X, \hat{R}_\ell) \otimes_R Q$$

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as the Q-linearization of the R-linear triangulated category $\mathrm{DM}_{h,c}(X,\hat{R}_\ell)$.[19] Then $\mathrm{D}^b_c(-,Q_\ell)$ is a motivic category which satisfies the absolute purity property (at least when restricted to quasi-excellent noetherian schemes of finite dimension).

As a final result, taking into account the fact the Q-localization functor is well behaved for h-motives (Corollary 5.4.11), we have a canonical identification, for any noetherian scheme of finite dimension,

$$(\mathrm{DM}_{\mathrm{h},c}(X,R)\otimes Q)^\sharp \simeq \mathrm{DM}_{\mathrm{h},c}(X,Q),$$

where the left-hand side denotes the pseudo-abelian completion of the Q-linearization of the R-linear triangulated category $\mathrm{DM}_{\mathrm{h},c}(X,R)$; see Appendix B. Note finally that, since the category $\mathrm{DM}_{h,c}(X,\hat{R}_\ell)$ has a bounded t-structure (using Proposition A.3.4, we may assume that the characteristic of the field R/ℓ is invertible in $\mathscr{O}_X$, and then apply Corollary 7.2.22), the category $\mathrm{D}^b_c(X,Q_\ell)$ is pseudo-abelian, by Corollary B.2.3.

Theorem 7.2.24. *The functor $\hat{\rho}^*_\ell$ (7.2.10.a) together with the equivalence of categories of Proposition 7.2.21 induce, for any noetherian scheme of finite dimension X, a Q-linear triangulated monoidal functor:*

$$\mathrm{DM}_{\mathrm{h},c}(X,Q)\to \mathrm{D}^b_c(X,Q_\ell)$$

(again, the ℓ-adic realization functor*).*

It is compatible with the six operations (when one restricts our attention to quasi-excellent noetherian schemes of finite dimension and morphisms of finite type between them).

Remark 7.2.25. As Q is a $\mathbf{Q}$-algebra, and taking into account Theorem 5.2.2, we have defined a morphism of premotivic categories

$$\hat{\rho}^*_\ell : \mathrm{DM}_{\text{Б},c}\to \mathrm{D}^b_c(-,Q_\ell)$$

which commutes with all of the six operations. Given (5.2.2.a) we see that this morphism induces in particular a cycle class in ℓ-adic étale cohomology, and even a higher cycle class. The compatibility of this realization with the six operations gives us all the required functoriality properties of this (higher) cycle class.

We like to think of $\hat{\rho}^*_\ell$ as a kind of *categorical cycle class* for ℓ-adic complexes.

The interest of the above theorem is to present the universal premotivic adjunction $\hat{\rho}^*_\ell$ as a *homotopy ℓ-adic completion*, which implies the non-trivial fact that it commutes with all of the six operations (i.e. with the right adjoint functors).

Remark 7.2.26. In the case where ℓ is a prime number invertible in the residue characteristics of the scheme X, in the triangulated categories $\mathrm{D}^b_c(X,\mathbf{Q}_\ell)$, there can be non-trivial extensions between objects of the form $p_!(\mathbf{Q}_\ell)(n)[2n]$, for $p:Y\to X$ proper and Y is regular, with $n\in\mathbf{Z}$. Indeed, in the case where X is the spectrum of an algebraically closed field k, this means for instance that the cohomology of smooth and proper k-schemes can be non-trivial in degree 1. In the case where X is the complement of a finite set of points in the spectrum of a ring of integers, examples are provided by Jannsen in [Jan90, Remarks 6.8.4].

Let us consider two (locally) constructible objects M and N in $\mathrm{DM}_\mathrm{h}(X,\mathbf{Z})$, and assume that

$$\mathrm{Hom}(M,N[i])=0 \quad \text{for } i>0. \tag{7.2.26.a}$$

[19] Under the assumption of Paragraph 7.2.20, and according to Proposition 7.2.21, this category is Ekedahl's derived category of ℓ-adic sheaves. Our definition has the advantage of having all the good properties without assuming any restriction on the residue characteristics of X.

This readily implies that

$$\mathrm{Hom}(M, N[i]) \otimes \mathbf{Z}/\ell^\nu \simeq \mathrm{Hom}(M, N/\ell^\nu[i])$$

for any non-negative integers ν and i. We thus have a Milnor short exact sequence

$$0 \to \varprojlim_\nu{}^1 \mathrm{Hom}(M, N) \otimes \mathbf{Z}/\ell^\nu \to \mathrm{Hom}(M_\ell, N_\ell[1]) \to \varprojlim_\nu \mathrm{Hom}(M, N[1]) \otimes \mathbf{Z}/\ell^\nu \to 0.$$

This proves

$$\mathrm{Hom}(M_\ell, N_\ell[1]) = 0. \tag{7.2.26.b}$$

In other words, if ever $\mathrm{DM}_\mathrm{h}(X, \mathbf{Z})$ has a suitable weight structure in the sense of Bondarko, there cannot be non-trivial extensions between ℓ-adic realizations of pure h-motives over X with integral coefficients. This shows that there is no hope to define a weight structure on $\mathrm{DM}_\mathrm{h}(X, \mathbf{Z})$ such that objects of the form $p_!(\mathbf{Z})(n)[2n]$ are pure for $p : Y \to X$ proper, Y is regular, and with $n \in \mathbf{Z}$, at least when X is a separably closed field, or the complement of a finite set of points in the spectrum of a ring of integers. Using the properties of continuity and of localization, it is a nice exercise to deduce from there that finite extensions of primary fields must be avoided as well. Remark that, in contrast, $\mathrm{DM}_\mathrm{h}(X, \mathbf{Q})$ carries a perfectly well-behaved theory of weights with a great level of generality; see [Héb11, Bon14].

Acknowledgements

We thank Giuseppe Ancona, Ofer Gabber, Annette Huber, Shane Kelly, Kobi Kremnitzer, and Jörg Wildeshaus for discussions, ideas and motivations shared during the long gestation of this project. We heartily thank the referee of this paper for his careful reading, suggestions and corrections. They lead us to go further in some aspects of our study, as well as clearing out the place of our results in the current literature.

Appendix A. Recall and complement on premotivic categories

A.1 Premotivic categories and morphisms

The following definition is a summary of the definitions in [CD09, § 1]. In this presentation, Sch is an arbitrary category of schemes.

Definition A.1.1. Let $\mathscr{P}$ be one of the classes: Ét, Sm, $\mathscr{S}^{\mathrm{ft}}$.

A *triangulated (*respectively *abelian) $\mathscr{P}$-premotivic category $\mathscr{M}$* is a fibered category over Sch satisfying the following properties.

(1) For any scheme S, $\mathscr{M}_S$ is a well-generated triangulated (respectively Grothendieck abelian) category with a closed monoidal structure.[20]

(2) For any morphism of schemes f, the functor f^* is triangulated (respectively additive), monoidal and admits a right adjoint denoted by f_*.

(3) For any morphism p in $\mathscr{P}$, the functor p^* admits a left adjoint denoted by $p_\sharp$.

(4) *$\mathscr{P}$-base change.* For any cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ T & \xrightarrow[p]{} & S \end{array}$$

there exists a canonical isomorphism: $\mathrm{Ex}(\Delta_\sharp^*) : q_\sharp g^* \to f^* p_\sharp$.

[20] In the triangulated case, we require that the bifunctor $\otimes$ is triangulated in each variable.

(5) *$\mathscr{P}$-projection formula.* For any morphism $p : T \to S$ in $\mathscr{P}$, and any object (M, N) of $\mathscr{M}_T \times \mathscr{M}_S$, there exists a canonical isomorphism:

$$\mathrm{Ex}(p_\sharp^*, \otimes) : p_\sharp(M \otimes_T p^*(N)) \to p_\sharp(M) \otimes_S N.$$

When $\mathscr{P} = \mathrm{Sm}$, we say simply *premotivic* instead of Sm-premotivic. Objects of $\mathscr{M}$ are generically called *premotives.*

Remark A.1.2. The isomorphisms appearing in properties (4) and (5) are particular instances of what is generically called an *exchange transformation* in [CD09].

Example A.1.3. Let $\mathscr{P}$ be one of the classes: Ét, Sm, $\mathscr{S}^{\mathrm{ft}}$.

Then the categories $\mathrm{Sh}_{\text{ét}}(\mathscr{P}_S, R)$ (respectively $\mathrm{Psh}(\mathscr{P}_S, R)$) of étale sheaves (respectively presheaves) of R-modules over $\mathscr{P}_S$ for various base schemes S form the fibers of an abelian premotivic category (see [CD12, Example 5.1.1]).

Moreover, the derived categories $\mathrm{D}(\mathrm{Sh}_{\text{ét}}(\mathscr{P}_S, R))$ (respectively $\mathrm{D}(\mathrm{Psh}(\mathscr{P}_S, R))$) for various schemes S form the fibers of a canonical triangulated premotivic category (see [CD12, Definition 5.1.17]).

A.1.4. Consider a premotivic triangulated category $\mathscr{T}$.

Given any smooth morphism $p : X \to S$, we define following Voevodsky the *(homological) premotive* associated with X/S as the object $M_S(X) := p_\sharp(\mathbb{1}_X)$. Then M_S is a covariant functor.

Let $p : \mathbf{P}^1_S \to S$ be the canonical projection. We define the *Tate premotive* as the kernel of the map $p_* : M_S(\mathbf{P}^1_S) \to \mathbb{1}_S$ shifted by -2. Given an integer n and an object M of $\mathscr{T}$, we define the *nth Tate twist* $M(n)$ of M as the nth tensor power of M by the object $\mathbb{1}(1)$, allowing negative n if $\mathbb{1}(1)$ is $\otimes$-invertible.

We associate with $\mathscr{T}$ a bigraded cohomology theory on Sch:

$$H^{i,n}_{\mathscr{T}}(S) := \mathrm{Hom}_{\mathscr{T}}(\mathbb{1}_S, \mathbb{1}_S(n)[i]).$$

One can isolate the following basic properties of $\mathscr{T}$ (see [CD12]).

Definition A.1.5. Consider the notations above. One introduces the following properties of the premotivic triangulated category $\mathscr{T}$.

(1) *Homotopy property.* For any scheme S, the canonical projection of the affine line over S induces an isomorphism $M_S(\mathbf{A}^1_S) \to \mathbb{1}_S$.

(2) *Stability property.* The Tate premotive $\mathbb{1}(1)$ is $\otimes$-invertible.

(3) *Orientation.* An *orientation of* $\mathscr{T}$ is natural transformation of contravariant functors

$$c_1 : \mathrm{Pic} \to H^{2,1}$$

(not necessarily additive).[21]

When $\mathscr{T}$ is equipped with an orientation one says $\mathscr{T}$ is *oriented.*

A.1.6. Recall that a cartesian functor $\varphi^* : \mathscr{T} \to \mathscr{T}'$ between fibered categories over Sch is the following data:

- for any base scheme S in Sch, a functor $\varphi^*_S : \mathscr{T}(S) \to \mathscr{T}'(S)$;
- for any morphism $f : T \to S$ in Sch, a natural isomorphism $c_f : f^*\varphi^*_S \xrightarrow{\sim} \varphi^*_T f^*$ satisfying the cocycle condition.

[21] However, the orientations which appear in this article are always additive.

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The following definition is a particular case of [CD12, Definition 1.4.6].

Definition A.1.7. Let $\mathscr{P}$ be one of the classes: Ét, Sm, $\mathscr{S}^{\mathrm{ft}}$.

A morphism $\varphi^* : \mathscr{M} \to \mathscr{M}'$ of *triangulated (*respectively *abelian) $\mathscr{P}$-premotivic categories* is a cartesian functor satisfying the following properties.

(1) For any scheme S, φ_S^* is triangulated (respectively additive), monoidal and admits a right adjoint denoted by φ_{S*}.
(2) For any morphism $p : T \to S$ in $\mathscr{P}$, there exists a canonical isomorphism: $\mathrm{Ex}(p_\sharp, \varphi^*) : p_\sharp \varphi_T^* \to \varphi_S^* p_\sharp$.

Sometimes, we refer to such a morphism as the *premotivic adjunction*

$$\varphi^* : \mathscr{M} \rightleftarrows \mathscr{M}' : \varphi_*.$$

A *sub-$\mathscr{P}$-premotivic triangulated (*respectively *abelian) category* $\mathscr{M}_0$ of $\mathscr{M}$ is a full triangulated (respectively additive) subcategory of $\mathscr{M}$ equipped with a $\mathscr{P}$-premotivic structure such that the inclusion $\mathscr{M}_0 \to \mathscr{M}$ is a morphism of $\mathscr{P}$-premotivic categories.

Remark A.1.8. Given a morphism of triangulated premotivic categories

$$\varphi^* : \mathscr{T} \to \mathscr{T}',$$

any orientation of $\mathscr{T}$ induces a canonical orientation of $\mathscr{T}'$. Indeed, we deduce from the preceding definitions that for any scheme X, the functor φ_X^* induces a morphism

$$H^{2,1}_{\mathscr{T}}(X) \to H^{2,1}_{\mathscr{T}'}(X)$$

contravariantly natural in X.

Example A.1.9. Consider the notations of Example A.1.3.

Recall from [CD12, Definition 5.2.16] the $\mathbf{A}^1$-localization $\mathrm{D}^{\mathrm{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(\mathscr{P}, R))$ of triangulated category $\mathrm{D}(\mathrm{Sh}_{\text{ét}}(\mathscr{P}, R))$, which is a $\mathscr{P}$-fibered category equipped with a localization morphism

$$\mathrm{D}(\mathrm{Sh}_{\text{ét}}(\mathscr{P}, R)) \to \mathrm{D}^{\mathrm{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(\mathscr{P}, R))$$

and satisfying the homotopy property.

When $\mathscr{P} = \mathrm{Sm}$, we will put $\mathrm{D}^{\mathrm{eff}}_{\mathbf{A}^1,\text{ét}}(S, R) = \mathrm{D}^{\mathrm{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}_S, R))$.

The main properties of a triangulated premotivic category can be summarized in the so-called Grothendieck's six functors formalism.

Definition A.1.10. A triangulated premotivic category $\mathscr{T}$ which is oriented satisfies *Grothendieck's six functors formalism* if it satisfies the stability property and for any separated morphism of finite type $f : Y \to X$ in Sch, there exists a pair of adjoint functors

$$f_! : \mathscr{T}(Y) \rightleftarrows \mathscr{T}(X) : f^!$$

such that the following hold.

(1) There exists a structure of a covariant (respectively contravariant) 2-functor on $f \mapsto f_!$ (respectively $f \mapsto f^!$).

(2) There exists a natural transformation $\alpha_f : f_! \to f_*$ which is an isomorphism when f is proper. Moreover, α is a morphism of 2-functors.

(3) For any smooth morphism $f : X \to S$ in Sch of relative dimension d, there are canonical natural isomorphisms

$$\mathfrak{p}_f : f_\sharp \longrightarrow f_!(d)[2d]$$
$$\mathfrak{p}'_f : f^* \longrightarrow f^!(-d)[-2d]$$

which are dual to each other.

(4) For any cartesian square in Sch

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\Delta"] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

such that f is separated of finite type, there exist natural isomorphisms

$$g^* f_! \xrightarrow{\sim} f'_! g'^*,$$
$$g'_* f'^! \xrightarrow{\sim} f^! g_*.$$

(5) For any separated morphism of finite type $f : Y \to X$, there exist natural isomorphisms

$$\mathrm{Ex}(f_!^*, \otimes) : (f_! K) \otimes_X L \xrightarrow{\sim} f_!(K \otimes_Y f^* L),$$
$$\underline{\mathrm{Hom}}_X(f_!(L), K) \xrightarrow{\sim} f_* \underline{\mathrm{Hom}}_Y(L, f^!(K)),$$
$$f^! \underline{\mathrm{Hom}}_X(L, M) \xrightarrow{\sim} \underline{\mathrm{Hom}}_Y(f^*(L), f^!(M)).$$

(5) For any closed immersion $i : Z \to S$ with complementary open immersion j, there exists distinguished triangles of natural transformations as follows:

$$j_! j^! \xrightarrow{\alpha'_j} 1 \xrightarrow{\alpha_i} i_* i^* \xrightarrow{\partial_i} j_! j^![1]$$
$$i_! i^! \xrightarrow{\alpha'_i} 1 \xrightarrow{\alpha_j} j_* j^* \xrightarrow{\tilde{\partial}_i} i_! i^![1]$$

where $\alpha'_?$ (respectively $\alpha_?$) denotes the counit (respectively unit) of the relevant adjunction.

A.1.11. In [CD12], we have studied some of these properties axiomatically, introducing the following definitions.

- Given a closed immersion i, the fact i_* is conservative and the existence of the first triangle in (6) is called the *localization property with respect to i*.
- The conjunction of properties (2) and (3) gives, for a smooth proper morphism f, an isomorphism $\mathfrak{p}_f : f_\sharp \to f_*(d)[2d]$. Under the stability and weak localization properties, when such an isomorphism exists, we say that f is *$\mathscr{T}$-pure* (or simply *pure* when $\mathscr{T}$ is clear).[22]

DEFINITION A.1.12. Consider the notations and assumptions above.

We say that $\mathscr{T}$ satisfies the *localization property* (respectively *weak localization property*) if it satisfies the localization property with respect to any closed immersion i (respectively which admits a smooth retraction).

[22] In fact, the isomorphism $\mathfrak{p}_f$ is canonical up to the choice of an orientation of $\mathscr{T}$. Moreover, we will define explicitly this isomorphism in the case where we need it; see (4.2.5.a).

We say that $\mathscr{T}$ satisfies the *purity property* (respectively *weak purity property*) if for any smooth proper morphism f (respectively for any scheme S and integer $n > 0$, the projection $p : \mathbf{P}^n_S \to S$) is $\mathscr{T}$-pure.

Building on the construction of Deligne of $f_!$ and on the work of Ayoub on cross functors, we have obtained in [CD12, Theorem 2.4.50] the following theorem which is little variation on a theorem of Ayoub.

Theorem A.1.13. *Assume that* Sch *is an adequate category of schemes in the sense of [CD12, 2.0].*[23] *The following conditions on a well-generated triangulated premotivic category $\mathscr{T}$ equipped with an orientation and satisfying the homotopy property are equivalent.*

(i) *The triangulated category $\mathscr{T}$ satisfies Grothendieck's six functors formalism.*

(ii) *The triangulated category $\mathscr{T}$ satisfies the stability and localization properties.*

Remark A.1.14. In fact, Ayoub in [Ayo07] proves this result with the following notable differences.

- One has to restrict oneself to a category of quasi-projective schemes over a scheme which admits an ample line bundle.
- The questions of orientation are not treated in [Ayo07]: this means one has to replace the Tate twist in property (3) above by the tensor product with a *Thom space*.
- The theorem of Ayoub is more general in the sense that it does not require an orientation on the category $\mathscr{T}$. In particular, it applies to the stable homotopy category of schemes, which does not admit an orientation.

Recall the following definition from [CD12].

Definition A.1.15. A triangulated premotivic category $\mathscr{T}$ which satisfies the stability and localization properties, and in which the functor $f^!$ exists for any proper morphism f in Sch, is called a triangulated motivic category.

A.1.16. Consider an adjunction

$$\varphi^* : \mathscr{T} \rightleftarrows \mathscr{T}' : \varphi_*$$

of triangulated premotivic categories which satisfies Grothendieck's six functors formalism. Then it is proved in [CD12] that φ^* commutes with $f_!$ for f separated of finite type. In fact, φ^* commutes with the left adjoint of the six-functors formalism while φ_* commutes with the right adjoint functors.

On the other hand, there are canonical exchange transformations:

$$\begin{aligned} &\varphi^* f_* \to f_* \varphi^*, f \text{ morphism in Sch}, \\ &\varphi^* f^! \to f^! \varphi^*, f \text{ separated morphism of finite type in Sch}, \\ &[\varphi^* \underline{\mathrm{Hom}}(-,-)] \longrightarrow [\underline{\mathrm{Hom}}(\varphi^*(-), \varphi^*(-))]. \end{aligned} \tag{A.1.16.a}$$

Definition A.1.17. In the above assumptions, one says the morphism φ^* *commutes with the six operations* if the exchange transformations (A.1.16.a) are all isomorphisms.

If $\mathscr{T}$ is a sub-premotivic triangulated category of $\mathscr{T}'$, one simply says $\mathscr{T}$ is *stable by the six operations* if the inclusion commutes with the six operations.

For example, if φ^* is an equivalence of premotivic triangulated categories, then it commutes with the six operations.

[23] Examples of an adequate category: noetherian (respectively and/or finite-dimensional, quasi-excellent, excellent) schemes (respectively Σ-schemes, eventually of finite type, for a noetherian base scheme Σ).

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A.2 Complement: the absolute purity property

In this section, we consider a triangulated premotivic category $\mathscr{T}$ which satisfies the hypothesis and equivalent conditions of Theorem A.1.13. We assume in addition that the motives of the form $M_S(X)(i)$ for a smooth S-scheme X and a Tate twist $i \in \mathbf{Z}$ form a family of generators of the category $\mathscr{T}(S)$.

A.2.1. As usual, a closed pair is a pair of schemes (X, Z) such that Z is a closed subscheme of X. We will consider abusively that to give such a closed pair is equivalent to give a closed immersion $i : Z \to X$. We will say (X, Z) is regular when i is regular.

A (cartesian) morphism of closed pairs $(f, g) : (Y, T) \to (X, Z)$ is a cartesian square of schemes.

$$\begin{tikzcd} T \arrow[r, hook, "k"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r, hook, "i"] & X \end{tikzcd} \tag{A.2.1.a}$$

We will usually denote it by f instead of (f, g).

Note the preceding diagram induces a unique map $C_T Y \to g^{-1}(C_Z X)$ on the underlying normal cones. We say f (or the above square) is *transversal* when this map is an isomorphism.

Definition A.2.2. Let (X, Z) be a closed pairs and $i : Z \to X$ be the canonical inclusion. For any pair of integers (n, m), we define the cohomology of X with support in Z as

$$H_Z^{n,m}(X) := \mathrm{Hom}_{\mathscr{T}(S)}(i_*(\mathbb{1}_Z), \mathbb{1}_S(m)[n]).$$

Equivalently,

$$H_Z^{n,m}(X) = \mathrm{Hom}_{\mathscr{T}(Z)}(\mathbb{1}_Z, i^!(\mathbb{1}_S)(m)[n]). \tag{A.2.2.a}$$

Moreover, using the first localization triangle for $\mathscr{T}$ with respect to i (point (6), Definition A.1.10), we get that it is contravariantly functorial with respect to morphism of closed pairs.

Remark A.2.3. (1) Using this localization triangle, this cohomology can be inserted in the usual localization long exact sequence (the twist m being the same for each group).

(2) Consider a morphism of closed pairs $f : (Y, T) \to (X, Z)$ defined by a cartesian square of the form (A.2.1.a). Using point (4) of Definition A.1.10 applied to this square, we can define the following *exchange transformation*:

$$\mathrm{Ex}^{*!} : g^* i^! \xrightarrow{ad(f_*, f^*)} g^* i^! f_* f^* \xrightarrow{\sim} g^* g_* k^! f^* \xrightarrow{ad'(g_*, g^*)} k^! f^*. \tag{A.2.3.a}$$

One can check that the functoriality property of $H_Z^{**}(X)$ is given by associating with a morphism $\rho : \mathbb{1}_Z \to i^!(\mathbb{1}_Z)(i)[n]$ the composite map

$$\mathbb{1}_T \xrightarrow{g^*(\rho)} g^* i^!(\mathbb{1}_Z)(i)[n] \xrightarrow{\mathrm{Ex}^{*!}} k^!(\mathbb{1}_T)(i)[n]$$

through the identification (A.2.2.a).

According to (A.2.2.a), the bigraded cohomology group $H^{**}(X)$ admits a structure of a bigraded module over the cohomology ring $H^{**}(Z)$. According to the preceding remark, this module structure is compatible with pullbacks.

Definition A.2.4. Let (X, Z) be a regular closed pair of codimension c. A *fundamental class* of Z in X is an element

$$\eta_X(Z) \in H_Z^{2c,c}(X)$$

which is a base of the $H^{**}(Z)$-module $H_Z^{**}(X)$.

In other words, the canonical map

$$H^{**}(Z) \to H^{**}_Z(X), \quad \lambda \mapsto \lambda \cdot \eta_X(Z) \tag{A.2.4.a}$$

is an isomorphism. Note that if such a fundamental class exists, it is unique up to an invertible element of $H^{00}(Z)$.

PROPOSITION A.2.5. *Consider a regular closed immersion $i : Z \to X$ of codimension c and a morphism in $\mathscr{T}(Z)$:*

$$\eta_X(Z) : \mathbb{1}_Z \to i^!(\mathbb{1}_X)(c)[2c].$$

The following conditions are equivalent.

(i) *The map $\eta_X(Z)$ is an isomorphism.*

(ii) *For any smooth morphism $f : Y \to X$, the cohomology class $f^*(\eta_X(Z))$, in the group $H^{2c,c}_{f^{-1}(T)}(Y)$, is a fundamental class.*

Proof. We first remark that for any smooth X-scheme Y, $T = Y \times_X Z$, and for any couple of integers $(n, r) \in \mathbf{Z}^2$, the map induced by $\eta_X(Z)$

$$\mathrm{Hom}(M_Z(T)(-r)[-n], \mathbb{1}_Z) \to \mathrm{Hom}(M_Z(T)(-r)[-n], i^!(\mathbb{1}_X)(c)[2c])$$

is isomorphic to the map

$$H^{n,r}(T) \to H^{n,r}_T(Y), \quad \lambda \mapsto \lambda.\eta_T(Y).$$

Then the equivalence between (i) and (ii) follows from the fact the family of motives of the form $M_Z(Y \times_Z X)(-r)[-n]$ generates the category $\mathscr{T}(Z)$ because of the following.

- We have assumed $\mathscr{T}$ it is generated by Tate twist as a triangulated premotivic category.
- i^* is essentially surjective according to the localization property. □

Using the arguments[24] of [Dég08a], one obtains that the orientation $c_1 : \mathrm{Pic} \to H^{2,1}$ can be extended canonically to a full theory of Chern classes and deduced the projective bundle formula. One gets in particular, following [Dég08a, Paragraph 4.4], the following proposition.

PROPOSITION A.2.6. *Let E be a vector bundle over a scheme X, $s : X \to E$ the zero section. Then s admits a canonical (depending only on the orientation c_1 of $\mathscr{T}$) fundamental class.*

This is the *Thom class* defined in [Dég08a, Paragraph 4.4]. In what follows we will denote it by $\mathrm{th}(E)$, as an element of $H^{2c,c}_X(E)$.

A.2.7. Let (X, Z) be a closed pair with inclusion $i : Z \to X$. Assume i is a regular closed immersion of codimension c.

Following the classical construction, one define the *deformation space* $D_Z X$ attached to (X, Z) as the complement of the blow-up $B_Z(X)$ in $B_Z(\mathbf{A}^1_X)$. Note it contains $\mathbf{A}^1_Z$ as a closed subscheme.

This space is fibered over $\mathbf{A}^1$, with fiber over 1 (respectively 0) being the scheme X (respectively the normal bundle $N_Z X$). In particular, we get morphisms of closed pairs

$$(X, Z) \xrightarrow{d_1} (D_Z X, \mathbf{A}^1_Z) \xleftarrow{d_0} (N_Z X, Z) \tag{A.2.7.a}$$

[24] In fact, if $\mathscr{T}$ is equipped with a premotivic morphism $\mathrm{D}(\mathrm{PSh}(-, R)) \to \mathscr{T}$, one can readily apply all the results of [Dég08a] to the category $\mathscr{T}(S)$ for any fixed base scheme S. All the premotivic triangulated categories considered in this paper will satisfy this hypothesis.

where d_0 (respectively d_1) means inclusion of the fiber over 0 (respectively 1). It is important to note that d_0 and d_1 are transversal.

For the next statement, we denote by $\mathscr{P}_{\mathrm{reg}}$ the class of closed pairs (X, Z) in Sch such that X and Z are regular.

THEOREM A.2.8. *The following conditions are equivalent.*

(i) *There exists a family*

$$(\eta_X(Z))_{(X,Z)\in\mathscr{P}_{\mathrm{reg}}}$$

such that the following hold.

- *For any closed pair (X, Z), $\eta_X(Z)$ is a fundamental class of (X, Z).*
- *For any transversal morphism $f : (Y, T) \to (X, Z)$ of closed pairs in $\mathscr{P}_{\mathrm{reg}}$, $f^*\eta_X(Z) = \eta_Y(T)$.*

(ii) *For any closed pair (X, Z) in $\mathscr{P}_{\mathrm{reg}}$, the deformation diagram (A.2.7.a) induces isomorphisms of bigraded cohomology groups:*

$$H^{**}_Z(X) \xleftarrow{d_1^*} H^{**}_{\mathbf{A}^1_Z}(D_Z X) \xrightarrow{d_0^*} H^{**}_Z(N_Z X).$$

Proof. The fact (i) implies (ii) follows from the homotopy property of $\mathscr{T}$, using the isomorphism of type (A.2.4.a) and the fact the morphisms of closed pairs d_0 and d_1 are transversal.

Reciprocally, given the isomorphisms which appear in (ii), one can put $\eta_X(Z) = d_1^*(d_0^*)^{-1}(\mathrm{th}(N_Z X))$, using Proposition A.2.6. This is a fundamental class for (X, Z) using once again the homotopy property for $\mathscr{T}$. The fact these classes are stable by transversal base change follows from the functoriality of the deformation diagram (A.2.7.a) with respect to transversal morphisms. □

DEFINITION A.2.9. We will say that $\mathscr{T}$ satisfies the *absolute purity property* if the equivalent properties of the preceding propositions are satisfied.

Example A.2.10. (1) The motivic category of Beilinson motives DM_{B} satisfies the absolute purity property according to [CD12, Theorem 14.4.1].

(2) According to the theorem of Gabber [Fuj02], the motivic category defined by the derived categories of étale sheaves of Λ-modules $X \mapsto \mathrm{D}(X_{\text{ét}}, \Lambda)$ satisfies the absolute purity property for any quasi-excellent scheme, with Λ a finite ring of order prime to the residue characteristics of X.

A.3 Torsion, homotopy and étale descent

Recall the following result, essentially proved in [Voe96], but formulated in the premotivic triangulated category of Example A.1.9.

PROPOSITION A.3.1. *For any scheme S of characteristic $p > 0$, the category $\mathrm{D}^{\mathrm{eff}}_{\mathbf{A}^1,\text{ét}}(S, \mathbf{Z})$ is $\mathbf{Z}[1/p]$-linear.*

Proof. The Artin–Schreier exact sequence [SGA4, IX, 3.5] can be written as an exact sequence of sheaves in $\mathrm{Sh}_{\text{ét}}(X, \mathbf{Z})$:

$$0 \to (\mathbf{Z}/p\mathbf{Z})_S \to \mathbf{G}_a \xrightarrow{F-1} \mathbf{G}_a \to 0$$

where F is the Frobenius morphism. But $\mathbf{G}_a$ is a strongly contractible sheaf, thus $F - 1$ induces an isomorphism in the $\mathbf{A}^1$-localized derived category $\mathrm{D}^{\mathrm{eff}}_{\mathbf{A}^1,\text{ét}}(S, \mathbf{Z})$. This implies $(\mathbf{Z}/p\mathbf{Z})_S = 0$ in the latter category which in turn implies $p \cdot \mathrm{Id}$ is an isomorphism, as required. □

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A.3.2. Let $\mathscr{T}$ be a triangulated premotivic category. If $\mathscr{T}$ is obtained by a localization of the derived category of an abelian premotivic category, it comes with a canonical premotivic adjunction

$$D(\mathrm{PSh}(S,\mathbf{Z})) \rightleftarrows \mathscr{T}.$$

Then, the fact $\mathscr{T}$ satisfies the homotopy and the étale descent properties is equivalent to the fact that the previous adjunction induces a premotivic adjunction of the form

$$\mathrm{D}^{\mathrm{eff}}_{\mathbf{A}^1,\text{ét}}(-,\mathbf{Z}) \rightleftarrows \mathscr{T} \tag{A.3.2.a}$$

(see [CD12, 5.1.2, 5.2.10, 5.2.19, and 5.3.23]).

Corollary A.3.3. *Let $\mathscr{T}$ be a premotivic triangulated category equipped with an adjunction of the form (A.3.2.a). Then for any scheme S of characteristic $p>0$, $\mathscr{T}(S)$ is $\mathbf{Z}[1/p]$-linear.*

Proposition A.3.4. *Let p be a prime number and $n=p^a$ be a power of p. Let $\mathscr{T}$ be a premotivic triangulated category equipped with a premotivic adjunction of the form*

$$t^*:\mathrm{D}^{\mathrm{eff}}_{\mathbf{A}^1,\text{ét}}(-,\mathbf{Z}/n\mathbf{Z}) \rightleftarrows \mathscr{T}:t_*.$$

Let S be a scheme. We put $S[1/p]=S\times\mathrm{Spec}(\mathbf{Z}[1/p])$ and consider the canonical open immersion $j:S[1/p]\to S$. Then the functor

$$j^*:\mathscr{T}(S)\to\mathscr{T}(S[1/p])$$

is an equivalence of categories.

Proof. Note that the proposition is obvious when $\mathscr{T}=\mathrm{D}^{\mathrm{eff}}_{\mathbf{A}^1,\text{ét}}(-,\mathbf{Z}/n\mathbf{Z})$ by the previous corollary and the localization property. In particular, for any object of the form $E=t^*(M)$ with M in $\mathrm{D}^{\mathrm{eff}}_{\mathbf{A}^1,\text{ét}}(-,\mathbf{Z}/n\mathbf{Z})$, we have $j_\sharp j^*(E)\simeq E$. In particular, we have $j_\sharp j^*(\mathbb{1}_S)\simeq\mathbb{1}_S$. Therefore, for any object E of $\mathscr{T}(S)$, one has

$$j_\sharp j^*(E)\simeq j_\sharp(j^*(\mathbb{1}_S)\otimes E)\simeq j_\sharp j^*(\mathbb{1}_S)\otimes E\simeq\mathbb{1}_S\otimes E.$$

As the functor $j_\sharp$ is fully faithful, this readily implies the proposition. □

Appendix B. Idempotents

B.1 Idempotents and localizations

B.1.1. In this section, we give some complements on localization of abstract triangulated categories.

For a triangulated category T, we shall denote by $T^\sharp$ its idempotent completion (with its canonical triangulated structure; see [BS01]).

Proposition B.1.2. *Let T be a triangulated category and $S\subset T$ a thick subcategory of T. Then $U^\sharp$ is a thick subcategory of $T^\sharp$ and the natural triangulated functor*

$$(T/U)^\sharp\to(T^\sharp/U^\sharp)^\sharp$$

is an equivalence of categories.

Proof. Both functors $T\to(T/U)^\sharp$ and $T\to(T^\sharp/U^\sharp)^\sharp$ share the same universal property, namely of being the universal functor from T to an idempotent complete triangulated category in which any object of U becomes null. □

Corollary B.1.3. *Given a triangulated category T and a thick subcategory U of T, an object of T belongs to U if and only if its image is isomorphic to zero in the triangulated category $(T^\sharp/U^\sharp)^\sharp$.*

Proof. As U is thick in T, an object of T is in U if and only if its image in the Verdier quotient T/U is trivial. On the other hand, the preceding proposition implies in particular that the natural functor

$$T/U \to (T^\sharp/U^\sharp)^\sharp$$

is fully faithful, which implies the assertion. □

B.1.4. We fix a commutative ring A and a multiplicative system $S \subset A$. Let T be an A-linear triangulated category. We define a new triangulated category $T \otimes_A S^{-1}A$ as follows. The objects of $T \otimes_A S^{-1}A$ are those of T, and morphisms from X to Y are given by the formula

$$\mathrm{Hom}_{T\otimes_A S^{-1}A}(X,Y) = \mathrm{Hom}_T(X,Y) \otimes_A S^{-1}A$$

with the obvious composition law. We have an obvious triangulated functor

$$T \to T \otimes_A S^{-1}A \tag{B.1.4.a}$$

which is the identity on objects and which is defined by the canonical maps

$$\mathrm{Hom}(X,Y) \to \mathrm{Hom}_T(X,Y) \otimes_A S^{-1}A$$

on arrows. The distinguished triangles of $T \otimes_A S^{-1}A$ are the triangles which are isomorphic to some image of a distinguished triangle of T by the functor (B.1.4.a).

Given an object X of T and an element $f \in S$, we write $f : X \to X$ for the map $f.1_X$, and we shall write X/f for some choice of its cone. We write $T_{S\text{-tors}}$ for the smallest thick subcategory of T which contains the cones of the form X/f for any object X and any f in S, the objects of which will be called *S-torsion objects of T*. The functor (B.1.4.a) clearly sends S-torsion objects to zero, and thus induces a canonical triangulated functor

$$T/T_{S\text{-tors}} \to T \otimes_A S^{-1}A. \tag{B.1.4.b}$$

Proposition B.1.5. *The functor (B.1.4.b) is an equivalence of categories.*

Proof. One readily checks that T is $S^{-1}A$-linear if and only if $T_{S\text{-tors}} \simeq 0$. Therefore, both functors $T \to T/T_{S\text{-tors}}$ and (B.1.4.a) share the same universal property: these are the universal A-linear triangulated functors from T to an $S^{-1}A$-linear triangulated category. □

Corollary B.1.6. *We have a canonical equivalence of A-linear triangulated categories*

$$(T \otimes_A S^{-1}A)^\sharp \simeq (T^\sharp \otimes_A S^{-1}A)^\sharp.$$

Proof. This follows again from the fact that, by virtue of Propositions B.1.2 and B.1.5, these two categories are the universal A-linear idempotent complete triangulated categories under T in which the S-torsion objects are trivial. □

Proposition B.1.7. *Let T be an A-linear triangulated category and U a thick subcategory of T. Given a prime ideal $\mathfrak{p}$ in A, we write $T_\mathfrak{p} = T \otimes_A A_\mathfrak{p}$. For an object X of T, the following conditions are equivalent.*

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(i) *The object X belongs to U.*

(ii) *For any maximal ideal $\mathfrak{m}$ in A, the image of X in $(T/U)_{\mathfrak{m}}$ is trivial.*

(iii) *For any maximal ideal $\mathfrak{m}$ of A, the image of X in $(T^{\sharp}_{\mathfrak{m}}/U^{\sharp}_{\mathfrak{m}})^{\sharp}$ is trivial.*

Proof. The equivalence between conditions (ii) and (iii) readily follows from Corollaries B.1.3 and B.1.6. The equivalence between conditions (i) and (ii) comes from the fact that the localizations $A_{\mathfrak{m}}$ form a covering for the flat topology and from the Yoneda lemma. □

B.2 Idempotents and t-structures

Proposition B.2.1. *Any triangulated category endowed with a bounded t-structure. is idempotent complete.*

Proof. Let $\mathscr{T}$ be a triangulated category endowed with a bounded t-structure given by the pair $(\mathscr{T}^{\leqslant 0}, \mathscr{T}^{\geqslant 0})$. We denote by $\mathscr{T}^{\sharp}$ the pseudo-abelianization of $\mathscr{T}$. By virtue of a result of Balmer and Schlichting [BS01, Theorem 1.12], the additive category $\mathscr{T}^{\sharp}$ is naturally endowed with the structure of a triangulated category: distinguished triangles of $\mathscr{T}^{\sharp}$ as those isomorphic to direct factors of distinguished triangles of $\mathscr{T}$. By definition, the embedding functor $\mathscr{T} \to \mathscr{T}^{\sharp}$ is then exact. Furthermore, one can define a t-structure $(\mathscr{T}^{\sharp\leqslant 0}, \mathscr{T}^{\sharp\geqslant 0})$ on $\mathscr{T}^{\sharp}$ as follows: an object of $\mathscr{T}^{\sharp}$ belongs to $\mathscr{T}^{\sharp\leqslant 0}$ (to $\mathscr{T}^{\sharp\geqslant 0}$) if it is a direct factor of an object of $\mathscr{T}^{\sharp\leqslant 0}$ (of $\mathscr{T}^{\sharp\geqslant 0}$, respectively). The truncation functors of the t-structure $(\mathscr{T}^{\leqslant 0}, \mathscr{T}^{\geqslant 0})$ extend uniquely to truncation functors for this t-structure on $\mathscr{T}^{\sharp}$. The embedding functor $\mathscr{T} \to \mathscr{T}^{\sharp}$ now is a t-exact functor. Let X be an object of $\mathscr{T}$ and $p : X \to X$ a projector with image Y in $\mathscr{T}^{\sharp}$. We will prove that Y belongs to $\mathscr{T}$ (by which we mean that it is isomorphic to an object of $\mathscr{T}$), by induction on the amplitude of X. We may assume that X belongs to $\mathscr{T}^{\geqslant 0}$. Let n be the smallest non-negative integer such that X belongs to $\mathscr{T}^{\leqslant n}$. If $n = 0$, then X belongs to the heart of the t-structure of $\mathscr{T}$, and any abelian category being in particular pseudo-abelian, this implies that the image of p, namely Y, is representable in $\mathscr{T}$. If $n > 0$, we then have a canonical distinguished triangle of the following form.

$$\tau^{<n}(Y) \to Y \to H^n(Y)[-n] \to \tau^{<n}(Y)[1]$$

We already know that $H^n(Y)[-n]$ belongs to $\mathscr{T}$, and, by induction, so does the truncation $\tau^{<n}(Y)$. Therefore, the object Y belongs to $\mathscr{T}$ as well. In other words, we have an equivalence of categories $\mathscr{T} \simeq \mathscr{T}^{\sharp}$, and the property of being idempotent complete being closed under equivalences of categories; this proves the proposition. □

Proposition B.2.2. *Let A be a commutative ring and $S \subset A$ a multiplicative system. Consider an A-linear triangulated category $\mathscr{T}$ endowed with a t-structure. Then there is a unique t-structure on the S-localization $S^{-1}\mathscr{T} = T \otimes_A S^{-1}A$ such that the canonical functor $\mathscr{T} \to S^{-1}\mathscr{T}$ is t-exact. In particular, if the t-structure of $\mathscr{T}$ is bounded, so is the t-structure of $S^{-1}\mathscr{T}$.*

Sketch of proof. We will consider the canonical functor $\mathscr{T} \to S^{-1}\mathscr{T}$ as the identity on objects. Let $(\mathscr{T}^{\leqslant 0}, \mathscr{T}^{\geqslant 0})$ be the given t-structure on $\mathscr{T}$. We define $(S^{-1}\mathscr{T}^{\leqslant 0}, S^{-1}\mathscr{T}^{\geqslant 0})$ as follows: a, object of $S^{-1}\mathscr{T}$ belongs to $S^{-1}\mathscr{T}^{\leqslant 0}$ (to $S^{-1}\mathscr{T}^{\geqslant 0}$) if it is isomorphic in $S^{-1}\mathscr{T}$ to the image of an object of $\mathscr{T}^{\leqslant 0}$ (of $\mathscr{T}^{\geqslant 0}$, respectively). For objects X and Y in $\mathscr{T}^{\leqslant 0}$ and $\mathscr{T}^{\geqslant 0}$, respectively, we have

$$\mathrm{Hom}_{S^{-1}\mathscr{T}}(X[i], Y) = S^{-1}\,\mathrm{Hom}_{\mathscr{T}}(X[i], Y) = 0$$

for $i > 0$. We leave the task of checking the axioms for a t-structure on $S^{-1}\mathscr{T}$ as an exercise for the reader. Once we know it is well defined, it is obvious that this t-structure on the S-localization

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is the unique one such that the canonical functor $\mathscr{T} \to S^{-1}\mathscr{T}$ is t-exact (because this functor is essentially surjective). For the same reason, it is also clear that, if the t-structure of $\mathscr{T}$ is bounded, so is the corresponding one on $S^{-1}\mathscr{T}$. □

The preceding two propositions thus give the following corollary.

Corollary B.2.3. *Let A be a commutative ring and consider an A-linear triangulated category $\mathscr{T}$, and suppose that there exists a bounded t-structure on $\mathscr{T}$. Then, for any multiplicative system $S \subset A$, the S-localization $S^{-1}\mathscr{T} = T \otimes_A S^{-1}A$ is idempotent complete.*

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