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BOOK REVIEWS

McCONNELL, J. C. and ROBSON, J. C., *Noncommutative Noetherian rings* (John Wiley & Sons, Chichester and New York 1987), xvi + 596 pp. 0 471 91550 5, £65.

Commutative Algebra grew out of Algebraic Geometry and Algebraic Number Theory. In particular, the geometry provided a great deal of motivation and insight, and even some language. Recall that there is a correspondence between algebraic varieties in $\mathbb{C}^n$ and ideals of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ with irreducible varieties corresponding to prime ideals. Localization at a prime ideal of a general commutative ring, having its origin in geometry as the name suggests, is a powerful tool in the commutative algebraist's armoury. As Noncommutative Algebra developed it is not surprising that it was influenced by the many deep and beautiful theorems already proved for commutative rings. But it was soon discovered that the localization technique was not automatically available. In fact not every noncommutative domain has a field of fractions, for example the free algebra on two generators over a field does not.

The discovery by A. W. Goldie in 1958 that localization is available in some situations, namely prime right Noetherian rings have a "field of fractions", called a right quotient ring which is simple Artinian, and the techniques used to prove this result, heralded the beginning of the modern theory of noncommutative Noetherian rings. Immediate subsequent developments, principally Goldie's extension of his theorem to the semiprime case, were encouraging, but localization in more general situations still proved difficult, indeed sometimes impossible. This led to the study of rings which were close to being commutative, and three principal areas were developed, namely (i) rings with polynomial identity (roughly subrings of rings of matrices over commutative rings), (ii) group rings of polycyclic-by-finite groups (already of interest to group theorists because of some beautiful work of P. Hall) and (iii) universal enveloping algebras of finite-dimensional solvable Lie algebras (at that time being developed because of their importance in the representation theory of solvable Lie algebras).

The experience gained in considering these special rings paid rich dividends when attention switched back to the general case, and a number of powerful theorems were proved. For example, localization in noncommutative Noetherian rings is rather well understood now. In fact, prime ideals need to be viewed not in isolation but in (possibly infinite) families called cliques. But, even with this understanding, in practice it is not generally possible to mimic proofs from the commutative case.

What then is the relationship between the commutative and noncommutative theories? The picture is patchy. Some theorems go over quite successfully but some do not. For example, the Forster–Swan–Eisenbud–Evans Theorem concerning the number of generators of modules, together with its consequences due to H. Bass and J.-P. Serre, have been successfully carried over to right Noetherian rings by J. T. Stafford and S. C. Coutinho. This requires localization to be defined out of the theorem, and the classical Krull dimension in terms of chains of prime ideals replaced by the Krull dimension of P. Gabriel and R. Rentschler defined in terms of chains of right ideals. By contrast, for example, A. V. Jategaonkar has exhibited principal right ideal domains with descending chains of prime ideals of arbitrary length. Indeed it is still not settled whether (right and left) Noetherian rings satisfy the descending chain condition on prime ideals.

From the foregoing you will have realised that the theory of commutative Noetherian rings has motivated a good deal of work in the noncommutative case, although as yet there is no underlying geometry to which to apply the results. Another major source of motivation is the study of the Weyl algebras which first appeared in Quantum Mechanics. This is not too surprising because they form a class of simple rings whose one-sided ideals behave nicely—of course commutative simple rings are fields so their ideal theory is of no interest. If k is a field of characteristic zero then the nth Weyl algebra $A_n(k)$ is the k-algebra with $2n$ generators $x_1,\ldots,x_n$, $y_1,\ldots,y_n$ and relations $x_iy_j-y_jx_i=\delta_{ij}$ (the Kronecker delta), $x_ix_j-x_jx_i=y_iy_{ij}-y_jy_i=0$, for all i,j. As a ring, $A_n(k)$ is a simple Noetherian domain, every one-sided ideal can be generated by two elements, $A_n(k)$ has Krull dimension n, global (homological) dimension n and Gelfand–Kirillov dimension $2n$. (Roughly speaking, Krull dimension gives a measure of how far a ring is from being Artinian, global dimension of how far it is from being semiprime Artinian and Gelfand–Kirillov dimension gives a measure of the rate of growth of a finitely generated algebra in terms of any generating set.) The subring $k[x_1,\ldots,x_n]$ of $A_n(k)$ has the same Krull, global and Gelfand–Kirillov dimensions, namely n, but has no bound on the number of generators of its ideals. The detailed study of the Weyl algebras has led to the study of related rings, for example skew polynomial rings $R[x;\sigma,\delta]$, where R is some given ring, and rings of differential operators on algebraic varieties (where geometry does make an appearance).

The third source of motivation for the theory of noncommutative Noetherian rings is in its applications. For example, in Group Theory, P. Hall showed that every finitely generated metabelian group satisfies the ascending chain condition on normal subgroups by using the Hilbert Basis Theorem and showed how to extend this fact to more general soluble groups. On the other hand, the algebras of Physics are of increasing interest. The vigorous development of the representation theory of finite dimensional Lie algebras, for both solvable and semisimple algebras, makes extensive use of their universal enveloping algebras which are Noetherian domains. The structure of these rings and their modules is being examined using the tools first introduced by Goldie thirty years ago together with later more sophisticated machinery.

Essentially starting with Goldie's Theorem this book traces the above development of noncommutative Noetherian rings in a very thorough and illuminating way. It is a veritable tour de force, encompassing a wide range of topics in some depth. Both authors have made significant contributions to this development (as has L. W. Small who assisted them somewhat) but they do not restrict themselves to these areas but range far and wide. The material has been very thoroughly digested with the resulting presentation being clear and precise. Some of the material is new, either in content or presentation. The lay-out and printing are excellent and always one has the feeling that a good deal of consideration has been given to the potential reader. In particular it is very easy to find information in this book. As well as a thorough treatment of the theory this text is full of illuminating examples which throw light on the definitions, theorems and even proofs. Another helpful feature is the number of calculations with specific examples. This text can be thought of as a mixture of encyclopaedia and handbook, and will surely prove an invaluable source, both for those working in the field and for those requiring rings and modules as tools.

The style is pleasant and remarkably uniform, given that there are two authors. They do not labour things. Where they need basic facts they clearly state what they are assuming the reader knows, giving references for this material. They are not frightened of technicalities but do not allow themselves to get bogged down in them. Results are proved in less generality if this will adequately illustrate the techniques of proof of the general result, but the general result is stated and a reference given. Every section begins with some indication of where the discussion will lead and at the close of each chapter there is a very helpful post script giving some history, references and additional comments. And, given the length of the book, the authors and publishers are to be congratulated on the remarkable lack of errors.

Although the treatment is very comprehensive there are some gaps. For example, Jategaonkar's examples cited above receive the briefest of mentions. Even more surprising is that J. E. Roseblade's work on the Nullstellensatz and on prime ideals in group rings of polycyclic-by-finite groups does not feature at all. (Group rings receive less attention than they warrant, deliberately,

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because of D. S. Passman's excellent exposition elsewhere, but I still think that Roseblade should have been mentioned.) In the discussion on enveloping algebras, only the solvable case is discused (as the authors admit because of lack of space), but it would have been useful to have at least some outline of the rapid developments in the semisimple case. In the chapter on K-theory, mention could have been made of D. R. Farkas and R. L. Snider's work on the Zero Divisor Question and its extension by J. A. Moody and others. My point is this: given that Ring Theory, both in its theorems and techniques, has proved so important in applications, the more applications that could have been included the better. To be fair, a number of applications are given and others are mentioned, for example Bernstein's Theorem on analytic continuation and Amitsur's non-crossed product division algebras.

The contents are as follows. 0. Preliminaries. 1. Some Noetherian rings. 2. Quotient rings and Goldie's theorem. 3. Structure of semiprime Goldie rings. 4. Semiprime ideals in Noetherian rings. 5. Some Dedekind-like rings. 6. Krull dimension. 7. Global dimension. 8. Gelfand–Kirillov dimension. 9. The Nullstellensatz. 10. Prime ideals in extension rings. 11. Stability. 12. K_0 and extension rings. 13. Polynomial identity rings. 14. Enveloping algebras of Lie algebras. 15. Rings of differential operators on algebraic varieties.

To conclude, this is an excellent account of the theory of noncommutative Noetherian rings which I recommend without qualification. Both authors were students of A. W. Goldie, as was the reviewer. Long live Alfred Goldie!

PATRICK SMITH

FREESE, R. and MCKENZIE, R., *Commutator theory for congruence modular varieties* (London Mathematical Society Lecture Note Series 125, Cambridge University Press 1987) 227 pp. 0 521 34832 3, £15.

In group theory a number of important concepts, such as Abelian, soluble and nilpotent groups, centre, centralizers, etc. are defined in terms of the commutator: $[x, y]=x^{-1}y^{-1}xy$. The commutator of normal subgroups can be defined as a normal subgroup and used for almost all these concepts. The idea of a commutator has been extended to ideals of a ring. Recently it has been extended much further.

One of the more interesting recent developments in general algebra theory has been the development of this general commutator theory. A variety V of algebras is called a congruence modular variety if the lattice of congruences of any algebra in the variety is modular. A commutator is then defined as a binary operation on the lattice of congruences of an algebra. It reduces to the classical commutator in the case of groups. The theory of these commutators is presented here.

There are fourteen chapters which provide the basic theory linking together several definitions of the commutator and presenting many very powerful and interesting applications. We now mention some of them. A centre is defined for arbitrary algebras and a theory of nilpotent algebras is developed. Certain rings are associated with Abelian varieties and their structure is determined. Strictly simple algebras are studied. A variety of Mal'tsev conditions are considered. These are equations, holding between terms in the algebra, which characterize properties of the congruence lattice. The last chapter presents a very powerful finite basis result concerning modular varieties of finite type.

Commutator theory is an interesting new development in general algebra and this book provides a good introduction to the subject. There are a number of exercises which contain some further results and some examples. Solutions to the exercises are given. There are also some interesting historical comments at several points. This is a worthy addition to the London Mathematical Society Lecture Note Series.

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