# FREE PRODUCTS WITH AMALGAMATION OF COMMUTATIVE INVERSE SEMIGROUPS*

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## 1. Introduction

A class of algebras $\mathscr{A}$ is said to have the *strong amalgamation property* if for any indexed set of algebras $\{A_i: i \in J\}$ from $\mathscr{A}$, each having an algebra $U \in \mathscr{A}$ as a subalgebra, there exist an algebra $B \in \mathscr{A}$ and monomorphisms $\phi_i: A_i \to B$ (for each $i \in J$) such that

(i) $\phi_i \mid U = \phi_j \mid U$ for all $i, j \in J$,

(ii) $\phi_i(A_i) \cap \phi_j(A_j) = \phi_i(U)$ for all $i, j \in J$ with $i \neq j$,

where $\phi_i \mid U$ denotes the restriction of $\phi_i$ to $U$. Omitting condition (ii) gives us the definition of the *weak amalgamation property.*

The development and importance of the (weak) amalgamation property in various branches of algebra are described in Jónsson (1965). In his paper (1975), Hall proves that the class of inverse semigroups has the strong amalgamation property. His result gives us strong hope that there may be other classes of semigroups which have the strong amalgamation property. The purpose of this paper is to prove that the class of commutative inverse semigroups has the strong amalgamation property. In Section 2 we prove it by constructing the free product amalgamating a common inverse subsemigroup, in the variety of commutative inverse semigroups. In Section 3 we show a simpler proof using the fact that the class of semilattices has the strong amalgamation property [Hall (1975), Remark 5].

It is clear that in a class of algebras closed under isomorphisms and the formation of the union of any ascending chain of algebras, the strong amalgamation property follows by transfinite induction from the case in which $|J| = 2$. Hence we shall consider in this paper only the case $|J| = 2$.

Let $S = \bigcup_{\alpha \in I} S_\alpha$ be a semigroup which is a semilattice $I$ of subsemigroups $S_\alpha$ of $S$. Hereafter "an element $x_\alpha$ of $S$" means "an element $x_\alpha$ of $S_\alpha$". Let $S$ be an inverse semigroup and $X$ be a subset. We let $\langle X \rangle$ denote the inverse subsemigroup of $S$ generated by $X$. The notations and terminologies are those of Clifford and Preston (1961, 1967) and Preston (1973), unless otherwise stated.

## 2. Strong amalgamation property

Let $S$ and $T$ be commutative inverse semigroups with a commom inverse subsemigroup $U$. We can assume without loss of generality that $S \cap T = U$. Let $E(S)$, $E(T)$ and $E(U)$ be the sets of idempotents of $S$, $T$ and $U$ respectively. Let $S^{(1)}$, $T^{(1)}$ and $U^{(1)}$ denote the commutative inverse semigroup $S \cup \{1\}$, $T \cup \{1\}$ and $U \cup \{1\}$ obtained by adjoining an identity 1 to each of $S$, $T$ and U (whether or not they already have identities). It is clear that $(S^{(1)} \times T^{(1)}) \setminus \{(1, 1)\}$ is isomorphic to the free commutative inverse semigroup product $S * T$, say, of $S$ and $T$. Hereafter we let $S * T$ denote $(S^{(1)} \times T^{(1)}) \setminus \{(1, 1)\}$. It follows from Exercise 6 of [Clifford and Preston (1961), Section 1.9] that $S^{(1)} = \bigcup_{\alpha \in E(S^{(1)})} S_\alpha$, $T^{(1)} = \bigcup_{\alpha \in E(T^{(1)})} T_\alpha$ and $U^{(1)} = \bigcup_{\alpha \in E(U^{(1)})} U_\alpha$ where $S_\alpha$, $T_\alpha$ and $U_\alpha$ are maximal subgroups, with identity $\alpha$, of $S^{(1)}$, $T^{(1)}$ and $U^{(1)}$ respectively. By the uniqueness of decomposition, $U_\alpha = S_\alpha \cap T_\alpha$ for each $\alpha \in E(U^{(1)})$.

Define a relation $\rho$ on $S * T$ as follows:

(1) For any elements $(x, y)$, $(x', y')$ of $S * T$, denote $(x, y)\rho_0(x', y')$ to mean that $(x, y) = (a, b)(u, 1)$ and $(x', y') = (a, b)(1, u)$ for some $(a, b) \in S^{(1)} \times T^{(1)}$ and $u \in U$. Let $\rho_1 = \rho_0 \cup \rho_0^{-1} \cup \iota$ and $\rho = \rho_1^t$.

Then of course $\rho$ is the congruence on $S * T$ generated by $\{((u, 1), (1, u)) \; u \in U\}$ [Clifford and Preston (1961), Section 1.5]. Let $(x, y)\rho$ denote the $\rho$-class containing $(x, y)$ in $S * T$.

LEMMA 1. *If $(x, 1)\rho(x', y')$, then there exists $\sigma \in E(U^{(1)})$ such that $\sigma y' \in U^{(1)}$ and $x = x'(\sigma y')$.*

PROOF. Let $(x, 1)\rho(x', y')$. By the definition (1), there exist $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$ in $S * T$ such that $(x, 1) = (a_1, b_1)$, $(x', y') = (a_n, b_n)$ and $(a_i, b_i)\rho_1(a_{i+1}, b_{i+1})$ for $i = 1, 2, \cdots, n - 1$.

We use induction on $n$. For $n = 1$, $(x, 1) = (x', y') = (a_1, b_1)$ and we take $\sigma = 1$. So we assume that the statement is true for $n - 1$. By the induction hypothesis, there exists $\sigma$ in $E(U^{(1)})$ such that $\sigma b_{n-1} \in U^{(1)}$ and $x = a^{n-1}(\sigma b_{n-1})$.

First we assume $(a_{n-1}, b_{n-1})\rho_0(a_n, b_n)$; then there exist $(c, d) \in S^{(1)} \times T^{(1)}$ and $u \in U^{(1)}$ such that $(a_{n-1}, b_{n-1}) = (c, d)(u, 1)$ and $(a_n, b_n) = (c, d)(1, u)$. Now $u \in U_\tau$ for some $\tau \in E(U^{(1)})$ and then

$$\tau\sigma b_n = \tau\sigma du = (\sigma b_{n-1})u \in U^{(1)}$$

since $\sigma b_{n-1}$, $u \in U^{(1)}$. Then because both belong to $S^{(1)}$, we can consider the product of $a_n$ and $\tau\sigma b_n$ in $S^{(1)}$, and

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$$a_n(\tau\sigma b_n) = c(\sigma b_{n-1})u = a_{n-1}(\sigma b_{n-1}) = x.$$

Similarly we may prove the statement in the case $(a_{n-1}, b_{n-1})\rho_0^{-1} \cup \iota(a_n, b_n)$. We obtain the lemma.

We define mappings $\phi_1: S \to (S * T)/\rho$ and $\phi_2: T \to (S * T)/\rho$ as follows:

$$\phi_1(x) = (x, 1)\rho \quad \text{for all} \quad x \in S,$$

$$\phi_2(y) = (1, y)\rho \quad \text{for all} \quad y \in T.$$

LEMMA 2. (i) *$\phi_1$ and $\phi_2$ are monomorphisms such that the following conditions are satisfied*:

$$\phi_1 \mid U = \phi_2 \mid U,$$

$$\phi_1(S) \cap \phi_2(T) = \phi_1(U)(= \phi_2(U)).$$

(ii) *$(S * T)/\rho$ is the free product $S *_U T$, say, of $S$ and $T$ amalgamating $U$ in the variety of commutative inverse semigroups.*

PROOF. It is clear that $\phi_1$ is a homomorphism.

Let $\phi_1(x) = \phi_1(x')$ where $x$ and $x'$ are elements of $S$. Then $x \in S_\alpha$, $x' \in S_\beta$ for some $\alpha, \beta \in E(S)$, and $(x, 1)\rho(x', 1)$. By Lemma 1, there exist $\sigma$ and $\tau$ in $E(U^{(1)})$ such that $x = x'\sigma$ and $x' = x\tau$. Then $\alpha = \beta \leqq \sigma\tau$ and $x = x'$. Hence $\phi_1$ is a monomorphism. Similarly $\phi_2$ is a monomorphism.

Since $(u, 1)\rho(1, u)$ for any $u$ in $U$, it is immediate that $\phi_1 \mid U = \phi_2 \mid U$.

Let $\phi_1(x) = \phi_2(y)$ where $x$ and $y$ are elements of $S$ and $T$ respectively. Then $(x, 1)\rho(1, y)$. By Lemma 1 and its dual, there exist $\sigma$, $\tau$ in $E(U^{(1)})$ such that $x = 1 \cdot (\sigma y) \in U^{(1)}$ and $y = 1 \cdot (\tau x) \in U^{(1)}$. Let $x \in S_\alpha$, $y \in T_\beta$ where $\alpha \in E(S)$ and $\beta \in E(T)$. Since $U_1 = \{1\}$, $\alpha, \beta \in E(U)$, $\alpha = \beta \leqq \sigma\tau$ and $x = y \in U$. Hence $\phi_1(S) \cap \phi_2(T) \subseteq \phi_1(U)$. It is trivial that $\phi_1(S) \cap \phi_2(T) \supseteq \phi_1(U)$. Thus $\phi_1(S) \cap \phi_2(T) = \phi_1(U)$, and (i) follows.

Part (ii) is of course obvious.

Now we have the following theorem, as an immediate consequence of Lemma 2(i).

THEOREM 1. *The class of commutative inverse semigroups has the strong amalgamation property.*

Since the direct product of two semilattices and its homomorphic images are semilattices, we have the following corollary, due to [Hall (1975), Remark 5].

COROLLARY 1. *The class of semilattices has the strong amalgamation property.*

The following corollaries are easy to prove directly or by using [Jónsson (1965), Theorems 3.4, 3.5].

COROLLARY 2. *Let $S_i$, $T_i$ and $A$ be commutative inverse semigroups such that $S_i$ is a subsemigroup of $T_i$ and $\phi_i$ is a monomorphism of $S_i$ into $A$, where $i$ ranges over an index set $I$. Then there exists a commutative inverse semigroup $B$ such that $A$ is a subsemigroup of $B$ and each $\phi_i$ extends ot a monomorphism of $T_i$ into $B$.*

COROLLARY 3. *For any commutative inverse semigroup $S$, there exists a commutative inverse semigroup $T$ such that $S$ is a subsemigroup of $T$ and any isomorphism between inverse subsemigroups of $S$ extends to an automorphism of $T$.*

## 3. Another proof

In this section, we assume [Hall (1975), Remark 5], that is, the class of semilattices has the strong amalgamation property. Let $S$ and $T$ be commutative inverse semigroups with a common inverse subsemigroup $U$. As we said in Section 2, we assume without loss of generality that $S \cap T = U$. By the assumption above there exists a semilattice $L$ such that $E(S)$ and $E(T)$ are embedded in $L$ and $E(S) \cap E(T) = E(U)$ in $L$. Then $L^{(1)}$ contains $E(S^{(1)})$ and $E(T^{(1)})$ and $E(S^{(1)}) \cap E(T^{(1)}) = E(U^{(1)})$ in $L^{(1)}$. We define a relation $\theta$, depending on $L$, on $S * T$ $(=(S^{(1)} \times T^{(1)}) \setminus \{(1,1)\})$ as follows:

(2) $(x_\alpha, y_\beta)\theta(x'_\gamma, y'_\delta)$ if and only if $\alpha\beta = \gamma\delta$ (in $L^{(1)}$) and there exists $u_\sigma \in U^{(1)}$ such that $\sigma \geqq \alpha\beta$ (in $L^{(1)}$) and $(u_\sigma, u_\sigma^{-1})(x_\alpha, y_\beta) = (\sigma, \sigma)(x'_\gamma, y'_\delta)$ (in $S^{(1)} \times T^{(1)}$).

It is routine to show that $\theta$ is a congruence on $S * T$. Thus $(S * T)/\theta$ is a commutative inverse semigroup. Let $(x, y)\theta$ be the $\theta$-class containing $(x, y)$.

We define mappings $\phi_1 \colon S \to (S * T)/\theta$ and $\phi_2 \colon T \to (S * T)/\theta$ as follows:

$$\phi_1(x) = (x, 1)\theta \quad \text{where} \quad x \in S,$$

$$\phi_2(y) = (1, y)\theta \quad \text{where} \quad y \in T.$$

LEMMA 3. (i) *$\phi_1$ and $\phi_2$ are monomorphisms such that the following donditions are satisfied:*

$$\phi_1 | U = \phi_2 | U,$$

$$\phi_1(S) \cap \phi_2(T) = \phi_1(U)(= \phi_2(U)).$$

(ii) *If $L$ is the free product $E(S) *_{E(U)} E(T)$, say, of $E(S)$ and $E(T)$ amalgamating $E(U)$ in the variety of semilattices, then $E((S * T)/\theta) \cong L$ and $(S * T)/\theta$ is the free product of $S$ and $T$ amalgamating $U$ in the variety of commutative inverse semigroups.*

PROOF. It is easy to see that $\phi_1$ is a homomorphism. We assume that $\phi_1(x_\alpha) = \phi_2(x'_\beta)$ where $x_\alpha$ and $x'_\beta$ are elements of $S$. Then $(x_\alpha, 1)\theta(x'_\beta, 1)$. By the definition (2), $\alpha = \beta$ and there exists $u_\sigma \in U^{(1)}$ such that $\sigma \geqq \alpha$ (in $L^{(1)}$) and $(u_\sigma, u_\sigma^{-1})(x_\sigma, 1) = (\sigma, \sigma)(x'_\alpha, 1)$ (in $S^{(1)} \times T^{(1)}$). Then $u_\sigma = \sigma$ and $x_\alpha = x'_\alpha$. Hence $\phi_1$ is a monomorphism. Similarly $\phi_2$ is a monomorphism.

Let $u_\alpha$ be an element of $U$. Since $(u_\alpha^{-1}, u_\alpha)(u_\alpha, 1) = (\alpha, \alpha)(1, u_\alpha)$,

$$\phi_1(u_\alpha) = (u_\alpha, 1)\theta = (1, u_\alpha)\theta = \phi_2(u_\alpha).$$

Thus $\phi_1 | U = \phi_2 | U$.

Let $\phi_1(x_\alpha) = \phi_2(y_\beta)$ where $x_\alpha \in S$ and $y_\beta \in T$. Then $(x_\alpha, 1)\theta(1, y_\beta)$. By the definition (2), $\alpha = \beta$ and there exists $u_\sigma \in U^{(1)}$ such that $\sigma \geqq \alpha$ (in $L^{(1)}$) and $(u_\sigma, u_\sigma^{-1})(x_\alpha, 1) = (\sigma, \sigma)(1, y_\alpha)$ (in $S^{(1)} \times T^{(1)}$). Then $\sigma = \alpha$ and $x_\alpha = y_\alpha = u_\alpha^{-1} \in U^{(1)} \cap S = U$. Hence $\phi_1(S) \cap \phi_2(T) \subseteq \phi_1(U)$. It is trivial that $\phi_1(S) \cap \phi_2(T) \supseteq \phi_1(U)$. Thus $\phi_1(S) \cap \phi_2(T) = \phi_1(U)$, and we have proved (i).

Let $\iota_1: E(S) \to L$ and $\iota_1: E(T) \to L$ be inclusion mappings. Since $L = E(S) *_{E(U)} E(T)$, there exists the unique homomorphism $\xi: L \to E((S * T)/\theta)$ such that $\xi\iota_i = \phi_i$ for $i = 1, 2$. Note that $E((S * T)/\theta) = \{(\alpha, \beta)\theta : (\alpha, \beta) \in E(S * T)\}$ from [Clifford and Preston (1967), Lemma 7.34]. Let $\eta: E((S * T)/\theta) \to L$ be a mapping defined by

$$\eta((\alpha, \beta)\theta) = \alpha\beta, \text{ for all } (\alpha, \beta) \in (E(S^{(1)}) \times E(T^{(1)})) \setminus \{(1, 1)\}.$$

It is routine to show that $\eta$ is well-defined and is a homomorphism and that $\xi\eta$ is the identity mapping on $E((S * T)/\theta)$. Since $\eta\xi$ maps $E(S)$ and $E(T)$ identically and $L = \langle E(S) \cup E(T) \rangle$, $\eta\xi$ is the identity mapping on $L$. Thus $\xi$ is an isomorphism and $L \cong E((S * T)/\theta)$.

Let $W$ be any commutative inverse semigroup and let $\psi_1: S \to W$ and $\psi_2: T \to W$ be homomorphisms such that $\psi_1 | U = \psi_2 | U$. Let $\psi_1^{(1)}: S^{(1)} \to W^{(1)}$ and $\psi_2^{(1)}: T^{(1)} \to W^{(1)}$ be extensions of $\psi_1$ and $\psi_2$ respectively. Let $\mu: (S * T)/\theta \to W$ be a mapping defined by $\mu((x, y)\theta) = \psi_1^{(1)}(x)\psi_2^{(1)}(y)$. In order to prove the latter part of (ii), it is sufficient to show that $\mu$ is well-defined. Let $(x_\alpha, y_\beta)\theta(x_\gamma, y_\delta)$ where $(x_\alpha, y_\beta)$ and $(x_\gamma, y_\delta)$ are elements of $S * T$. By the definition (2), $\alpha\beta = \gamma\delta$ (in $L^{(1)}$) and there exists $u_\sigma \in U^{(1)}$ such that $\sigma \geqq \alpha\beta$ (in $L^{(1)}$) and $(u_\sigma, u_\sigma^{-1})(x_\alpha, y_\beta) = (\sigma, \sigma)(x_\gamma, y_\delta)$ (in $S^{(1)} \times T^{(1)}$). If $\sigma = 1$, then $(x_\alpha, y_\beta) = (x_\gamma, y_\delta)$ and $\mu((x_\alpha, y_\beta)\theta) = \mu((x_\gamma, y_\delta)\theta)$. Let us assume $\sigma \neq 1$. Since $\xi: L \to E((S * T)/\theta)$ is a homomorhpism, $(\sigma, \sigma)\theta \geqq (\alpha, \beta)\theta = (\gamma, \delta)\theta$. Then

$$\begin{aligned}\mu((x_\alpha, y_\beta)\theta) &= \mu((\sigma, \sigma)\theta(x_\alpha, y_\beta)\theta) = \mu((u_\sigma, u_\sigma^{-1})\theta(x_\alpha, y_\beta)\theta) \\ &= \mu((\sigma, \sigma)\theta(x_\gamma, y_\delta)\theta) = \mu((x_\gamma, y_\delta)\theta).\end{aligned}$$

Thus $\mu$ is well-defined and we obtain the lemma.

From Lemma 3(i), Theorem 1 follows immediately, so the proof of Lemma 3(i) is our second, simpler proof that the class of commutative inverse semigroups has the strong amalgamation property.

REMARK. If $L$ is not the free product of $E(S)$ and $E(T)$ amalgamating $E(U)$ in the variety of semilattices, then $E((S * T)/\theta)$ is not necessarily isomorphic to $L$.

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