Canad. Math. Bull. Vol. **29** (3), 1986

A METHOD FOR CONSTRUCTING SQUARE ROOTS IN FINITE FULL TRANSFORMATION SEMIGROUPS

BY

PETER M. HIGGINS

ABSTRACT. Let $\boldsymbol{T}_n$ denote the full transformation semigroup on the set $X = \{1, 2, \ldots, n\}$, that is the set of all mappings from X to X, the semigroup operation being composition of mappings. The aim of this paper is to provide a method for the construction of all square roots of an arbitrary element $\alpha \in \boldsymbol{T}_n$, by employing a representation of the members of $\boldsymbol{T}_n$ as special directed graphs.

1. **Preliminaries**. Square roots of members of $\boldsymbol{T}_n$ have been characterized by Snowden and Howie [2]. However, the criterion established there for the existence of a square root of $\alpha \in \boldsymbol{T}_n$ is "disappointingly complicated". Indeed the authors suggest the alternative approach adopted here: to visualise the elements of $\boldsymbol{T}_n$ as digraphs and discover a method for constructing square roots by inspection of the digraph of a typical member $\alpha \in \boldsymbol{T}_n$ and its square.

The following graph theoretic definitions and results come from [1]. For more background on digraphs the reader is referred to Chapter 16 in particular. A digraph is *weak* if it is connected when viewed as a graph. A *functional digraph* is a weak digraph in which every point has outdegree 1. An *in-tree* is a digraph with a *sink* (point of outdegree 0) which is a tree when regarded as a graph.

RESULT 1 ([1], Theorem 16.5). The following are equivalent for a weak digraph D.

1. D is functional.

2. D has exactly one cycle, the removal of whose arcs results in a digraph in which each component is an in-tree with its sink in the cycle.

3. D has exactly one cycle Z, and the removal of any arc of Z results in an in-tree.

RESULT 2 ([1], Theorem 16.4). A weak digraph is an in-tree if and only if exactly one point has outdegree 0 and all others have outdegree 1.

A tree is *rooted* if it has a distinguished point, called the *root*. An in-tree has a natural root in its sink.

We associate with $\alpha \in \boldsymbol{T}_n$ a digraph (which we shall also call α) on n labelled points, where ij is an arc if $i\alpha = j$. Every point of α has outdegree 1 so that the components of α are functional. Each component A of α can be pictured as a cycle Z_A, together with

Received by the editors January 15, 1985, and, in revised form, April 15, 1985.

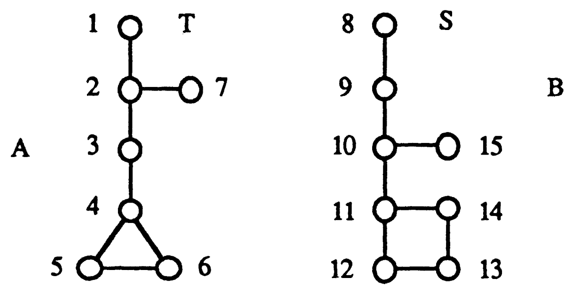
AMS Subject Classification (1980): 20M20, 05C20.

https://doi.org/10.4153/CMB-1986-053-2 Published online by Cambridge University Press

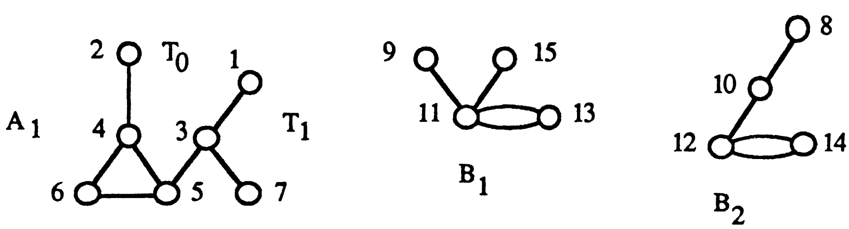
a family of in-trees rooted at the points of Z_A. For two points on a digraph, i and j, the *distance* between i and j, denoted by $d(i,j)$, is the length of a minimal directed path from i to j (if such exists). For an in-tree T, the *radius* of T is the greatest distance from a point of T to the sink. It is easy to prove by induction on the radius that the direction on the arcs of an in-tree are implicitly defined once the sink has been specified. Hence if we adopt the convention that the cycles of $\alpha \in \boldsymbol{T}_n$ are directed counterclockwise, then the arrows may be deleted from the picture of α with the exception that the picture must provide indication of all cycles of order one in order to avoid ambiguity. For example, for the member of $\boldsymbol{T}_{15}$

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 2 & 3 & 4 & 5 & 6 & 4 & 2 & 9 & 10 & 11 & 12 & 13 & 14 & 11 & 10 \end{pmatrix}$$

the corresponding digraph is



Although we aim to find a method of constructing square roots in $\boldsymbol{T}_n$, we first consider α^2 in order to discover how to recognize squares. The graph of α^2 is

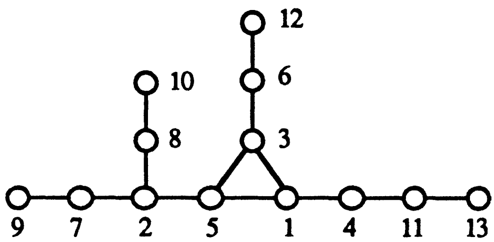


Observe that the component A, whose cycle is of odd order, has given rise to one component in α^2 whose cycle is of the same odd order. The tree T has given rise to a pair of trees T_0, T_1, each rooted on the cycle of A_1. The tree T_0 has the same root as T and its points are the points of T whose distance from the root is even. Similarly T_1 has as its points all the points of T an odd distance from the root, plus another point of the cycle as root. We shall not spell out the precise relationship between T_0 and T_1 yet, but note that the radius of the "odd" tree T_1 will always be equal to, or one greater than, the radius of the corresponding "even" tree T_0.

In contrast the component B, upon squaring, gives rise to a pair of components B_1, B_2. This occurs because the cycle of B has even order. The tree S "splits" into an even−odd pair of trees in a similar fashion to T, but the even and odd trees lie on different components.

The foregoing casual analysis does contain all the ideas involved in the solution of the problem. Indeed we can already say that the map α as given above has no square root by arguing as follows. Suppose that $\beta \in \boldsymbol{T}_{15}$ and $\beta^2 = \alpha$. The component A of α must have arisen from the squaring of a component of β with a 3-cycle, as the only other way a 3-cycle could be introduced is by squaring a 6-cycle, which of course would create two 3-cycles. The tree T would then be half of an even−odd pair of trees whose partner would also lie on the cycle (456). In the absence of this partner, we conclude no such β exists. The argument is even quicker if we focus our attention on the component B, for in a square the components with cycles of even order must occur in pairs. Hence α is not a square as it has but one component with a 4-cycle.

As another example consider the member α of $\boldsymbol{T}_{13}$ given by



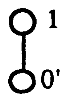
We can state immediately that α is not a square, as there are an odd number of trees rooted on its cycle, and so they may not be associated in even−odd pairs (we must be a little more careful, a tree with one arrow gives rise, upon squaring, to a pair in which the "even tree" has no arrows, however, this is not a possibility here as α has no single-arrowed trees). This example first appears in [2], where the Snowden-Howie characterisation requires a page of ancillary calculation in order to show that α has no square root.

2. **The construction of parent trees**. For a component A [tree T] of α ($\alpha \in \boldsymbol{T}_n$) we shall write $A^2[T^2]$ for the corresponding subgraph in α^2.

To examine the relationship between a tree T and its square, we introduce the idea of even−odd offspring. Let T be a tree with sink 0 and other points $1, 2, \ldots, m$ say. We define the *even−odd offspring* of T as an ordered pair of trees (T_0, T_1). The points of T_0 (the *even tree*) are the points of T an even distance from the sink (including the sink) and jk is an arc of T_0 if $d(j,k) = 2$ (in T). The points of the *odd-tree* T_1 are the points of T an odd distance from the sink, together with a new point $0'$, and jk is an arc of T_1 if $d(j,k) = 2$ (in T) or $k = 0'$ and $d(j,0) = 1$ (in T). We call T a *parent tree* of the pair (T_0, T_1).

One of the constructions that will need to be performed in order to find all square roots of a given $\alpha \in \boldsymbol{T}_n$ will be the construction of all parent trees (if any) of a given pair of trees (T_0, T_1). To this end we investigate the relationship between a tree T and its offspring.

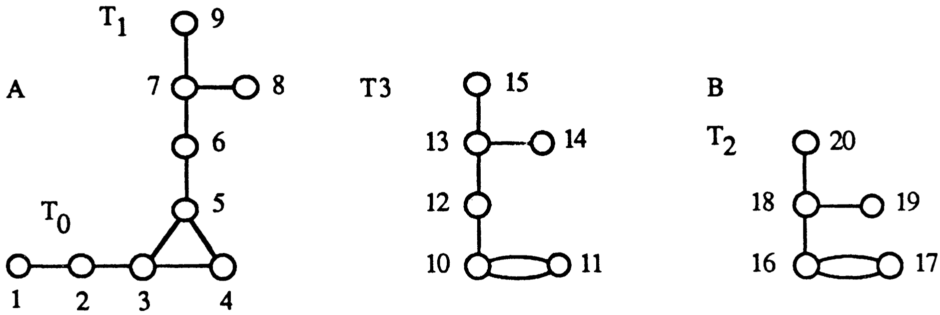
Take a maximal directed path P of T from an endpoint u of T to the sink and label the points of P by $k, k-1, \ldots, 0$ where $d(u, 0) = k \geq 1$. The path P corresponds to maximal directed paths (P_0, P_1) in (T_0, T_1) respectively, in which either $|P_1| = |P_0|$ or $|P_1| = |P_0| + 1$ according as k is even or odd ($|P|$ denotes the length of the path P). Now consider a sub-tree T' of T rooted at the point $2r$ on P $(0 \leq 2r \leq k-1)$. Now T' corresponds to a pair of trees (T_0', T_1') rooted on (P_0, P_1) respectively. The pair (T_0', T_1') is the even–odd offspring of T', each member of the pair is rooted at a distance r from the sink of P_0 and P_1 respectively. On the other hand a tree T' rooted at a point $2r+1$ of P $(1 \leq 2r+1 \leq k-1)$ gives rise to a pair of trees (T_0', T_1') rooted on (P_1, P_0) respectively. Furthermore T_0' is rooted a distance $r+1$ from the sink of P_1 while T_1' is rooted a distance r from the sink of P_0; the pair (T_0', T_1') is again an even–odd offspring pair of T'.

These observations allow us to construct all parent trees of a given pair (T_0, T_1) of trees with no common points. We assume inductively that we may construct all parent trees of any such pair (T_0', T_1') for which the total number of points is less than that of (T_0, T_1). (There is no difficulty starting this induction for the pair ($\bigcirc_0$, ) has a unique parent in .)

If (T_0, T_1) is an even–odd offspring pair of some tree T, it must be possible to choose maximal paths P_0, P_1 to the sinks 0 and $0'$ of T_0 and T_1 respectively, such that $|P_1| = |P_0|$ or $|P_1| = |P_0| + 1$. Furthermore it must be possible to make this choice so that the rooted trees of (P_0, P_1) can be listed in even–odd offspring pairs so that for any such pair (T_0', T_1') either T_0' is rooted on P_0, T_1' is rooted on P_1 at a distance r from the respective roots $(r \geq 0)$, or T_0' is rooted on P_1 at a distance $r+1$ from $0'$ and T_1' is rooted on P_0 at a distance r from 0 $(r \geq 0)$.

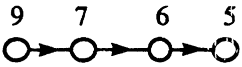
We then construct a path P from (P_0, P_1) as follows: label the points of P_0 by $0, 1, 2, \ldots, k$ (where $k = |P_0|$) and those of P_1 by $0', 1', \ldots,$ up to either k' or $(k+1)'$ as the case may be. The points of P from the sink outwards are then $0, 1', 1, 2', 2, \ldots$ ending either k', k or $k', k, (k+1)'$ as the case may be. The trees of P_0 and P_1 have been paired in offspring pairs according to the criterion of the previous paragraph. For each such pair (T_0', T_1') construct a parent tree T_2' which will then have its sink on P at either the point r or r' as the case may be. The tree T so constructed is then a parent of (T_0, T_1) and all such parent trees can be so constructed.

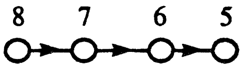
The theory developed will be illustrated by means of the following example. Let $\alpha \in \boldsymbol{T}_{20}$ be defined by the digraph:

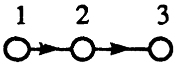


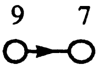
We shall find all square roots of α, but for now let us calculate the parent trees of (T_0, T_1) and (T_2, T_3) beginning with the former pair.

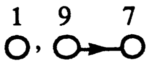
There are two choices for a maximal path P_1 from T_1:



or


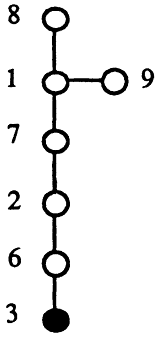
. If we choose the latter then the only possible choice for P_0 is


. There is only one non-trivial tree on P_1 or P_0, the tree


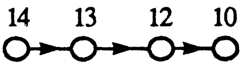
occurring at the point of P_1 labelled 7, which can be paired with the trivial tree at the point of P_0 labelled 1, to give an even–odd pair (


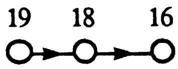
) in accord with the criteria laid down above. Our parent tree is then

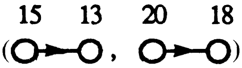
T,



where the root 3 is shaded. This parent tree is unique as the other choice for P_1 leads also to T.

For the pair (T_2, T_3) from the components C and B respectively of our example, we choose the path


for our P_1 (the other choice leads to the same set of parents). There are two possible choices for P_0, we choose


We may now regard (


) as an even–odd pair positioned at the second point of P_1 and the first of P_0, whence our parent tree is

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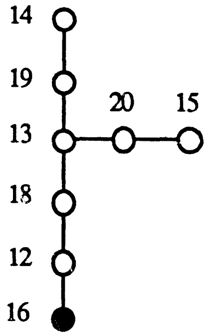
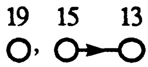
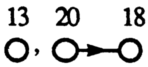
T

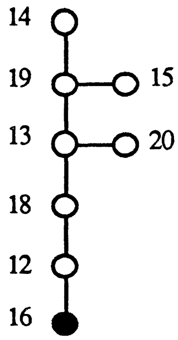


Alternatively, we may regard (

) and (

) as even–odd pairs giving the parent tree

T',



The alternative choice for P_0 gives another two parent trees, making four in all.

3. **The construction of square roots**. Let $\alpha \in T_n$. We shall say that a component A of α is *odd* (*even*) if its cycle, which we denote by Z_A, is of odd (even) order. We examine the relationship between the components of α and those of α^2.

Let A be an odd component of α. As observed before, A^2 is also an odd component of α^2. Each tree T rooted on Z_A gives rise to an even–odd offspring pair (T_0, T_1) on A^2. The remaining question is to determine the point $0'$, the sink of T_1. Clearly, if we label the sink of T by 0, then $0'$ is the point of Z_A adjacent to 0, travelling counterclockwise. If Z_A has $2t - 1$ points $(t \geq 1)$ then 0 and $0'$ will be t points apart on Z_{A^2} (travelling counterclockwise). We shall call such a positioning of the roots of T_0 and T_1 around Z_{A^2} *consistent*.

Finally, let A be an even component of α with Z_A of order $2t$ $(t \geq 1)$. Then A^2 consists of two components A_0, A_1 each of whose cycles has order t. A tree T of A gives rise to even–odd offspring (T_0, T_1) situated in different components. Note that given the roots of T_0 and T_1, the cycle Z_A can be uniquely reconstructed: if T_0 and T_1 are rooted at $0, 0'$ on Z_{A_0} and Z_{A_1} respectively with $Z_{A_0} = (0, 1, \ldots, t)$, $Z_{A_1} = (0', 1', \ldots, t')$, then $Z_A = (0, 0', 1, 1', \ldots, t, t')$. Hence all the offspring pairs of the trees of A must be rooted on A_0 and A_1 so as to determine the same cycle Z_A. We call a list of pairs of the trees of A_0, A_1 *consistent* if each pair determines the same cycle Z_A.

THEOREM. *Let $\alpha \in \boldsymbol{T}_n$. Then α is a square if and only if the components of α can be grouped in pairs, (A_0, A_1) such that either:*

(i) $A_0 = A_1 = A$ say, A is odd and the trees of A can be consistently listed in offspring pairs; or

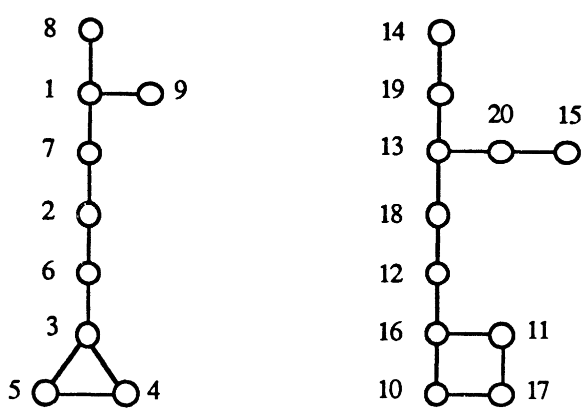
(ii) $A_0 \neq A_1$, $|Z_A| = |Z_{A_1}|$, and the trees of A_0, A_1 can be grouped consistently in offspring pairs. Furthermore each such grouping allows construction of a square root and all square roots can be so constructed.

PROOF. It remains to check that given α and such a grouping of its components we may construct a square root. Suppose an odd component is paired with itself as in condition (i). The cycle Z_A of A, which we take as $(1\ 2\ 3 \ldots 2t-1)$, has a unique square root in $Z_{\bar{A}} = (1\ t+1\ 2\ t+2 \ldots t\ 2t-1)$. For each pair of trees (T_0, T_1) construct a parent tree T. We then construct the component $\bar{A}$ with cycle $Z_{\bar{A}}$ and one parent tree for each offspring pair. The consistency of the pairing guarantees that the reconstructed component $\bar{A}$ is such that $\bar{A}^2 = A$.

Finally suppose A_0, A_1 are paired in accordance with (ii). Take an offspring pair (T_0, T_1) and construct the unique cycle $Z_{\bar{A}}$ whose square is Z_A and such that the roots of T_0 and T_1 are counterclockwise adjacent on $Z_{\bar{A}}$. Consistency of the pairing allows construction of a component $\bar{A}$ with cycle $Z_{\bar{A}}$ whose square is the pair (A_0, A_1).

Therefore a square root of α may be constructed, and we get distinct roots for each choice of pairings of components and of trees.

We calculate all the square roots of α as given in Section 2. The only possible pairing of components is (A, A) and (B, C). For the (A, A) case the only possible pairing of the trees of A is (T_0, T_1). Note that this pairing is consistent (if T_0 was rooted at the point 4, the pairing would be inconsistent and we would conclude that α was not a square). The unique parent tree was calculated in Section 2. For the (B, C) pairing the only possible pairing of the trees is (T_2, T_3). Since there is just one pair to consider, consistency is automatic. The four parent trees of (T_2, T_3) were calculated in Section 2, giving $1 \times 4 = 4$ square roots of α in all, one of which is



where we have chosen the tree labelled T in Section 2 as the parent tree of (T_2, T_3).

Our characterisation makes it relatively easy to calculate the number of squares in $\boldsymbol{T}_n$ for small n. Write down all possible forms for the digraph of $\alpha \in \boldsymbol{T}_n$, and decide which forms represent squares. The number of members of $\boldsymbol{T}_n$ with a given form of digraph can be calculated by elementary combinatorial arguments.

Our results can be used to construct all square roots of $\alpha \in \boldsymbol{PT}_n$ (where $\boldsymbol{PT}_n$ is the semigroup of all partial maps of $\{1, 2, \ldots, n\}$ under composition). As is well known, $\boldsymbol{PT}_n$ is isomorphic to the subsemigroup of $\boldsymbol{T}_{\{0, 1, \ldots, n\}}$ consisting of all maps which fix 0. Therefore to calculate the square roots of $\alpha \in \boldsymbol{PT}_n$, we calculate the square roots of α, regarding it as a member of $\boldsymbol{T}_{\{0, 1, \ldots, n\}}$, but only roots which fix 0 need be considered.

The method described here for extracting square roots for members of $\boldsymbol{T}_n$ can be extended to the problem of finding all pth roots for any prime p, which would then allow mth roots to be found for any positive integer m.

References

1. F. Harary, *Graph Theory*, Addison-Wesley, 1969.

2. M. Snowden and J. M. Howie, *Square roots in finite full transformation semigroups*, Glasgow Math. J., **23** (1982), pp. 137–149.

California State University, Chico
Chico, California 95929

https://doi.org/10.4153/CMB-1986-053-2 Published online by Cambridge University Press