The Functional Equation of Zeta Distributions Associated With Non-Euclidean Jordan Algebras

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Abstract. This paper is devoted to the study of certain zeta distributions associated with simple non-Euclidean Jordan algebras. An explicit form of the corresponding functional equation and Bernstein-type identities is obtained.

1 Introduction

In 1859 B. Riemann published his only paper on number theory which introduces the use of complex analysis into the subject. Riemann's main goal was to outline the eventual proof of the prime number theorem by counting the primes using complex integration. In doing this, he introduced what is nowadays called the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, for $\mathrm{Re}(s) > -1$. Riemann proved that his ζ-function extends meromorphically to the entire complex plane and satisfies the functional equation

$$\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s) = \pi^{(1-s)/2}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s).$$

This is the symmetric form of the functional equation, which represents one of the fundamental results of zeta function theory.

Since then, there have been many generalizations of $\zeta(s)$, which are also known as zeta functions. These functions arise naturally in many branches of analytic and algebraic number theory, and their study has many important applications. Many of these functions admit an analogue of the functional equation for $\zeta(s)$, and bear other similarities to $\zeta(s)$. Probably the most important generalization of $\zeta(s)$ is due to Hecke, where the functional equation of the twisted zeta function of algebraic number fields was proved. In his celebrated 1950 Ph.D. thesis [20], Tate lifted the zeta function to a zeta integral defined on an adelic space and reinterpreted Hecke's results in terms of Fourier analysis on the adelic object and its quotients, and the right mixture of their multiplicative and additive structures. Tate's method is powerful enough to uniformly reprove the analytic continuation and functional equations of Hecke's L-functions.

In the Archimedean case, the general theory of zeta integrals really begins around 1970 when Bernstein and Gel'fand jointly [5], and Atiyah independently [1], proved

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the fundamental theorem of complex powers. This theorem states that the zeta integrals have a meromorphic continuation to the whole complex plane with poles in a finite number of arithmetic progressions of negative rational numbers. Bernstein later proved the theorem again using his theory of Bernstein polynomials [4].

In 1972, Godment and Jacquet [11] generalized Tate's results obtaining functional equation for zeta integrals on the space of square matrices $M(n, \mathbb{F})$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. We mention that the study of zeta integrals and their functional equations in the setting of prehomogeneous vector spaces has a long and rich history [19, 17, 18, 16, 6, 8]. (Of course, this list of references is not complete.)

The purpose of this paper is to give an explicit expression for the Fourier transform of a new class of zeta integrals associated to a certain class of Lie groups defined by Jordan algebras. We consider Lie groups which are related to conformal groups of a non-Euclidean Jordan algebra. See the tables below for a precise list of the groups under consideration. Bernstein-type identities are also obtained. Our results generalize the functional equation and the Bernstein polynomials obtained in [3]. The Euclidean case was treated earlier in [16].

To be more precise, let $\mathbb{V}$ be a non-Euclidean Jordan algebra of dimension m and rank n, and let L be the structure group of $\mathbb{V}$. According to the tables below, there exists one and only one open and dense L-orbit $\mathcal{O}$ in $\mathbb{V}$ (cf. [9]).

Let ν be a Cartan involution of $\mathbb{V}$, and set $\overline{\mathbb{V}} = \nu(\mathbb{V})$. For $\mathbf{s} = (s_1, s_2, \ldots, s_n) \in \mathbb{C}^n$, let $\nabla_{\mathbf{s}}$ (resp. $\overline{\nabla}_{\mathbf{s}}$) be a type of generalized power function defined on $\mathbb{V}$ (resp. $\overline{\mathbb{V}}$). For instance, when $\mathbb{V}$ is the Jordan algebra of complex symmetric matrices, we have $\nabla_{\mathbf{s}} = |\nabla_1|^{s_1} |\nabla_2|^{s_2-s_1} \cdots |\nabla_n|^{s_n-s_{n-1}}$ where ∇_j is the determinant of a block matrix of size $(n-j+1) \times (n-j+1)$. Let $\mathcal{S}(\mathbb{V})$ be the space of Schwartz functions on $\mathbb{V}$. For $h \in \mathcal{S}(\mathbb{V})$ and $f \in \mathcal{S}(\overline{\mathbb{V}})$, define the following zeta integrals

$$Z(h, \mathbf{s}) = \int_{\mathcal{O}} \nabla_{\mathbf{s}}(X) h(X) \, dX, \quad \overline{Z}(f, \mathbf{s}) = \int_{\overline{\mathcal{O}}} \overline{\nabla}_{\mathbf{s}}(Y) f(Y) \, dY.$$

Here $\overline{\mathcal{O}}$ is the open and dense L-orbit in $\overline{\mathbb{V}}$. The $\mathcal{S}'$-distributions defined by analytic continuation of these two integrals are called zeta distributions. The main result of the paper is to give, in a uniform way, an explicit expression of the functional equation involving Z and $\overline{Z}$ for non-Euclidean Jordan algebras. The existence of such a functional equation can be found in [7]. In order to do so, it is enough to find the functional equation for a special subclass of functions in $\mathcal{S}(\mathbb{V})$. Using a decomposition theorem proved in [13], we prove that

$$(1) \qquad \overline{Z}(f, \mathbf{s}) = \pi^{-\frac{m}{2}} \pi^{-|\mathbf{s}|} \prod_{j=1}^{n} \frac{\Gamma\left(\frac{s_j + (e+1) + d(n-j)}{2}\right)}{\Gamma\left(\frac{-s_j - d(n-j)}{2}\right)} Z\left(\mathcal{F}(f), -\mathbf{s}^* - \frac{m}{n}\right),$$

where $-\mathbf{s}^* - \frac{m}{n} = \left(-s_n - \frac{m}{n}, -s_{n-1} - \frac{m}{n}, \ldots, -s_1 - \frac{m}{n}\right)$, $|\mathbf{s}| = s_1 + \cdots + s_n$, and $\mathcal{F}(f)$ is the Fourier transform of f. Here d and e are certain integers which depend on $\mathbb{V}$. In [6], the authors prove (1) where $\mathbb{V}$ is the complexification of a Euclidean Jordan algebra. Further, in the particular case where $s_1 = s_2 = \cdots = s_n$, equation (1) coincides with the functional equation proved in [3].

Using (1), we prove certain Bernstein-type identities for $\nabla_{\mathbf{s}}$, in the sense that

$$\nabla_\ell (\partial_X)^2 \nabla_{\mathbf{s}}(X) = \prod_{\kappa=\ell}^{n} \big(s_{n-\kappa+1} + d(\kappa-1) \big) \big(s_{n-\kappa+1} + (e-1) + d(\kappa-1) \big) \nabla_{\mathbf{s}^{(\ell)}}(X)$$

where $\mathbf{s}^{(\ell)} = (s_1 - 2, \ldots, s_{n-\ell+1} - 2, s_{n-\ell+2}, \ldots, s_n)$ for $1 \leq \ell \leq n$.

2 Preliminary Results

Let $\mathbb{V}$ be a simple Jordan algebra. The structure group L of $\mathbb{V}$ is the group of linear transformations g of $\mathbb{V}$ for which the polynomial $\det(X)$, the determinant of $X \in \mathbb{V}$, is semi-invariant, *i.e.,*

$$\det(gX) = \chi(g) \det(X),$$

where χ is the $\mathbb{R}^*$-valued character of L. Denote by N the group of transformations

$$n_X \colon v \mapsto v + X, \quad \text{for } v \in \mathbb{V},$$

and let $P := L \ltimes N$. Let j be the rational transformation of $\mathbb{V}$ defined by $j(X) = -X^{-1}$, the inverse in Jordan algebra sense. The conformal group G, or Kantor–Koecher–Tits group of $\mathbb{V}$, is the group of rational transformations of $\mathbb{V}$ generated by P and j. It is a Lie group and P is a maximal parabolic subgroup of G, where N is abelian and isomorphic to $\mathbb{V}$. Hence, the Lie algebra $\mathfrak{n} := \mathrm{Lie}(N)$ has a natural structure of a Jordan algebra. Next we shall make precise the list of the groups G under consideration.

For $X, Y \in \mathbb{V}$, let $\mathcal{L}(X)Y := XY$, and define $\tau(X, Y) := \mathrm{tr}(\mathcal{L}(XY))$. Set ν to be a Cartan involution of $\mathbb{V}$, *i.e.,* ν is an involutive automorphism of $\mathbb{V}$ such that the bilinear form $\tau(\nu(X), Y)$ is positive definite. The involution

$$\theta(g) := \nu \circ j \circ g \circ j \circ \nu, \quad g \in G$$

is a Cartan involution for G. By abuse of notation, we also use θ to indicate its differential.

Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{l} = \mathrm{Lie}(L)$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be its Cartan decomposition with respect to θ. The Lie algebra $\mathfrak{g}$ is called the Kantor–Koecher–Tits algebra of the Jordan algebra $\mathbb{V}$. Fix a maximal toral subalgebra $\mathfrak{t}$ in the orthogonal complement of $\mathfrak{k} \cap \mathfrak{l}$ in $\mathfrak{k}$. The rank of $\mathbb{V}$ is $n := \dim_{\mathbb{R}}(\mathfrak{t})$. The roots of $\mathfrak{t}_{\mathbb{C}}$, the complexification of $\mathfrak{t}$, in $\mathfrak{g}_{\mathbb{C}}$ form a root system of type C_n, and we fix a basis $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ of $\mathfrak{t}^*$ such that

$$\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\pm(\gamma_i \pm \gamma_j)/2, \pm\gamma_j\}.$$

For the subsystem $\Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, there are three possibilities

$$A_{n-1} = \{\pm(\gamma_i - \gamma_j)/2\}, \quad D_n = \{\pm(\gamma_i \pm \gamma_j)/2\}, \quad \text{and} \quad C_n.$$

The A_{n-1} case arises when $\mathbb{V}$ is a Euclidean Jordan algebra.

Now we turn our attention to $\mathfrak{g}_{\mathbb{C}}$. For the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, we define

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$X = \frac{1}{2} \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}, \quad Y = \frac{1}{2} \begin{bmatrix} -i & 1 \\ 1 & i \end{bmatrix}, \quad H = i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The Cayley transform is the automorphism of $\mathfrak{sl}_2(\mathbb{C})$ given by

$$c = \mathrm{Ad}\, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}.$$

It satisfies

$$c(X) = x, \quad c(Y) = y, \quad c(H) = h.$$

For $1 \le j \le n$, let Ψ_j be the holomorphic homomorphisms

$$\Psi_j \colon \mathfrak{sl}_2(\mathbb{C}) \longrightarrow \mathfrak{g}_{\mathbb{C}},$$

such that $\Psi_j(X)$ spans the root space $\mathfrak{p}_{\gamma_j}$, and write

$$X_j = \Psi_j(X), \quad x_j = \Psi_j(x), \quad h_j = \Psi(h), \dots.$$

The Cayley transform of $\mathfrak{g}_{\mathbb{C}}$ is the product

$$\mathbf{c} = \prod_{j=1}^{n} \exp\{\mathrm{ad}\, \frac{i\pi}{4}(x_j + y_j)\}.$$

Thus we obtain an $\mathbb{R}$-split toral subalgebra $\mathfrak{a}$ defined by

$$\mathfrak{a} = \mathbf{c}(i\mathfrak{t}) = \mathbb{R}h_1 \oplus \cdots \oplus \mathbb{R}h_n.$$

The roots of $\mathfrak{a}$ in $\mathfrak{g}$ are

$$\Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm \epsilon_i \pm \epsilon_j \,;\, \pm 2\epsilon_j\} \quad \text{where} \quad \epsilon_i = \frac{1}{2}\gamma_i \circ \mathbf{c}^{-1}.$$

Let $2d$ be the multiplicity of the short roots, and let $(e + 1)$ be the multiplicity of the long roots. If $m := \dim(\mathbb{V})$, then $m = n((e + 1) + d(n - 1))$. The integers d and e are listed in the tables below.

Recall that for $X, Y \in \mathbb{V}$, $\mathcal{L}(X)Y = XY$. If x is a primitive idempotent element, then the possible eigenvalues of $\mathcal{L}(x)$ are $1, 1/2$, and 0, and the Peirce decomposition of $\mathbb{V}$ associated to x is given by $\mathbb{V} = \mathbb{V}(x, 1) \oplus \mathbb{V}(x, \frac{1}{2}) \oplus \mathbb{V}(x, 0)$, where $\mathbb{V}(x, \lambda) := \{v \in \mathbb{V} \mid \mathcal{L}(x)v = \lambda v\}$. For $\ell = 1, \dots n$, write $\mathbb{V}_{\mathrm{r}}^{(n-\ell+1)} := \mathbb{V}(x_\ell + \cdots + x_n, 1)$. Each $\mathbb{V}_{\mathrm{r}}^{(n-\ell+1)}$ is a Jordan algebra of rank $n - \ell + 1$, and $\mathbb{V}_{\mathrm{r}}^{(1)} \subset \mathbb{V}_{\mathrm{r}}^{(2)} \subset \cdots \subset \mathbb{V}_{\mathrm{r}}^{(n)} = \mathbb{V}$. Also, one can define the subalgebras $\mathbb{V}_1^{(\ell)} := \mathbb{V}(x_1 + \cdots + x_\ell, 1)$ of rank ℓ.

Let $\mathbb{G}$ be the set of conformal transformations defined at 0. Then $\mathbb{G}$ is an open dense set in G and

$$\{g \in \mathbb{G} \mid g(0) = 0\} = \overline{P},$$

where $\overline{P} := L \ltimes \overline{N}$ and $\overline{N} = \theta(N)$. Moreover, $\mathbb{G} = NL\overline{N}$, and the map $\psi \colon N \times L \times \overline{N} \to \mathbb{G}$ is a diffeomorphism. The latter decomposition is called the Gelfand–Naimark decomposition. One can prove that if $g \in G$ and $X \in \mathbb{V}$, then $gn_X \in NL\overline{N}$, and its Gelfand–Naimark decomposition is given by

$$gn_X = n_{g \cdot X} Dg(X) \overline{n}',$$

where $Dg(X) \in L$ is the differential of the map $X \mapsto g \cdot X$, and $\overline{n}' \in \overline{N}$ (*cf.* [14, Proposition 1.2.2]).

Set $\overline{\mathbb{V}} := \nu(\mathbb{V})$. Observe that the Lie algebra $\mathfrak{n}$ (resp., $\overline{\mathfrak{n}}$) of N (resp., $\overline{N}$) can be identified with $\mathbb{V}$ (resp., $\overline{\mathbb{V}}$).

Define $\Lambda_\ell := \sum_{k=\ell}^{n} \epsilon_k$, and let $\mathbb{V}^{(n-\ell+1)}$ (resp., $\overline{\mathbb{V}}^{(n-\ell+1)}$) be a Jordan algebra in $\mathbb{V}$ (resp., $\overline{\mathbb{V}}$) of rank $n - \ell + 1$. On open and dense subsets of $\mathbb{V}^{(n-\ell+1)}$ and $\overline{\mathbb{V}}^{(n-\ell+1)}$, respectively, we may define the functions

$$\nabla_\ell(X) := e^{\Lambda_\ell \log(Dj(X^{-1}))},$$

for $X \in \mathbb{V}$ and $1 \le \ell \le n$, and the functions

$$\overline{\nabla}_1(X) := \nabla_1(\nu(X)), \quad \overline{\nabla}_\ell(X) := \frac{\nabla_1(\nu(X))}{\nabla_{n-\ell+2}(\nu(X))},$$

for $X \in \overline{\mathbb{V}}$ and $2 \le \ell \le n$. We mention that our definition of ∇_ℓ and $\overline{\nabla}_\ell$ is different from the one in [6]. (See also Remark 3.7 below.) Using [10, Section II.3], one can see that ∇_ℓ and $\overline{\nabla}_\ell$ are given in terms of the principal minors associated to $\mathbb{V}$ and $\overline{\mathbb{V}}$, respectively. See Tables 1–3 for more information. Moreover, ∇_ℓ^2 and $\overline{\nabla}_\ell^2$ are two homogeneous polynomials of degree $2(n - \ell + 1)$. Thus, both ∇_ℓ and $\overline{\nabla}_\ell$ extend to well defined functions on $\mathbb{V}^{(n-\ell+1)}$ and $\overline{\mathbb{V}}^{(n-\ell+1)}$, respectively.

For a given simple non-Euclidean Jordan algebra, the Kantor–Koecher–Tits group and algebra are not quite unique. Groups with different centers or fundamental groups may arise. Below we give the precise list of the Lie groups G under consideration. We shall also use the following notations: for $1 \le \ell \le n$, we denote by $\det_\ell$ the determinant of the block matrix

$$\begin{bmatrix} a_{\ell,\ell} & \cdots & a_{\ell,n} \\ \vdots & & \vdots \\ a_{n,\ell} & \cdots & a_{n,n} \end{bmatrix}.$$

For $X = (X_1, \ldots, X_n) \in \mathbb{C}^n$, we shall write $\|X\|_1 = |X_1|^2 - |X_2|^2 - \cdots - |X_n|^2$, and $\|X\|_2 = |X_1| + |X_n|$. Further, we will view the quaternionic matrices as complex matrices of the form

$$\begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix}.$$

$\mathbb{V}$	G	L	∇_ℓ	d	e
$\mathrm{Sym}_n(\mathbb{C})$	$\mathrm{Sp}_n(\mathbb{C})$	$\mathrm{GL}_n(\mathbb{C})$	$\lvert\det_\ell\rvert$	1	1
$M_n(\mathbb{C})$	$\mathrm{SL}_{2n}(\mathbb{C})$	$S(\mathrm{GL}_n(\mathbb{C})^2)$	$\lvert\det_\ell\rvert$	2	1
$\mathrm{Skew}_{2n}(\mathbb{C})$	$\mathrm{SO}_{4n}(\mathbb{C})$	$\mathrm{GL}_{2n}(\mathbb{C})$	$\lvert\mathrm{Pfaffian}_\ell\rvert$	4	1
$\mathrm{Herm}_3(\mathbb{O})_\mathbb{C}$	$E_{7,\mathbb{C}}$	$E_{6,\mathbb{C}}\times\mathbb{C}^*$	$\lvert\deg.(4-\ell)\mathbb{C}-\mathrm{poly.}\rvert$	8	1
$\mathbb{C}^n$	$\mathrm{SO}_{n+1}(\mathbb{C})$	$O_{n-1}\times\mathbb{R}^+$	$\lVert\cdot\rVert_\ell$	$n-2$	1

Table 1: Non-Euclidean Jordan algebras of type I: the complex cases

$\mathbb{V}$	G	L	∇_ℓ	d	e
$\mathrm{Sym}_{2n}(\mathbb{C})\cap M_n(\mathbb{H})$	$\mathrm{Sp}_{n,n}$	$\mathrm{GL}_n(\mathbb{H})$	$\lvert\det_\ell(A+iB)\rvert$	2	2
$M_n(\mathbb{H})$	$\mathrm{SL}_{2n}(\mathbb{H})$	$\mathrm{GL}_n(\mathbb{H})^2$	$\lvert\det_\ell(A+iB)\rvert$	4	3
$\mathbb{R}^{n\times1}$	$SO_{n,1}$	$O_{n-1}\times\mathbb{R}^*$	$\lVert\cdot\rVert_\ell^{1/2}$	0	$n-1$

Table 2: Non-Euclidean Jordan algebras of type I: the non-complex cases

$\mathbb{V}$	G	L	∇_ℓ	d	e
$M_n(\mathbb{R})$	$\mathrm{GL}_{2n}(\mathbb{R})$	$\mathrm{GL}_{2n}(\mathbb{R})^2$	$\lvert\det_\ell\rvert$	1	0
$\mathrm{Skew}_{2n}(\mathbb{R})$	$O_{2n,2n}$	$\mathrm{GL}_n(\mathbb{R})$	$\mathrm{Pfaffian}_\ell$	2	0
$\mathrm{Herm}_3(\mathbb{O}_s)$	$E_7(7)$	$E_6(6)\times\mathbb{R}^+$	$\lvert\deg.(4-\ell)\mathbb{R}\text{-poly.}\rvert$	4	0
$\mathbb{R}^{p-1}\times\mathbb{R}^{q-1}$	$O_{p,q},\ p,q\geq3$	$SO_{p-1,q-1}\times\mathbb{R}^+$	$\lVert\cdot\rVert_\ell$	$(p+q-4)/2$	0

Table 3: Non-Euclidean Jordan algebras of type II

3   The Functional Equation

Next, we turn our attention to the functional equation. For that we need to introduce some notation for the Fourier transform. Recall that $\tau(X,Y)=\mathrm{tr}(\mathcal{L}(XY))$, and ν is an involution of $\mathbb{V}$ such that $\tau(\nu(X),Y)$ is positive definite. We consider the inner product on $\mathbb{V}$ defined by

$$(2)\qquad\qquad\langle X,Y\rangle:=\frac{n}{m}\tau(X,Y).$$

The corresponding norms are

$$\|X\|^2 := \langle \nu(X), X \rangle \quad \text{for } X \in \mathbb{V}, \quad \text{and} \quad \|Y\|^2 := \langle \nu(Y), Y \rangle \quad \text{for } Y \in \overline{\mathbb{V}}.$$

These norms determine the Lebesgue measure dX on $\mathbb{V}$ and dY on $\overline{\mathbb{V}}$, in a standard way.

Recall that $\mathfrak{g}$ denotes the Lie algebra of the conformal group G. Set

$$\langle\!\langle \, \cdot \, , \, \cdot \, \rangle\!\rangle_{\mathfrak{g}} = \frac{n}{4m} B(\, \cdot \, , \, \cdot \,)$$

to be the non-degenerate pairing between $\mathfrak{n} \cong \mathbb{V}$ and $\overline{\mathfrak{n}} \cong \overline{\mathbb{V}}$. Here B is the Killing form of $\mathfrak{g}$.

Lemma 3.1   *Let $\mathbb{V}^{(\ell)} \subset \mathbb{V}$ be a smaller Jordan algebra of rank $\ell \leq n$. If $\mathfrak{g}_\ell$ denotes the Lie algebra of the conformal group G_ℓ associated to $\mathbb{V}^{(\ell)}$, then the restriction of $\langle\!\langle \, \cdot \, , \, \cdot \, \rangle\!\rangle_{\mathfrak{g}}$ to $\mathfrak{g}_\ell \times \mathfrak{g}_\ell$ coincides with $\langle\!\langle \, \cdot \, , \, \cdot \, \rangle\!\rangle_{\mathfrak{g}_\ell}$.*

Proof   Since $\mathfrak{g}_\ell$ is simple and $\langle\!\langle \, \cdot \, , \, \cdot \, \rangle\!\rangle_{\mathfrak{g}}$ is $\mathfrak{g}_\ell$-invariant, there exists a nonzero constant c such that $\langle\!\langle \, \cdot \, , \, \cdot \, \rangle\!\rangle_{\mathfrak{g}} = c\langle\!\langle \, \cdot \, , \, \cdot \, \rangle\!\rangle_{\mathfrak{g}_\ell}$ on $\mathfrak{g}_\ell \times \mathfrak{g}_\ell$. To see that $c = 1$, compute

$$\langle\!\langle H_1, H_1 \rangle\!\rangle_{\mathfrak{g}} = \operatorname{tr}(\operatorname{ad}(H_1)^2) = \frac{n}{4m} 8(d(n-1) + (e+1)) = 2,$$

and, if $m_\ell = \dim(\mathbb{V}^{(\ell)})$, then

$$\langle\!\langle H_1, H_1 \rangle\!\rangle_{\mathfrak{g}_\ell} = \operatorname{tr}\big(\operatorname{ad}(H_1)^2_{|\mathfrak{g}_\ell}\big) = \frac{\ell}{4m_\ell} 8(d(\ell-1) + (e+1)) = 2. \qquad \blacksquare$$

The significance of the above lemma is that when we pass from $\mathbb{V}$ to a smaller Jordan algebra $\mathbb{V}^{(\ell)}$ in the induction argument below, the pairing (2) is unchanged.

Set $\mathcal{O}_\xi^{(n)}$ to be the open L-orbit of the element $\sum_{j=1}^n \xi_j x_j$ in a simple Jordan algebra $\mathbb{V}$, where $\xi_j = \pm 1$. If $\mathbb{V}$ is a Euclidean Jordan algebra, then $\xi_j = 1$ for $1 \leq j \leq p$ and $\xi_j = -1$ for $p+1 \leq j \leq n$ and there are $n+1$ open L-orbits. If $\mathbb{V}$ is a Jordan algebra of type I or of type II, and for the choices of G we work with, we have $\xi_j = 1$ for all $1 \leq j \leq n$ (*cf.* [9]). Denote by $\mathcal{O}_1^{(n)}$ the orbit of the element $\sum_{j=1}^n x_j$, which is open and dense in $\mathbb{V}$, and is a semisimple symmetric space.

For $\mathbf{s} = (s_1, s_2, \ldots, s_n) \in \mathbb{C}^n$, define

$$\nabla_{\mathbf{s}}(X) := \nabla_1(X)^{s_1} \nabla_2(X)^{s_2 - s_1} \cdots \nabla_n(X)^{s_n - s_{n-1}}, \quad \text{for } X \in \mathbb{V},$$

and

$$\overline{\nabla}_{\mathbf{s}}(Y) := \overline{\nabla}_1(Y)^{s_1} \overline{\nabla}_2(Y)^{s_2 - s_1} \cdots \overline{\nabla}_n(Y)^{s_n - s_{n-1}}, \quad \text{for } Y \in \overline{\mathbb{V}}.$$

Denote by $\mathcal{S}(\mathbb{V})$ and $\mathcal{S}(\overline{\mathbb{V}})$ the spaces of Schwartz functions on $\mathbb{V}$ and $\overline{\mathbb{V}}$, respectively. For $f \in \mathcal{S}(\overline{\mathbb{V}})$ and $h \in \mathcal{S}(\mathbb{V})$, define

$$\overline{Z}(f, \mathbf{s}) := \int_{\mathcal{O}_\xi^{(n)}} f(Y) \overline{\nabla}_{\mathbf{s}}(Y) \, dY, \quad Z(h, \mathbf{s}) := \int_{\mathcal{O}_\xi^{(n)}} h(X) \nabla_{\mathbf{s}}(X) \, dX.$$

For the case when $\mathbb{V}$ is a Euclidean Jordan algebra, a functional equation involving $\overline{Z}(\cdot, \mathbf{s})$ and $Z(\cdot, \mathbf{s})$ was proved in [16, Theorem 1]. Thus, we shall restrict our attention to the cases where $\mathbb{V}$ is of type I and type II.

In the set $\mathbf{s} \in \mathbb{C}^n$ for which $\overline{\nabla}_{\mathbf{s}}$ and $\nabla_{\mathbf{s}}$ are absolutely convergent on $\mathbb{V}$ and $\overline{\mathbb{V}}$, respectively, the integrals $\overline{Z}(f, \mathbf{s})$ and $Z(h, \mathbf{s})$ are complex meromorphic functions of $\mathbf{s} \in \mathbb{C}^n$. Next, we will show that there is a meromorphic continuation to all of $\mathbb{C}^n$, and a functional equation relating the two integrals, via the Fourier transform, holds. A general reason for the existence of the functional equation and the meromorphic continuation can be found in [7]. Since $\mathcal{O}_1^{(n)}$ is dense in $\mathbb{V}$, we may rewrite $\overline{Z}(f, \mathbf{s})$ and $Z(h, \mathbf{s})$ as integrals over $\overline{\mathbb{V}}$ and $\mathbb{V}$, respectively.

For $p, q \in \{0, \pm 1\}$, let

$$\mathfrak{g}(p, q; \ell) = \left\{ X \in \mathfrak{g} \mid \mathrm{ad}(h_1 + \cdots + h_\ell)X = \frac{p}{2}X, \mathrm{ad}(h_{\ell+1} + \cdots + h_n)X = \frac{q}{2}X \right\}.$$

By [13], we have $\mathbb{V} \cong \mathbb{V}_1^{(\ell)} \oplus \mathfrak{g}(1, -1; \ell) \oplus \mathbb{V}_r^{(n-\ell)}$. Fix ℓ. For the smaller Jordan algebras $\mathbb{V}_1^{(\ell)}$ and $\mathbb{V}_r^{(n-\ell)}$, we will denote the analogs of ∇_j by $\nabla_{1,j}$ and $\nabla_{r,j}$. Similarly, we write $\overline{\nabla}_{1,j}$ and $\overline{\nabla}_{r,j}$ for $\overline{\mathbb{V}}_1^{(\ell)}$ and $\overline{\mathbb{V}}_r^{(n-\ell)}$.

If $h \in \mathcal{L}^1(\mathbb{V}, dX)$ and $f \in \mathcal{L}^1(\overline{\mathbb{V}}, dY)$, then by [13] we have

$$(3) \quad \int_{\mathbb{V}} h(X)dX = \int_{\mathfrak{g}(1,-1;\ell)} \int_{\mathbb{V}_r^{(n-\ell)}} \int_{\mathbb{V}_1^{(\ell)}} h\left(\exp(u)(X + X')\right) \nabla_{1,1}(X)^{2d(n-\ell)} \, dX dX' du,$$

and

$$(4) \quad \int_{\overline{\mathbb{V}}} f(Y)dY = \int_{\mathfrak{g}(1,-1;\ell)} \int_{\overline{\mathbb{V}}_r^{(n-\ell)}} \int_{\overline{\mathbb{V}}_1^{(\ell)}} f\left(\exp(v)(Y + Y')\right) \overline{\nabla}_{r,1}(Y')^{2d\ell} \, dY dY' dv,$$

(see [3] for the proof). Here we give the subspaces $\mathfrak{g}(1, -1; \ell), \mathbb{V}_1^{(\ell)}, \mathbb{V}_r^{(n-\ell)}, \overline{\mathbb{V}}_1^{(\ell)}$, and $\overline{\mathbb{V}}_r^{(n-\ell)}$ the Lebesgue measures determined by the restriction of $\langle \cdot, \cdot \rangle$ to each subspace, and we normalize them in the same way as the measure dX on $\mathbb{V}$ and dY on $\overline{\mathbb{V}}$.

Remark 3.2   (i) For $u \in \mathfrak{g}(1, -1; \ell)$

$$\mathrm{ad}(u) \colon \mathbb{V}_1^{(\ell)} \to \mathfrak{g}(1, 1; \ell), \quad \mathrm{ad}(u) \colon \mathfrak{g}(1, 1; \ell) \to \mathbb{V}_r^{(n-\ell)}, \quad \mathrm{ad}(u) \colon \mathbb{V}_r^{(n-\ell)} \to 0.$$

Hence $\mathrm{Ad}(\exp(u)) = I + \mathrm{ad}(u) + \frac{1}{2}\mathrm{ad}(u)^2$ on $\mathbb{V}_1^{(\ell)}$, and $\mathrm{ad}(u)^2 \colon \mathbb{V}_1^{(\ell)} \to \mathbb{V}_r^{(n-\ell)}$.

(ii) For $u \in \mathfrak{g}(1, -1; \ell)$ and X in a compact set in $\mathbb{V}_1^{(\ell)}$, such that $\nabla_{1,1}(X) \neq 0$, there exist two constants C and C' such that $C\|u\| \leq \|\mathrm{ad}(u)(X)\| \leq C'\|u\|$.

For $f \in \mathcal{S}(\overline{\mathbb{V}})$ and $h \in \mathcal{S}(\mathbb{V})$, define the Fourier transforms by

$$\mathcal{F}(f)(X) = \int_{\overline{\mathbb{V}}} f(Y)e^{-2\pi i \tau(X,Y)} \, dY, \quad \mathcal{F}(h)(Y) = \int_{\mathbb{V}} h(X)e^{-2\pi i \tau(X,Y)} \, dX.$$

We normalize the Lebesgue measures on $\overline{\mathbb{V}}$ and $\mathbb{V}$ by

$$\mathcal{F}^2(f)(X) = f(-X) \quad \text{and} \quad \mathcal{F}^2(h)(Y) = h(-Y).$$

Further, we define $\overline{T}_f \in \mathbb{C}^\infty(\overline{\mathbb{V}}_1^{(\ell)} \times \overline{\mathbb{V}}_r^{(n-\ell)})$ and $T_h \in \mathbb{C}^\infty(\mathbb{V}_1^{(\ell)} \times \mathbb{V}_r^{(n-\ell)})$ by

$$\overline{T}_f(Y, Y') = \int_{\mathfrak{g}(1,-1;\ell)} f(\exp(u)(Y + Y')) \, du,$$

$$T_h(X, X') = \int_{\mathfrak{g}(1,-1;\ell)} h(\exp(v)(X + X')) \, dv.$$

The functions $\overline{T}_f$ and T_h are not defined everywhere. For example the integral defining $T_h(0,0)$ is not convergent when $h(0,0) \neq 0$. However, among other things, we can show that $\overline{T}_f$ and T_h are defined almost everywhere for the Schwartz functions f and h.

Henceforth we fix $\ell = 1$. For $\mathbb{V}_1^{(1)}$, set

$$\mathcal{O}_1^{(1)} := \{X \in \mathbb{V}_1^{(1)} \mid \nabla_{1,1}(X) \neq 0\}.$$

The orbit $\mathcal{O}_1^{(1)}$ is open and dense in $\mathbb{V}_1^{(1)}$.

Lemma 3.3 *Let $h \in \mathcal{S}(\mathbb{V})$ and $f \in \mathcal{S}(\overline{\mathbb{V}})$.*

(i) *For fixed $X' \in \mathbb{V}_r^{(n-1)}$ (resp., $Y \in \overline{\mathbb{V}}_1^{(1)}$), the integral defining $T_h(X, X')$ (resp., $\overline{T}_f(Y, Y')$) is finite when $\nabla_{1,1}(X) \neq 0$ (resp., $\overline{\nabla}_{r,1}(Y') \neq 0$), i.e., T_h and $\overline{T}_f$ are defined almost everywhere.*

(ii) *For a compact $\mathcal{K}$ in $\mathcal{O}_1^{(1)}$ and an integer N, there exists a constant C such that*

$$T_h(X, X') \leq C(1 + \|X'\|)^{-N}, \quad \text{for } X \in \mathcal{K}.$$

Further, for $X \in \mathcal{O}_1^{(1)}$, $T_h(X, \cdot) \in \mathcal{S}(\mathbb{V}_r^{(n-1)})$.

Proof (i) Recall that $\mathrm{Ad}(\exp(u)) = I + \mathrm{ad}(u) + \frac{1}{2}\,\mathrm{ad}(u)^2$ on $\mathbb{V}_1^{(1)}$. Since h is a Schwartz function in each variable, then for an integer N, there exists a constant C such that

$$\begin{aligned} \left| h\big(\exp(u)(X + X')\big) \right| &= \left| h\big(X + \mathrm{ad}(u)X + X' + \tfrac{1}{2}\,\mathrm{ad}(u)^2(X)\big) \right| \\ &\leq C(1 + \|\,\mathrm{ad}(u)X\|)^{-N} \\ &\leq C'(1 + \|u\|)^{-N}. \end{aligned}$$

Now if N is big enough, then $(1 + \|u\|)^{-N}$ is an $\mathcal{L}^1$-function on $\mathfrak{g}(1, -1; 1)$. The corresponding statement for $\overline{T}_f$ follows in a similar way.

(ii) Again, since h is a Schwartz function in each variable, then for $M, N \in \mathbb{N}$, there exists a constant C such that

$$\left| h\big(\exp(u)(X + X')\big) \right| \leq C\big(1 + \|X' + \tfrac{1}{2}\,\mathrm{ad}(u)^2(X)\|\big)^{-N}\big(1 + \|\,\mathrm{ad}(u)X\|\big)^{-M}.$$

Using Remark 3.2 and the fact that

$$\frac{1 + \|X'\|}{1 + \|X' + \frac{1}{2}\,\mathrm{ad}(u)^2(X)\|} \leq 1 + \left\| \tfrac{1}{2}\,\mathrm{ad}(u)^2(X) \right\|,$$

we can see that for $X \in \mathcal{K}$, $T_h(X, X') \leq C'(1 + \|X'\|)^{-N}$. The last claim in (ii) follows by using the same method applied to the derivatives of h with respect to the variable in $\mathbb{V}_r^{(n-1)}$. ∎

Let $\mathcal{O} := \left\{ X \in \mathbb{V} \mid \text{Proj}(X) \in \mathcal{O}_1^{(1)} \right\}$ where $\text{Proj} \colon \mathbb{V} \to \mathbb{V}_1^{(1)}$ is the orthogonal projection. For an arbitrary domain Ξ, we denote by $\mathcal{D}(\Xi)$ the set of smooth functions having compact support in Ξ.

Lemma 3.4 *If $h \in \mathcal{D}(\mathcal{O})$, then $T_h \in \mathcal{D}\left(\mathcal{O}_1^{(1)} \times \mathbb{V}_r^{(n-1)}\right)$.*

Proof The change of coordinates $(X, u, X') \mapsto \exp(u)(X + X')$ is a smooth one-to-one map $\mathbb{V}_1^{(1)} \times \mathfrak{g}(1, -1; 1) \times \mathbb{V}_r^{(n-1)} \to \mathbb{V}$, and is a diffeomorphism from $\mathcal{O}_1^{(1)} \times \mathfrak{g}(1, -1; 1) \times \mathbb{V}_r^{(n-1)}$ onto $\mathcal{O}$, with Jacobian $\nabla_{1,1}(X)^{2d(n-1)}$ as proved in [3, Lemma 3.14]. Then $h(\exp(u)(X + X')) = h(X + \text{ad}(u)X + X' + \frac{1}{2}\text{ad}(u)^2 X)$ is smooth and has a compact support contained in $\mathcal{O}_1^{(1)} \times \mathfrak{g}(1, -1; 1) \times \mathbb{V}_r^{(n-1)}$. Hence, the lemma holds. ∎

Let $\mathbb{V}^\sharp := \{ X \in \mathbb{V} \mid \nabla_1(X) \cdots \nabla_n(X) \neq 0 \} \subset \mathcal{O}_1^{(n)}$ (recall that $\mathcal{O}_1^{(n)}$ is the open dense L-orbit in $\mathbb{V}$). For the zeta distributions Z and $\overline{Z}$, we shall use the subscript ℓ to indicate the rank. Further, observe that the integers d and e for a Jordan algebra $\mathbb{V}^{(\ell)} \subset \mathbb{V}$ of rank $\ell \leq n$ are the same as for $\mathbb{V}$, unless $\ell = 1$ where $d = 0$ for $\mathbb{V}^{(1)}$.

Proposition 3.5 *Let $h \in \mathcal{D}(\mathcal{O} \cap \mathbb{V}^\sharp)$. If $\mathbf{s} = (s_1; \tilde{\mathbf{s}})$ where $\tilde{\mathbf{s}} := (s_2, \ldots, s_n)$, then*

(i) *$Z_{n-1}(T_h(X, \cdot), \tilde{\mathbf{s}})$ is a smooth function with compact support in the X-variable.*

(ii) *As meromorphic functions of $\mathbf{s}$*

$$Z(h, \mathbf{s}) = Z_1(\nabla_{1,1}(\cdot)^{2d(n-1)} Z_{n-1}(T_h, \tilde{\mathbf{s}}), s_1).$$

Proof (i) Notice that if $\exp(u)(X + X') \in \mathcal{O} \cap \mathbb{V}^\sharp$, then

$$\nabla_{r,\tilde{\mathbf{s}}}(X') := \nabla_{r,1}(X')^{s_2} \nabla_{r,2}(X')^{s_3 - s_2} \cdots \nabla_{r,n-1}(X')^{s_n - s_{n-1}} \neq 0.$$

Thus

$$Z_{n-1}(T_h(X, \cdot), \tilde{\mathbf{s}}) = \int_{\mathbb{V}_r^{(n-1)}} T_h(X, X') \nabla_{r,\tilde{\mathbf{s}}}(X') \, dX'$$

for all $X \in \mathbb{V}_1^{(1)}$ and $\tilde{\mathbf{s}} \in \mathbb{C}^{n-1}$. By differentiating inside the integral, we can see that $Z_{n-1}(T_h(X, \cdot), \tilde{\mathbf{s}})$ is smooth and, by the previous lemma, has compact support.

(ii) Start with $\mathbf{s}$ such that $\nabla_{\mathbf{s}}$ is absolutely convergent. Using the fact that

$$\nabla_1(X + X') = \nabla_{1,1}(X) \nabla_{r,1}(X')$$

$$\nabla_\ell(X + X') = \nabla_{r,\ell-1}(X'), \quad 2 \leq \ell \leq n,$$

for $X \in \mathbb{V}_1^{(1)}$ and $X' \in \mathbb{V}_r^{(n-1)}$, and the integral formula (3), we obtain

$$
\begin{aligned}
Z(h, \mathbf{s}) &= \int_{\mathbb{V}} h(X) \nabla_{\mathbf{s}}(X) \, dX \\
&= \int_{\mathbb{V}_1^{(1)}} \int_{\mathbb{V}_r^{(n-1)}} T_h(X, X') \nabla_{\mathbf{s}}(X + X') \nabla_{1,1}(X)^{2d(n-1)} \, dX' dX \\
&= \int_{\mathbb{V}_1^{(1)}} \int_{\mathbb{V}_r^{(n-1)}} T_h(X, X') \nabla_{r,\tilde{\mathbf{s}}}(X') \nabla_{1,1}(X)^{s_1} \nabla_{1,1}(X)^{2d(n-1)} \, dX' dX \\
&= \int_{\mathbb{V}_1^{(1)}} Z_{n-1}\big(T_h(X, \cdot), \tilde{\mathbf{s}} \big) \nabla_{1,1}(X)^{2d(n-1)} \nabla_{1,1}(X)^{s_1} \, dX \\
&= Z_1(\nabla_{1,1}(\cdot)^{2d(n-1)} Z_{n-1}(T_h, \tilde{\mathbf{s}}), s_1).
\end{aligned}
$$

Now, by meromorphic continuation, the statement holds for general $\mathbf{s}$. ∎

Notice that if m_ℓ denotes the dimension of a smaller Jordan algebra of rank $\ell \leq n$, then $\frac{m}{n} = \frac{m_\ell}{\ell} + d(n - \ell)$. The dimension of a Jordan algebra of rank 1 is $(e + 1)$.

Theorem 3.6   *Let $\mathbf{s} \in \mathbb{C}^n$ and $f \in \mathcal{S}(\overline{\mathbb{V}})$. As meromorphic functions*

$$
\overline{Z}(f, \mathbf{s}) = \pi^{-\frac{m}{2}} \pi^{-|\mathbf{s}|} \prod_{j=1}^{n} \frac{\Gamma\left(\frac{s_j + (e+1) + d(n-j)}{2} \right)}{\Gamma\left(\frac{-s_j - d(n-j)}{2} \right)} Z(\mathcal{F}(f), t(\mathbf{s}))
$$

where

$$
t(\mathbf{s}) = \left(-s_n - \frac{m}{n}, -s_{n-1} - \frac{m}{n}, \ldots, -s_1 - \frac{m}{n} \right),
$$

and $|\mathbf{s}| = s_1 + \cdots + s_n$. Here Γ denotes the Euler gamma function.

Proof   We shall proceed by induction on the rank n of $\mathbb{V}$. For $n = 1$, the Jordan algebra $\mathbb{V} \equiv \overline{\mathbb{V}} \equiv \mathbb{R}^m$. Let $s \in \mathbb{C}$ be such that $-m < \mathrm{Re}(s) < 0$. For s in this range, both $|x|^s$ and $|x|^{-s-m}$ are locally $\mathcal{L}^1$-functions on $\mathbb{R}^m$. Since f and $e^{-\pi t |\cdot|^2}$ for $t > 0$ are absolutely convergent, then

$$
\int_{\mathbb{R}^m} f(x) t^{-\frac{m}{2}} e^{-\frac{\pi |x|^2}{t}} \, dx = \int_{\mathbb{R}^m} \mathcal{F}(f)(y) e^{-\pi t |y|^2} \, dy.
$$

Multiplying both sides by $t^{\frac{m+s}{2} - 1}$ and integrating over $t > 0$, we get

$$
(5) \quad \int_0^\infty \int_{\mathbb{R}^m} t^{\frac{s}{2} - 1} f(x) e^{-\frac{\pi |x|^2}{t}} \, dx dt = \int_0^\infty \int_{\mathbb{R}^m} t^{\frac{m+s}{2} - 1} \mathcal{F}(f)(y) e^{-\pi t |y|^2} \, dy dt.
$$

To evaluate the right-hand side of (5), we switch the order of integration, and using the substitution $v = \pi t |y|^2$ instead of t, we obtain the right-hand side of the functional equation. For the left-hand side of (5), we again use the Fubini–Tonelli theorem to switch the order of integration and use the change of variable $u = \frac{\pi |x|^2}{t}$ instead of t. Then we obtain the left-hand side of the functional equation.

Next, we assume that the statement holds for every non-Euclidean Jordan algebra of rank less than n.

By [7], there exists a constant $c(t(\mathbf{s}))$ such that $\overline{Z}(\mathcal{F}(f), t(\mathbf{s})) = c(t(\mathbf{s}))Z(f, \mathbf{s})$, for $f \in \mathcal{S}(\mathbb{V})$. To explicitly compute the constant $c(t(\mathbf{s}))$, it is enough to consider a subclass of functions f in $\mathcal{S}(\mathbb{V})$. Assume that $\mathrm{Re}(s_1) \ll \mathrm{Re}(s_2) \ll \cdots \ll \mathrm{Re}(s_n) \ll 0$, so the integrals stated below converge. For $Y \in \overline{\mathbb{V}}_l^{(1)}$ and $Y' \in \overline{\mathbb{V}}_r^{(n-1)}$, we have

$$\overline{\nabla}_\ell(Y + Y') = \overline{\nabla}_{l,1}(Y)\overline{\nabla}_{r,\ell}(Y'), \quad \text{for } 1 \le \ell \le n - 1,$$
$$\overline{\nabla}_n(Y + Y') = \overline{\nabla}_{l,1}(Y).$$

Using the integral formula (4), Lemma 3.3, and Lemma 3.4, we obtain

$$\overline{Z}(\mathcal{F}(f), t(\mathbf{s})) = \int_{\overline{\mathbb{V}}} \mathcal{F}(f)(X)\overline{\nabla}_{t(\mathbf{s})}(X)\, dX$$

$$= \int_{\overline{\mathbb{V}}_l^{(1)}} \int_{\overline{\mathbb{V}}_r^{(n-1)}} \int_{\mathfrak{g}(1,-1;1)} \mathcal{F}(f)(\exp(u)(X + X'))$$
$$\overline{\nabla}_{t(\mathbf{s})}(X + X')\overline{\nabla}_{r,1}(X')^{2d}\, du dX' dX$$

$$= \int_{\overline{\mathbb{V}}_l^{(1)}} \int_{\overline{\mathbb{V}}_r^{(n-1)}} \overline{T}_{\mathcal{F}(f)}(X, X')\overline{\nabla}_{r,1}(X')^d \overline{\nabla}_{l,1}(X)^{-s_1 - \frac{m}{n}}$$
$$\overline{\nabla}_{r,1}(X')^{-s_n - \frac{m}{n} + d}\overline{\nabla}_{r,2}(X')^{s_n - s_{n-1}} \cdots \overline{\nabla}_{r,n}(X')^{s_3 - s_2}\, dX' dX$$

$$= \int_{\overline{\mathbb{V}}_l^{(1)}} \int_{\overline{\mathbb{V}}_r^{(n-1)}} \overline{T}_{\mathcal{F}(f)}(X, X')\overline{\nabla}_{r,1}(X')^d \overline{\nabla}_{l,1}(X)^{-s_1 - \frac{m}{n}}\overline{\nabla}_{r,t(\bar{\mathbf{s}})}(X')\, dX' dX$$

where

$$t(\bar{\mathbf{s}}) = \left(-s_n - \frac{m_{n-1}}{n-1}, -s_{n-1} - \frac{m_{n-1}}{n-1}, \ldots, -s_2 - \frac{m_{n-1}}{n-1} \right).$$

Using Proposition 3.5 and the following fact (*cf.* [13])

$$\overline{T}_{\mathcal{F}(f)}(X, X')\overline{\nabla}_{r,1}(X')^d = \mathcal{F}_1\left(\nabla_{l,1}(\,\cdot\,)^{d(n-1)}\mathcal{F}_{n-1}(T_f) \right)(X, X'),$$

where $\mathcal{F}_1$ (resp., $\mathcal{F}_{n-1}$) is the Fourier transform on $\overline{\mathbb{V}}_l^{(1)}$ (resp., $\overline{\mathbb{V}}_r^{(n-1)}$), we conclude

that

$$\overline{Z}(\mathcal{F}(f), t(\mathbf{s})) = \int_{\overline{\mathbb{V}}_1^{(1)}} \int_{\overline{\mathbb{V}}_r^{(n-1)}} \mathcal{F}_1\big(\nabla_{l,1}(\,\cdot\,)^{d(n-1)} \mathcal{F}_{n-1}(T_f)\big)$$

$$\times (X, X') \overline{\nabla}_{l,1}(X)^{-s_1 - \frac{m}{n}} \overline{\nabla}_{r, t(\tilde{\mathbf{s}})}(X') \, dX' dX$$

$$= \int_{\overline{\mathbb{V}}_1^{(1)}} \mathcal{F}_1\big(\nabla_{l,1}(\,\cdot\,)^{d(n-1)} \overline{Z}_{n-1}(\mathcal{F}_{n-1}(T_f), t(\tilde{\mathbf{s}}))\big) \overline{\nabla}_{l,1}(X)^{-s_1 - \frac{m}{n}} \, dX$$

$$= \overline{Z}_1\big(\mathcal{F}_1\big(\nabla_{l,1}(\,\cdot\,)^{d(n-1)} \overline{Z}_{n-1}(\mathcal{F}_{n-1}(T_f), t(\tilde{\mathbf{s}}))\big), -s_1 - d(n-1) - (e+1)\big)$$

$$= \pi^{-\frac{(e+1)(n-1)}{2} - \frac{d(n-1)(n-2)}{2}} \pi^{-|t(\tilde{\mathbf{s}})|} \pi^{-\frac{(e+1)}{2}} \pi^{s_1 + d(n-1) + (e+1)} \frac{\Gamma\big(\frac{-s_1 - d(n-1)}{2}\big)}{\Gamma\big(\frac{s_1 + d(n-1) + (e+1)}{2}\big)}$$

$$\times \prod_{j=1}^{n-1} \frac{\Gamma\big(\frac{-s_{j+1} - d(n-1-j)}{2}\big)}{\Gamma\big(\frac{s_{j+1} + d(n-1-j) + (e+1)}{2}\big)} Z_1\big(\nabla_{l,1}(\,\cdot\,)^{d(n-1)} Z_{n-1}(T_f, \tilde{\mathbf{s}}), s_1 + d(n-1)\big)$$

$$= \pi^{-\frac{m}{2}} \pi^{-|t(\mathbf{s})|} \prod_{j=1}^{n} \frac{\Gamma\big(\frac{-s_j - d(n-j)}{2}\big)}{\Gamma\big(\frac{s_j + d(n-j) + (e+1)}{2}\big)} Z_1\big(Z_{n-1}(T_f, \tilde{\mathbf{s}}), s_1 + 2d(n-1)\big)$$

$$= \pi^{-\frac{m}{2}} \pi^{-|t(\mathbf{s})|} \prod_{j=1}^{n} \frac{\Gamma\big(\frac{-s_j - d(n-j)}{2}\big)}{\Gamma\big(\frac{s_j + d(n-j) + (e+1)}{2}\big)} Z(f, \mathbf{s}).$$

Now the theorem follows by meromorphic continuation. Notice that $t(t(\mathbf{s})) = \mathbf{s}$. ∎

Remark 3.7   In [6], the authors consider the case of complex Jordan algebras where they also prove the functional equation. Our definitions of ∇_ℓ and $\overline{\nabla}_\ell$ on $\mathbb{V}$ and $\overline{\mathbb{V}}$, correspond to their definitions of $|\Delta_\ell|^{1/2}$ and $|\nabla_\ell^*|^{1/2}$ on $\overline{\mathbb{V}}$ and $\mathbb{V}$, respectively. The fact that these definitions are reversed explains the shift between the constants in [6] and our constants.

4   Bernstein Identities for $\overline{\nabla}_{\mathbf{s}}$

Recall that $P_\ell(X) := \nabla_\ell(X)^2$ and $\overline{P}_\ell(Y) := \overline{\nabla}_\ell(Y)^2$ are two homogeneous polynomials of degree $2(n-\ell+1)$. Then, define the constant coefficient operators characterized by

$$P_\ell(\partial_X) e^{\tau(X,Y)} = \overline{P}_\ell(Y) e^{\tau(X,Y)} \quad \text{and} \quad \overline{P}_\ell(\partial_Y) e^{\tau(X,Y)} = P_\ell(X) e^{\tau(X,Y)}.$$

Theorem 4.1   *There exists a polynomial $b_\ell(\mathbf{s})$ on $\mathbf{s}$ so that*

$$\overline{P}_\ell(\partial_Y) \overline{\nabla}_{\mathbf{s}}(Y) = b_\ell(\mathbf{s}) \overline{\nabla}_{\mathbf{s}^{(\ell)}}(Y),$$

where

$$\mathbf{s}^{(\ell)} = (s_1 - 2, s_2 - 2, \ldots, s_{n-\ell+1} - 2, s_{n-\ell+2}, \ldots, s_n)$$

and

$$b_\ell(\mathbf{s}) = \prod_{\kappa=\ell}^{n}(s_{n-\kappa+1} + d(\kappa - 1))(s_{n-\kappa+1} + (e - 1) + d(\kappa - 1)).$$

The polynomials b_ℓ are the so-called Bernstein polynomials of $\overline{\nabla}_\mathbf{s}$.

Proof Denote by

$$c(\mathbf{s}) = \pi^{-\frac{m}{2}}\pi^{-|\mathbf{s}|} \prod_{\kappa=1}^{n} \frac{\Gamma\left(\frac{s_\kappa + (e+1) + d(n-\kappa)}{2}\right)}{\Gamma\left(\frac{-s_\kappa - d(n-\kappa)}{2}\right)}.$$

By Theorem 3.6 we have

$$\int_{\overline{\mathbb{V}}} f(Y)\overline{\nabla}_\mathbf{s}(Y)\,dY = c(\mathbf{s}) \int_{\mathbb{V}} \mathcal{F}(f)(X)\nabla_{t(\mathbf{s})}(X)\,dX.$$

Changing f by $\overline{P}_\ell(\partial_Y)f$ in the above equation, and using the fact that

$$\mathcal{F}(\overline{P}_\ell(\partial_Y)f)(X) = (-1)^{n-\ell+1}(2\pi)^{2(n-\ell+1)}P_\ell(X)\mathcal{F}(f)(X),$$

we obtain

(6) $$\int_{\overline{\mathbb{V}}} \overline{P}_\ell(\partial_Y)f(Y)\overline{\nabla}_\mathbf{s}(Y)\,dY = (-1)^{n-\ell+1}(2\pi)^{2(n-\ell+1)}c(\mathbf{s})$$

$$\int_{\mathbb{V}} \mathcal{F}(f)(X)P_\ell(X)\nabla_{t(\mathbf{s})}(X)\,dX.$$

On the other hand, if $\mathbf{2}^{(n-\ell+1)}$ denotes the vector

$$(0,\ldots,0,\underbrace{2,\ldots,2}_{n-\ell+1}),$$

we can rewrite $P_\ell(X)\nabla_{t(\mathbf{s})}(X)$ as $\nabla_{t(\mathbf{s})+\mathbf{2}^{(n-\ell+1)}}(X)$, where

$$\nabla_{t(\mathbf{s})+\mathbf{2}^{(n-\ell+1)}} = \nabla_{t(\mathbf{s}^{(\ell)})}.$$

Here $\mathbf{s}^{(\ell)}$ is the parameter introduced in the statement of the theorem. Using integration by parts on the left-hand side of (6), we get

$$\int_{\overline{\mathbb{V}}} f(Y)\overline{P}_\ell(\partial_Y)\overline{\nabla}_\mathbf{s}(Y)\,dY = (-1)^{n-\ell+1}(2\pi)^{2(n-\ell+1)}c(\mathbf{s}) \int_{\mathbb{V}} \mathcal{F}(f)(X)\nabla_{t(\mathbf{s}^{(\ell)})}(X)\,dX$$

$$= (-1)^{n-\ell+1}(2\pi)^{2(n-\ell+1)}\frac{c(\mathbf{s})}{c(\mathbf{s}^{(\ell)})} \int_{\overline{\mathbb{V}}} f(Y)\overline{\nabla}_{\mathbf{s}^{(\ell)}}(Y)\,dY.$$

Hence

$$\overline{P}_\ell(\partial_Y)\overline{\nabla}_\mathbf{s}(Y) = (-1)^{n-\ell+1}(2\pi)^{2(n-\ell+1)}\frac{c(\mathbf{s})}{c(\mathbf{s}^{(\ell)})}\overline{\nabla}_{\mathbf{s}^{(\ell)}}(Y).$$

An easy computation shows that

$$\frac{c(\mathbf{s})}{c(\mathbf{s}^{(\ell)})} = \frac{1}{\pi^{2(n-\ell+1)}} \prod_{\kappa=1}^{n-\ell+1} \frac{\Gamma\left(\frac{s_\kappa+(e+1)+d(n-\kappa)}{2}\right)}{\Gamma\left(\frac{-s_\kappa-d(n-\kappa)}{2}\right)} \prod_{\kappa=1}^{n-\ell+1} \frac{\Gamma\left(\frac{-s_\kappa-d(n-\kappa)}{2}+1\right)}{\Gamma\left(\frac{s_\kappa+(e+1)+d(n-\kappa)}{2}-1\right)}.$$

Using the fact that $\Gamma(z+1) = z\Gamma(z)$, we can see that

$$\frac{c(\mathbf{s})}{c(\mathbf{s}^{(\ell)})} = \frac{1}{\pi^{2(n-\ell+1)}} \prod_{\kappa=1}^{n-\ell+1} \left(\frac{s_\kappa+(e+1)+d(n-\kappa)}{2}-1\right)\left(\frac{-s_\kappa-d(n-\kappa)}{2}\right)$$

$$= \frac{(-1)^{n-\ell+1}}{(2\pi)^{2(n-\ell+1)}} \prod_{\kappa=1}^{n-\ell+1} (s_\kappa+d(n-\kappa))(s_\kappa+(e-1)+d(n-\kappa)).$$

∎

For $\mathbf{t} \in \mathbb{N}^n$ such that $t_n \geq t_{n-1} \geq \cdots \geq t_1$, we set

$$\overline{P}_{\mathbf{t}}(\partial_Y) := \overline{P}_1(\partial_Y)^{t_1} \overline{P}_2(\partial_Y)^{t_2-t_1} \cdots \overline{P}_n(\partial_Y)^{t_n-t_{n-1}}.$$

Theorem 4.2   *For $\mathbf{t} \in \mathbb{N}^n$ such that $t_n \geq t_{n-1} \geq \cdots \geq t_1$, the following holds*

$$\overline{P}_{\mathbf{t}}(\partial_Y)\overline{\nabla}_{\mathbf{s}}(Y) = b_{\mathbf{t}}(\mathbf{s})\overline{\nabla}_{\mathbf{s}-2\mathbf{t}^*}(Y),$$

where

$$b_{\mathbf{t}}(\mathbf{s}) = \prod_{\kappa=1}^{n} (s_\kappa + \frac{m}{n} - d(\kappa-1) - 2t_\kappa)_{t_\kappa,\text{even}}(s_\kappa + d(n-\kappa))_{t_\kappa,\text{even}},$$

and $\mathbf{t}^* = (t_n, \ldots, t_1)$. Here $(a)_{t,\text{even}} := a(a+2)(a+4)\cdots(a+2t)$.

Proof   Let $|\mathbf{t}| = t_1 + \cdots + t_n$. Since $\mathcal{F}(\overline{P}_{\mathbf{t}}(\partial_Y)f)(X) = (-1)^{|\mathbf{t}|}(2\pi)^{2|\mathbf{t}|}\overline{P}_{\mathbf{t}}(X)\mathcal{F}(f)(X)$, we can employ the same argument used in the proof of Theorem 4.1 with $2|\mathbf{t}|$ integrations by parts to obtain

$$\overline{P}_{\mathbf{t}}(\partial_Y)\overline{\nabla}_{\mathbf{s}}(Y) = (-1)^{|\mathbf{t}|}(2\pi)^{2|\mathbf{t}|}\frac{c(\mathbf{s})}{c(\mathbf{s}-2\mathbf{t}^*)}\overline{\nabla}_{\mathbf{s}-2\mathbf{t}^*}(Y).$$

∎

Next we shall use $\overline{\left(-\frac{m}{n}\right)}$ to denote $\left(-\frac{m}{n}, \ldots, -\frac{m}{n}\right)$.

Corollary 4.3   *For any $f \in \mathcal{S}(\overline{\mathbb{V}})$*

$$\frac{\overline{Z}(f,\mathbf{s})}{\prod_{j=1}^{n} \Gamma\left(\frac{s_j+(m/n)-d(j-1)}{2}\right)} \Bigg|_{\mathbf{s}=\overline{\left(-\frac{m}{n}\right)}} = \frac{\pi^{m/2}}{\prod_{j=1}^{n} \Gamma\left(\frac{m/n-d(j-1)}{2}\right)} f(0).$$

Proof By the functional equation, we have

$$\frac{\overline{Z}(f, \mathbf{s})}{\prod_{j=1}^{n} \Gamma\left(\frac{s_j + (e+1) + d(n-j)}{2}\right)} = \frac{\pi^{-m/2} \pi^{-|\mathbf{s}|} Z(\mathcal{F}(f), t(\mathbf{s}))}{\prod_{j=1}^{n} \Gamma\left(\frac{-s_j - d(n-j)}{2}\right)}.$$

Since

$$\prod_{j=1}^{n} \Gamma\left(\frac{s_j + (e+1) + d(n-j)}{2}\right) = \prod_{j=1}^{n} \Gamma\left(\frac{s_j + \frac{m}{n} - d(j-1)}{2}\right),$$

the corollary follows by Fourier inversion. ∎

The next corollary is a consequence of Corollary 4.3 and the fact that

$$\overline{Z}(\overline{P}_{\mathbf{t}}(\partial_Y) f, \mathbf{s}) = b_{\mathbf{t}}(\mathbf{s}) \overline{Z}(f, \mathbf{s} - 2\mathbf{t}^*).$$

Corollary 4.4

$$\left(\pi^{-\frac{m}{2}} b_{\mathbf{t}}(\mathbf{s}) \prod_{j=1}^{n} \frac{\Gamma\left(\frac{-s_j - d(n-j)}{2}\right)}{\Gamma\left(\frac{s_j + (m/n) - d(j-1)}{2}\right)} \overline{Z}(f, \mathbf{s} - 2\mathbf{t}^*) \right) \Big|_{\mathbf{s} = \overline{(-\frac{m}{n})}} = \overline{P}_{\mathbf{t}}(\partial_Y) \delta$$

where δ denotes the Dirac function at 0.

Remark 4.5 This remark is an extension of [2, Théorème III.10], which can be derived by an almost straightforward modification of the proof in [2]. To simplify the presentation of our statement, we shall slightly change the definition of $\nabla_{\mathbf{s}}$. For $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$, set $\nabla^{\boldsymbol{\alpha}} := \nabla_1^{\alpha_1} \nabla_2^{\alpha_2} \cdots \nabla_n^{\alpha_n}$. The relation between $\nabla^{\boldsymbol{\alpha}}$ and $\nabla_{\mathbf{s}}$ is that $\nabla^{\boldsymbol{\alpha}} = \nabla_{\mathbf{s}}$ if and only if $\alpha_1 = s_1$ and $\alpha_i = s_i - s_{i-1}$ for $2 \leq i \leq n$. For $\boldsymbol{\alpha} \in \mathbb{C}^n$, such that $\mathrm{Re}(\alpha_i) \gg 0$ for all $1 \leq i \leq n$, the function $\nabla^{\boldsymbol{\alpha}}$ is absolutely convergent. We claim that if $f \in \mathcal{S}(\mathbb{V})$, then the function

$$\boldsymbol{\alpha} \mapsto R_{\boldsymbol{\alpha}}(f),$$

where

$$R_{\boldsymbol{\alpha}}(f) := \frac{\int_{\mathbb{V}} f(X) \nabla^{\boldsymbol{\alpha}}(X) \, dX}{\prod_{1 \leq i \leq j \leq n} \Gamma(\alpha_i + \cdots + \alpha_j + d(n-j) + 1) \Gamma(\alpha_i + \cdots + \alpha_j + d(n-j) + e)},$$

extends holomorphically to all of $\mathbb{C}^n$. We shall only present the main steps of the argument. In this setting, $R_{\boldsymbol{\alpha}}$ deserves the name of Riesz distribution associated to the non-Euclidean Jordan algebra $\mathbb{V}$. From Theorem 4.1, one can see that if we apply the polynomial differential operator $P_\ell(\partial_X)$ to $\nabla^{\boldsymbol{\alpha}}$, then the shift is only negative in the power of the first factor in $\nabla^{\boldsymbol{\alpha}}$, for every ℓ. In order to shift the power of the other factors in $\nabla^{\boldsymbol{\alpha}}$ we will use different differential operators.

For $X \in \mathbb{V}^{\sharp} = \{X \in \mathbb{V} \mid \nabla_1(X) \cdots \nabla_n(X) \neq 0\}$ and for $1 \leq \ell \leq n$, set

$$\mathbb{D}_{\ell}(\boldsymbol{\alpha}, \partial_X) = \nabla_1(X)^{2+\alpha_1} \cdots \nabla_{\ell-1}(X)^{2+\alpha_{\ell-1}} \circ P_{\ell}(\partial_X) \circ \nabla_1(X)^{-\alpha_1} \cdots \nabla_{\ell-1}(X)^{-\alpha_{\ell-1}}.$$

Using Theorem 4.1, we obtain

$$\mathbb{D}_{\ell}(\boldsymbol{\alpha}, \partial_X) \nabla(X)^{\boldsymbol{\alpha}} = \prod_{\kappa=1}^{n-\ell+1} \left(\alpha_{\ell} + \cdots + \alpha_{n-\kappa+1} + d(\kappa-1) \right)$$

$$\left(\alpha_{\ell} + \cdots + \alpha_{n-\kappa+1} + d(\kappa-1) + (e-1) \right) \nabla(X)^{\boldsymbol{\alpha}-2_{\ell}+2_1+\cdots+2_{\ell-1}},$$

where $2_j = (0, \ldots, 0, 2, 0, \ldots, 0)$ and

$$\boldsymbol{\alpha} - 2_{\ell} + 2_1 + \cdots + 2_{\ell-1} = (\alpha_1 + 2, \ldots, \alpha_{\ell-1} + 2, \alpha_{\ell} - 2, \alpha_{\ell+1}, \ldots, \alpha_n).$$

Now we shall construct new differential operators which will allow us to shift down the power of any factor in $\nabla^{\boldsymbol{\alpha}}$.

For $1 \leq \ell \leq n$ and $1 \leq j \leq \ell - 1$, let

$$\mathbb{E}_{j,\ell}(\boldsymbol{\alpha}, \partial_X) = \prod_{i=1}^{2^{\ell-1-j}} \mathbb{D}_j(a(\boldsymbol{\alpha}, \ell, i, j), \partial_X)$$

where

$$a(\boldsymbol{\alpha}, \ell, i, j) = \boldsymbol{\alpha} - 2_{\ell} + \left(2^{\ell-1-j} - (i-1) \right) 2_j + \left\{ (i-1) + 2^{\ell-1-j} \right\} \left(2_1 + \cdots + 2_{j-1} \right).$$

The operator $\mathbb{P}_{\ell}(\boldsymbol{\alpha}, \partial_X) := \mathbb{E}_{1,\ell}(\boldsymbol{\alpha}, \partial_X) \circ \cdots \circ \mathbb{E}_{\ell-1,\ell}(\boldsymbol{\alpha}, \partial_X) \circ \mathbb{D}_{\ell}(\boldsymbol{\alpha}, \partial_X)$ satisfies

$$(7) \quad \mathbb{P}_{\ell}(\boldsymbol{\alpha}, \partial_X) \nabla(X)^{\boldsymbol{\alpha}} = \prod_{\kappa=1}^{n-\ell+1} N_{\ell,\kappa}(\boldsymbol{\alpha}) \prod_{j=1}^{\ell-1} \prod_{i=1}^{2^{\ell-1-j}}$$

$$\left\{ \prod_{\kappa=1}^{n-\ell+1} N_{j,\kappa}(\boldsymbol{\alpha} + 2^{\ell-1-j} - 2i) \prod_{\kappa=n-\ell+2}^{n-j+1} N_{j,\kappa}(\boldsymbol{\alpha} + 2^{\ell-1-j} - 2(i-1)) \right\} \nabla(X)^{\boldsymbol{\alpha}-2_{\ell}},$$

where

$$N_{j,\kappa}(\boldsymbol{\alpha}) := (\alpha_j + \cdots + \alpha_{n-\kappa+1} + d(\kappa-1))(\alpha_j + \cdots + \alpha_{n-\kappa+1} + d(\kappa-1) + (e-1)).$$

Now, we consider a second family of operators defined by

$$\mathbb{F}_{j,\ell}(\boldsymbol{\alpha}, \partial_X) = \prod_{i=1}^{2^{\ell-1-j}} \mathbb{D}_j(b(\boldsymbol{\alpha}, \ell, i, j), \partial_X)$$

where $b(\boldsymbol{\alpha}, \ell, i, j) = \boldsymbol{\alpha} - (i-1)2_j + \{i - 1 - 2^{\ell-j}\}(2_1 + \cdots + 2_{j-1})$ for $1 \leq j \leq \ell - 1$. The operator

$$\mathbb{P}'_\ell(\boldsymbol{\alpha}, \partial_X) := \mathbb{D}_\ell\big(\alpha - (2_1 + \cdots + 2_{\ell-1}), \partial_X\big) \circ \mathbb{F}_{\ell-1,\ell}(\boldsymbol{\alpha}, \partial_X) \circ \cdots \circ \mathbb{F}_{1,\ell}(\boldsymbol{\alpha}, \partial_X)$$

satisfies

$$(8) \quad \mathbb{P}'_\ell(\boldsymbol{\alpha}, \partial_X)\nabla(X)^{\boldsymbol{\alpha}} = \prod_{\kappa=1}^{n-\ell+1} N_{\ell,\kappa}(\boldsymbol{\alpha}) \prod_{j=1}^{\ell-1} \prod_{i=1}^{2^{\ell-1-j}} \prod_{\kappa=1}^{n-j+1} N_{j,\kappa}(\boldsymbol{\alpha} - 2(i-1))\nabla(X)^{\boldsymbol{\alpha}-2\ell}.$$

The purpose of $\mathbb{P}_\ell$ and $\mathbb{P}'_\ell$ is to obtain, as nearly as we can, the set of possible poles of $\nabla^{\boldsymbol{\alpha}}$. Indeed, by [2, Théorème III.10], only the common zeros of the Bernstein polynomials in (7) and (8) can give a potential pole. Further, for every fixed $\ell \leq n$, one can check that the common factors in (7) and (8) are

$$(9) \quad \prod_{1 \leq j \leq \ell \leq i \leq n} \big(\alpha_j + \cdots + \alpha_i + d(n-i)\big)\big(\alpha_j + \cdots + \alpha_i + d(n-i) + (e-1)\big).$$

Now we come to the last step in the proof of the claim above. For $\boldsymbol{\alpha} \in \mathbb{C}^n$ such that $\operatorname{Re} \alpha_i \gg 0$, for $1 \leq i \leq n$, the function $\nabla^{\boldsymbol{\alpha}}$ is absolutely convergent. By [2, Théorème III.10], the factor (9) is the greatest common divisor of each Bernstein-type polynomial $b(\boldsymbol{\alpha})$ that satisfies $D_\ell(\boldsymbol{\alpha}, \partial_X)\nabla^{\boldsymbol{\alpha}} = b(\boldsymbol{\alpha})\nabla^{\boldsymbol{\alpha}-2\ell}$, for all $1 \leq \ell \leq n$ and for all polynomial differential operators $D_\ell(\boldsymbol{\alpha}, \partial_X)$. Hence, the set of potential poles of $Z(f, \boldsymbol{\alpha})$ is included in the set of those of

$$\prod_{1 \leq j \leq i \leq n} \Gamma\big(\alpha_j + \cdots + \alpha_i + d(n-i) + 1\big) \Gamma\big(\alpha_j + \cdots + \alpha_i + d(n-i) + (e-1) + 1\big).$$

Remark 4.6 The distributions $R_{\boldsymbol{\alpha}}$ are related to the theory of classical Riesz distributions and generalizations. For instance, when $\mathbb{V}$ is a simple Euclidean Jordan algebra with symmetric cone Ω, then families of Riesz distributions are associated to Ω. We refer to Faraut–Korányi's book [10] for a thorough study on Riesz distributions for symmetric cones; see also [12]. In [10], the authors investigate the set of multi-parameters $\boldsymbol{\alpha}$ for which $R_{\boldsymbol{\alpha}}$ is of positive type, *i.e.*, $R_{\boldsymbol{\alpha}}$ is a positive measure. For example, on the real line, and in Faraut–Korányi's setting, $R_{\boldsymbol{\alpha}}$ coincides with the Riemann–Liouville distribution $\frac{1}{\Gamma(\alpha)}x_+^{\alpha-1}$, which is positive if and only if $\alpha \geq 0$. In the framework of non-Euclidean Jordan algebras, it is a natural question to ask for conditions on $\boldsymbol{\alpha}$ such that $R_{\boldsymbol{\alpha}}$ is of positive type. In the case where $\alpha_1 \neq 0$ and $\alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$, this problem has a solution in [3], but the general case remains open.

Appendix

For the convenience of the reader, we briefly give some details on the notations that we used in this paper for the case $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$. Assume that $\mathfrak{g}$ is the Lie algebra $\mathfrak{sp}(n, \mathbb{C})$ which is given by

$$\mathfrak{sp}(n, \mathbb{C}) = \left\{ \begin{bmatrix} U & V \\ W & -U^t \end{bmatrix} \;\middle|\; V, W \in \operatorname{Sym}(n, \mathbb{C}), U \in M(n, \mathbb{C}) \right\},$$

where $\text{Sym}(n, \mathbb{C})$ denotes the subspace of symmetric matrices in $M(n, \mathbb{C})$. The Lie algebra $\mathfrak{g}$ can be written as $\mathfrak{g} = \mathbb{V} \oplus \mathfrak{l} \oplus \overline{\mathbb{V}}$ where

$$\mathbb{V} = \left\{ \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix} \middle| V \in \text{Sym}(n, \mathbb{C}) \right\},$$

$$\overline{\mathbb{V}} = \left\{ \begin{bmatrix} 0 & 0 \\ W & 0 \end{bmatrix} \middle| W \in \text{Sym}(n, \mathbb{C}) \right\},$$

$$\mathfrak{l} = \left\{ \begin{bmatrix} U & 0 \\ 0 & -U^t \end{bmatrix} \middle| U \in M(n, \mathbb{C}) \right\}.$$

In this example, $m_{\mathbb{R}} = n(n+1)$, $e = 1$, and $d = 1$. On $\mathbb{V}$, the involution ν is given by $\nu \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ V & 0 \end{bmatrix}$. The action of the structure group L on $\mathbb{V}$ is given by

$$\begin{bmatrix} u & 0 \\ 0 & \{u^t\}^{-1} \end{bmatrix} \cdot \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & uVu^t \\ 0 & 0 \end{bmatrix}.$$

For $g_1, g_2 \in \mathfrak{g}$, the pairing $\langle\!\langle g_1, g_2 \rangle\!\rangle_{\mathfrak{g}} = \text{Re}(\text{Tr}(g_1 g_2))$. We choose the Lebesgue measures dX on $\mathbb{V}$ and dY on $\overline{\mathbb{V}}$ to be

$$dX = (\sqrt{2})^{n(n-1)} \prod_{i \leq j} dv_{i,j}, \quad X = \begin{bmatrix} 0 & (v_{i,j})_{i,j} \\ 0 & 0 \end{bmatrix} \in \mathbb{V},$$

$$dY = (\sqrt{2})^{n(n-1)} \prod_{i \leq j} dw_{i,j}, \quad Y = \begin{bmatrix} 0 & 0 \\ (w_{i,j})_{i,j} & 0 \end{bmatrix} \in \overline{\mathbb{V}}.$$

For $1 \leq \ell \leq n$, we have the following subalgebras

$$\mathbb{V}_{\text{l}}^{(\ell)} = \left\{ \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \middle| A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, a \in \text{Sym}(\ell, \mathbb{C}) \right\} \subset \mathbb{V},$$

$$\mathbb{V}_{\text{r}}^{(n-\ell)} = \left\{ \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} \middle| C = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}, c \in \text{Sym}(n - \ell, \mathbb{C}) \right\} \subset \mathbb{V},$$

$$\mathfrak{g}(1, -1; \ell) = \left\{ \begin{bmatrix} B & 0 \\ 0 & -B^t \end{bmatrix} \middle| B = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}, b \in M(n - \ell, \ell; \mathbb{C}) \right\} \subset \mathfrak{l}.$$

The change of coordinates

$$\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} B & 0 \\ 0 & -B^t \end{bmatrix}; \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} \right) \longmapsto \begin{bmatrix} 0 & \begin{bmatrix} a & ab^t \\ ba & bab^t + c \end{bmatrix} \\ 0 & 0 \end{bmatrix},$$

is a diffeomerphism from $\mathcal{O}_{\text{l}}^{(\ell)} \times \mathfrak{g}(1, -1; \ell) \times \mathbb{V}_{\text{r}}^{(n-\ell)} \to \mathcal{O}^{(\ell)}$. Here $\mathcal{O}_{\text{l}}^{(\ell)}$ is the set of elements in $\mathbb{V}_{\text{l}}^{(\ell)}$ such that the matrix a is invertible, and $\mathcal{O}^{(\ell)}$ is the set of elements $\begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$ in $\mathbb{V}$ where the principal minor $\det_{\ell}(V) \neq 0$. We have a similar change of coordinates for $\overline{\mathbb{V}}$.

Finally, we shall write the integral formula (3), where we identify $\mathbb{V}$ with $\mathrm{Sym}(n, \mathbb{C})$, $\mathbb{V}_\mathrm{l}^{(1)}$ with $\mathbb{C}$, $\mathbb{V}_\mathrm{r}^{(n-1)}$ with $\mathrm{Sym}(n-1, \mathbb{C})$, and $\mathfrak{g}(1, -1; 1)$ with $\mathbb{C}^{n-1}$. Thus, for $f \in \mathcal{L}^1(\mathrm{Sym}(n, \mathbb{C}), dX)$, we have

$$\int_{\mathrm{Sym}(n,\mathbb{C})} f(X) dX = \int_{\mathrm{Sym}(n-1,\mathbb{C})} \int_{\mathbb{C}^{n-1}} \int_{\mathbb{C}} f\left(\begin{bmatrix} a & ab^t \\ ba & bab^t + c \end{bmatrix} \right) |a|^{2(n-1)} \, da \, db \, dc.$$

A similar formula for $\overline{\mathbb{V}}$ holds.

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