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ALEXANDER POLYNOMIALS OF TWO-BRIDGE LINKS

TAIZO KANENOBU

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Abstract

We provide an algorithm for calculating the Alexander polynomial of a two-bridge link by putting every two-bridge link in a special type of Conway diagram. Using this algorithm, some necessary conditions for a polynomial to be the Alexander polynomial of a two-bridge link are given, in particular, certain alternating and monotonicity conditions on the coefficients, analogous to corresponding known properties of the reduced Alexander polynomial.

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Hartley [4] gave a necessary condition for a polynomial to be the Alexander polynomial of a two-bridge knot or the reduced Alexander polynomial of a two-bridge link. He showed how the coefficients of the polynomial may be read straight from the extended diagram, which is derived from Schubert's normal form of a two-bridge knot or link, and showed the following theorem: If $\Delta(t) = \Sigma_{i=0}^{n}(-1)^i a_i t^i$ where $a_i > 0$, is the Alexander polynomial of a two-bridge knot or the reduced Alexander polynomial of a two-bridge link, then for some integer s, $a_0 < a_1 < \cdots < a_s = a_{s+1} = \cdots = a_{n-s} > \cdots > a_n$. On the other hand, using surgery techniques, Bailey [1] presented an algorithm for calculating the Alexander polynomial of a two-bridge link from Conway diagram. As a corollary to this he proved a conjecture of Kidwell about the linking complexity or geometric intersection numbers of a link in the special case of two-bridge links.

The main results of this paper are Theorems 1 and 3. The former provides another algorithm for calculating the Alexander polynomial of a two-bridge link from a special type of Conway diagram. The latter gives some necessary conditions for a polynomial to be the Alexander polynomial of a two-bridge link. These

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conditions are analogous to Hartley's theorem above. Theorem 2 and Corollary 1 also give some properties of the Alexander polynomial of a two-bridge link, including the Torres condition [8]. Corollary 2 is the above-mentioned conjecture of Kidwell in the case of two-bridge links.

In Section 2, we give some lemmas for Theorems 1 and 2. In Section 3, we summarize some properties of two-bridge links. In Section 4, we state the above-mentioned results. In Section 5, we prove Theorem 3.

1. Preliminaries

In this paper, a link L will mean a piecewise linear embedding of two oriented circles K_1 and K_2 in the 3-sphere S^3. Two links L and L' are called equivalent, if there is an orientation preserving autohomeomorphism of S^3, which maps L onto L'. The Alexander polynomial $\Delta(x, y)$ of L is an element of the polynomial ring $Z[x, x^{-1}, y, y^{-1}] = \Lambda$, and is determined only up to multiplication by a unit $\pm x^i y^j$. Let $G = \pi_1(S^3 - L)$, and let G' be its commutator subgroup. Then $\Lambda = Z[G/G']$; the basis $\{x, y\}$ of G/G' is always taken to be represented by the meridians of K_1 and K_2 respectively.

Throughout this paper, we will often abbreviate a polynomial $f(x, y)$ in Λ to f and will use the following notation;

$$F_n(x, y) = \begin{cases} \sum_{i=0}^{n-1} (xy)^i & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\sum_{i=n}^{-1} (xy)^i & \text{if } n < 0. \end{cases}$$

In the figures of this paper we will use the concept of a tangle [2], which is a portion of the link diagram containing two arcs. An integral tangle, which is represented by a circle labeled "i" or "$-i$", where i is a non-negative integer, is a 2-braid with i or $-i$ crossings, in the manner indicated in Figure 1.

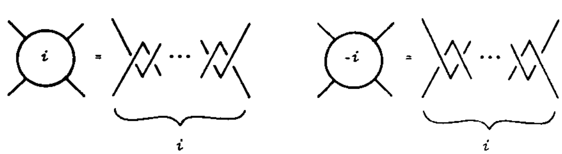

FIGURE 1

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2. Lemmas

LEMMA 1. *Let $L(q, r, s, t)$ be a link as shown in Figure 2, where T is any tangle. Let $\Delta^{(q,r,s,t)}$ be the Alexander polynomial of $L(q, r, s, t)$. If we set $\Delta = \Delta^{(q,r,s,t)}$, $\Delta_0 = \Delta^{(q,r,0,0)}$ and $\Delta_{00} = \Delta^{(0,0,0,0)}$, then*

$$(2.1) \qquad \Delta = \{s(x-1)(y-1)F_t + 1\}\Delta_0 + \frac{F_t}{F_r}(xy)^r(\Delta_0 - \Delta_{00}),$$

where $r \neq 0$.

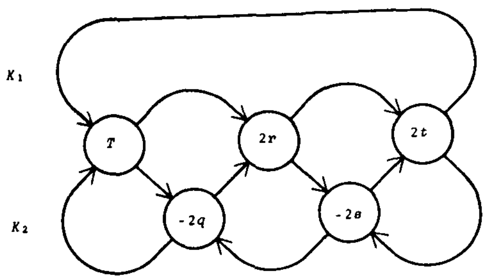

FIGURE 2

LEMMA 2. *Besides the notation in Lemma 1, let $\Delta'_0 = \Delta^{(q,r,0,t)}$ and $\Delta^{(t_0)} = \Delta^{(q,r,s,t_0)}$. Then*

$$(2.2) \qquad \Delta = s(x-1)(y-1)F_t\Delta_0 + \Delta'_0;$$

$$(2.3) \qquad \Delta^{(t)} = F_t\Delta^{(1)} - xyF_{t-1}\Delta_0;$$

$$(2.4) \qquad \Delta^{(t)} + xy\Delta^{(t-2)} = (1+xy)\Delta^{(t-1)}.$$

REMARK. (1) In the above notation $\Delta^{(t)} = \Delta$ and $\Delta^{(0)} = \Delta_0$.

(2) (2.4) is a special case of Conway's result [2, page 338], see also [5, page 462].

Lemma 1 can be shown by using Fox's free differential calculus, see [3], [8]. The proofs of these lemmas are standard, so we omit them.

3. Two-bridge links

According to Conway [2], every two-bridge link can be put in the form as shown in Figure 3. It will be denoted by $C(a_1, a_2, \ldots, a_n)$, including the indicated orientation of each component. The diagram is slightly different in the cases $n = 2k$ and $n = 2k + 1$, as indicated in Figure 3. From this projection we can see that a two-bridge link is a link with two components each of which is a trivial

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knot. Moreover a two-bridge link is interchangeable, that is, there is an isotopy of S^3 which interchanges the two components. This follows immediately from Schubert's normal form [6], or Bailey [1, page 48] also proves this using Conway's diagram.

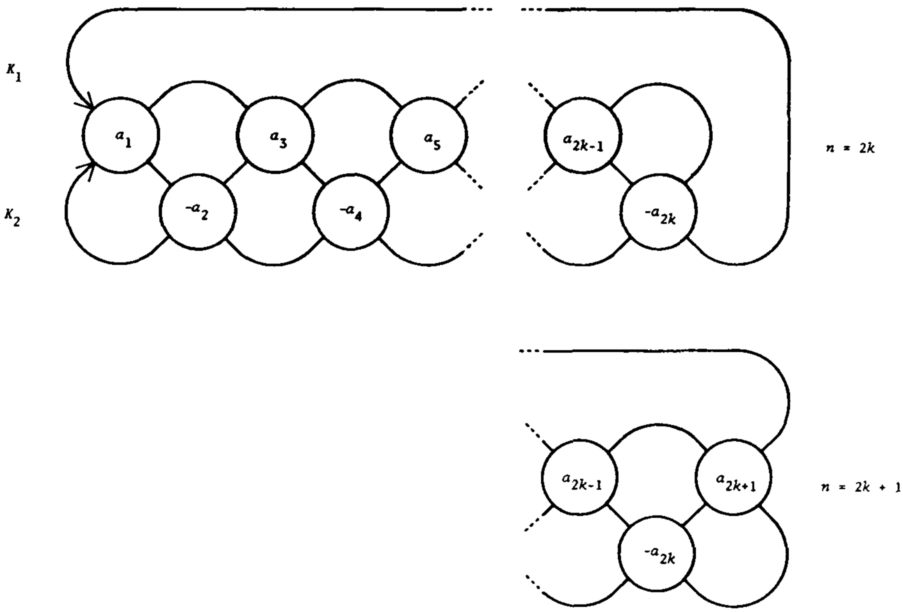

FIGURE 3

Let $\alpha(>0)$ and β be the coprime integers computed from the continued fraction:

$$\frac{\alpha}{\beta} = a_1 + \frac{1}{a_2} + \cdots + \frac{1}{a_n}.$$

Then α is even and $0 < |\beta| < \alpha$. This link is equivalent to the link with Schubert's normal form (α, β), denoted by $S(\alpha, \beta)$ endowed with suitable orientations. According to Schubert [6, page 144], $S(\alpha, \beta)$ and $S(\alpha', \beta')$ are equivalent if and only if $\alpha = \alpha'$ and $\beta^{\pm 1} \equiv \beta' \pmod{2\alpha}$. Furthermore, if $\beta' \equiv \beta + \alpha \pmod{2\alpha}$ or $\beta\beta' \equiv \alpha + 1 \pmod{2\alpha}$, then $S(\alpha, \beta)$ differs from $S(\alpha, \beta')$ only by the orientation of one of the components (see [7, page 7]).

The two-fold cover of S^3 branched over $S(\alpha, \beta)$ is the lens space $L(\alpha, \beta)$, see [2], [6], [7]. If we neglect the difference between $S(\alpha, \beta)$ and $S(\alpha, -\beta)$ and the orientations of $S(\alpha, \beta)$, this sets up a one-to-one correspondence between two-bridge links and the lens spaces up to homeomorphism.

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We can obtain easily another continued fraction:

$$\frac{\alpha}{\beta} = 2b_1 + \frac{1}{2b_2} + \cdots + \frac{1}{2b_m},$$

where m is odd. $C(2b_1, 2b_2, \ldots, 2b_m)$ is then equivalent to $C(a_1, a_2, \ldots, a_n)$ and will be denoted by $D(b_1, b_2, \ldots, b_m)$. In the following we will consider every two-bridge link to be put in this form (see [7, page 13]).

4. Main theorems

From Lemma 1, we have

THEOREM 1. *Let $L_0 = D(0)$ and for $n \geqslant 1$ let*

$$L_n = D(p_1, q_1, p_2, q_2, \ldots, p_{n-1}, q_{n-1}, p_n),$$

where $\prod_{i=1}^{n} p_i \prod_{j=1}^{n-1} q_j \neq 0$. Let $\Delta_n(x, y)$ be the polynomial inductively defined as follows:

(4.1)
$$\Delta_0 = 0;$$
$$\Delta_1 = F_{p_1};$$
$$\Delta_n = \{q_{n-1}(x-1)(y-1)F_{p_n} + 1\}\Delta_{n-1} + (xy)^{p_{n-1}} \frac{F_{p_n}}{F_{p_{n-1}}}(\Delta_{n-1} - \Delta_{n-2}), \quad \textit{for } n \geqslant 2.$$

Then $\Delta_n(x, y)$ is the Alexander polynomial of L_n.

In the following, by the Alexander polynomial of a two-bridge link we mean the polynomial defined in Theorem 1 and we will use the following notation besides that in Theorem 1. Let $\Delta_n^{(p)}$ be the Alexander polynomial of $D(p_1, q_1, p_2, q_2, \ldots, p_{n-1}, q_{n-1}, p)$; thus $\Delta_n^{(p_n)} = \Delta_n$ and $\Delta_n^{(0)} = \Delta_{n-1}$. Let $l_n = \sum_{i=1}^{n} p_i$, that is, the linking number of L_n. Let $\tilde{l}_n = \sum_{i=1}^{n} |p_i|$.

From Lemma 2, we have

THEOREM 2.

(4.2)
$$\Delta_n = q_{n-1}(x-1)(y-1)F_{p_n}\Delta_{n-1} + \Delta_{n-1}^{(p_{n-1}+p_n)};$$

(4.3)
$$\Delta_n^{(p)} = F_p \Delta_n^{(1)} - xyF_{p-1}\Delta_{n-1};$$

(4.4)
$$\Delta_n^{(p)} + xy\Delta_n^{(p-2)} = (1 + xy)\Delta_n^{(p-1)}.$$

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Using (4.4) or Theorem 1 we can easily prove each of the following formulae.

COROLLARY 1.

(4.5) $$\Delta_n(x, y) = \Delta_n(y, x);$$

(4.6) $$\Delta_n(x, y) \equiv F_{l_n}(x, y) \mod (x-1)(y-1);$$

(4.7) $$\Delta_n(x, y) = (xy)^{l_n - 1}\Delta_n(x^{-1}, y^{-1}).$$

The fact that a two-bridge link is interchangeable assures us of (4.5). From (4.6), we have immediately

(4.8) $$\Delta_n(x, 1) = F_{l_n}(x, 1).$$

(4.7) and (4.8) constitute the Torres conditions [8] for two-bridge links.

DEFINITION 1. Let $f(x, y)$ be a polynomial in Λ. If $f(x, y) \neq 0$, then $\deg_x f =$ (maximum x-power of any term of f) minus (minimum x-power of any term of f). If $f(x, y) = 0$, then $\deg_x f = -1$. We define $\deg_y f$ in the same way.

DEFINITION 2. $\Lambda^{+1}(r, s)$ denotes the set of all polynomials $f(x, y) = \sum_{r \leqslant i, j \leqslant s} a_{ij} x^i y^j$ in Λ satisfying the following conditions.

(i) $\deg_x f = \deg_y f = s - r$.

(ii) Both

$$\begin{bmatrix} a_{sr} & \cdots & a_{ss} \\ \vdots & & \vdots \\ a_{rr} & \cdots & a_{rs} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{rr} & \cdots & a_{rs} \\ \vdots & & \vdots \\ a_{sr} & \cdots & a_{ss} \end{bmatrix}$$

are symmetric matrices.

(iii) $a_{ij} \geqslant 0$ if $i + j$ is even, and $a_{ij} \leqslant 0$ if $i + j$ is odd.

(iv) Let $b_{ij} = a_{i+r, j+r}$. Then $|b_{k,0}| \leqslant |b_{k-1,1}| \leqslant \cdots \leqslant |b_{k-u,u}|$, and $|b_{k,0}| \leqslant |b_{k+1,1}| \leqslant \cdots \leqslant |b_{k+v,v}|$ for $0 \leqslant k \leqslant s - r$, where $u = [k/2]$* and $v = [(s - r - k)/2]$.

Furthermore $\Lambda^{-1}(r, s)$ denotes the set of all polynomials $f(x, y)$ in Λ such that $-f(x, y) \in \Lambda^{+1}(r, s)$.

THEOREM 3. *For* $n \geqslant 1$, $\Delta_n \in \Lambda^{\varepsilon_n}(r_n, s_n)$, *where*

$$\varepsilon_n \doteq \prod_{i=1}^{n} \frac{p_i}{|p_i|} \prod_{j=1}^{n-1} \frac{q_j}{|q_j|}, \qquad r_n = \frac{l_n - \tilde{l}_n}{2} \quad \text{and} \quad s_n = \frac{l_n + \tilde{l}_n}{2} - 1.$$

*[ ] denotes the Gaussian symbol.

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Note that $r_n \leqslant 0 \leqslant s_n$, $r_n - r_{n-1} = \frac{1}{2}(p_n - |p_n|)$ and $s_n - s_{n-1} = \frac{1}{2}(p_n + |p_n|)$. The proof of Theorem 3 will be given in Section 5.

Let $\Delta(t) = \sum_{i=0}^{m}(-1)^i a_i t^i$, where m is odd, be the reduced Alexander polynomial of L_n. Since $\Delta(t) = \varepsilon_n t^{-2r_n}(1-t)\Delta_n(t,t)$, we have $0 < a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{(m-1)/2}$ and $a_k = -a_{m-k}$ from Theorem 3. This is a weaker result than that of Hartley [4] stated in the beginning of this paper.

For the sake of Corollary 2 below, we need some preliminaries.

DEFINITION 3. Let $L = K_1 \cup K_2$ be a link and S be a Seifert surface for K_1 with S and K_2 in general position. If γ_S = 2(genus of S) plus (the number of times K_2 intersects S), then $\gamma_1 = \min_S \gamma_S$ is the *linking complexity* of K_2 with K_1. We define γ_2 in the same way. We call the ordered pair (γ_1, γ_2) the *linking complexity of the link L.*

This definition follows Bailey [1, page 45], see also [5].

PROPOSITION. (Kidwell) *If $\Delta(x,y)$ is the Alexander polynomial of a link L with linking complexity (γ_1, γ_2), then $\gamma_1 - 1 \geqslant \deg_x \Delta(x,y)$.*

PROOF. See [1, page 46].

COROLLARY 2. *Let (γ_1, γ_2) be the linking complexity of L_n. Then*

$$\gamma_1 = \gamma_2; \tag{4.9}$$

$$\deg_x \Delta_n(x,y) + 1 = \gamma_1 = \tilde{l}_n. \tag{4.10}$$

REMARK. The first equality of (4.10) is Proposition 6 of [1, page 57].

PROOF. (4.9) follows from interchangeability of a two-bridge link or (4.10). For (4.10), from the diagram of L_n, we see that $\gamma_1 \leqslant \tilde{l}_n$. By Theorem 3, $\deg_x \Delta_n + 1 = \tilde{l}_n$ and by Proposition, $\gamma_1 \geqslant \deg_x \Delta_n + 1$.

5. Proof of Theorem 3

In this section we use the following trivial lemma without mention.

LEMMA 3. *Let $f \in \Lambda^\varepsilon(r,s)$ and $g \in \Lambda^\varepsilon(r-k, s+k)$ $(k \geqslant 0)$. Then $f + g \in \Lambda^\varepsilon(r-k, s+k)$.*

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LEMMA 4. *Let* $f \in \Lambda^{\varepsilon}(r, s)$. *Then*

$$F_n f \in \begin{cases} \Lambda^{\varepsilon}(r, s+n-1) & \text{if} \quad n > 0, \\ \Lambda^{-\varepsilon}(r+n, s-1) & \text{if} \quad n < 0, \end{cases}$$

$$G_n f \in \Lambda^{(-1)^{n-1}\varepsilon}(r, s+n-1) \quad \text{if} \quad n > 0,$$

where $G_n(x, y) = x^{n-1}F_n(x^{-1}, y)$.

PROOF. We show that $f \in \Lambda^{+1}(r, s)$ implies $F_n f \in \Lambda^{+1}(r, s+n-1)$ if $n > 0$. The other cases can be proved similarly. It is clear that $F_n f$ satisfies the conditions (i), (ii), (iii) and the first inequality of (iv) in Definition 2 for $\Lambda^{+1}(r, s+n-1)$. The second inequality of (iv) can be reduced to the sublemma below.

SUBLEMMA. *Let* $f(x) = \sum_{i=0}^{n} a_i x^i$, *where* $a_i = a_{n-i}$ *and* $0 < a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{[n/2]}$. *Let* $(\sum_{j=0}^{m} x^j)f(x) = \sum_{k=0}^{m+n} b_k x^k$. *Then* $b_k = b_{m+n-k}$ *and* $0 < b_0 \leqslant b_1 \leqslant \cdots \leqslant b_{[(m+n)/2]}$.

We omit the proof, as it is straightforward to prove it directly.

LEMMA 5. *If* $\Delta_{n-1} \in \Lambda^{-\varepsilon}(r, s-1)$ *and* $\Delta_n^{(1)} \in \Lambda^{\varepsilon}(r, s)$, *then*

$$\Delta_n^{(p)} \in \begin{cases} \Lambda^{\varepsilon}(r, s+p-1) & \text{if} \quad p > 0, \\ \Lambda^{-\varepsilon}(r+p, s-1) & \text{if} \quad p < 0. \end{cases}$$

PROOF. (4.2) in Theorem 2 states that $\Delta_n^{(p)} = F_p\Delta_n^{(1)} - xyF_{p-1}\Delta_{n-1}$. The case $p = 1$ is the hypothesis. If $p \geqslant 2$, then using Lemma 4, $F_p\Delta_n^{(1)} \in \Lambda^{\varepsilon}(r, s+p-1)$ and $-xyF_{p-1}\Delta_{n-1} \in \Lambda^{\varepsilon}(r+1, s+p-2)$. Thus $\Delta_n^{(p)} \in \Lambda^{\varepsilon}(r, s+p-1)$. If $p \leqslant -1$, then $F_p\Delta_n^{(1)}, -xyF_{p-1}\Delta_{n-1} \in \Lambda^{-\varepsilon}(r+p, s-1)$, so $\Delta_n^{(p)} \in \Lambda^{-\varepsilon}(r+p, s-1)$.

LEMMA 6. *Let* $\Delta_n^{\langle m \rangle}$ *be the Alexander polynomial of*

$$D(p_1, q_1, \ldots, p_{n-m}, q_{n-m}, 1, q_{n-m+1}, 1, \ldots, q_{n-1}, 1).$$

Then we have

$$\Delta_n^{\langle m \rangle} = G_{m+1}\Delta_{n-m} - xyG_m\Delta_{n-m}^{(p_{n-m}-1)} + (x-1)(y-1)\sum_{k=1}^{m}(q_{n-k}+1)G_k\Delta_{n-k}, \tag{5.1}$$

where the last term denotes zero if $m = 0$.

PROOF. We prove (5.1) by induction on m. For $m = 0$, it is clear that $\Delta_n^{\langle 0 \rangle} = \Delta_n$. Assume that (5.1) is proved for $m - 1$. Substituting $p_{n-m+1} = 1$ in

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$\Delta_n^{\langle m-1\rangle}$ we have

$$\Delta_n^{\langle m\rangle} = G_m\Delta^{(1)}_{n-m+1} - xyG_{m-1}\Delta^{(0)}_{n-m+1} + (x-1)(y-1)\sum_{k=1}^{m-1}(q_{n-k}+1)G_k\Delta_{n-k}.$$

By (4.2), $\Delta^{(1)}_{n-m+1} = q_{n-m}(x-1)(y-1)\Delta_{n-m} + \Delta_{n-m}^{(p_{n-m}+1)}$. Thus we have

$$\Delta_n^{\langle m\rangle} = G_m\{-(x-1)(y-1)\Delta_{n-m} + \Delta_{n-m}^{(p_{n-m}+1)}\} - xyG_{m-1}\Delta_{n-m} + (x-1)(y-1)\sum_{k=1}^{m}(q_{n-k}+1)G_k\Delta_{n-k}.$$

By (4.4), $\Delta_{n-m}^{(p_{n-m}+1)} = (xy+1)\Delta_{n-m} - xy\Delta_{n-m}^{(p_{n-m}-1)}$. Thus we have

$$\Delta_n^{\langle m\rangle} = \{(x+y)G_m - xyG_{m-1}\}\Delta_{n-m} - xyG_m\Delta_{n-m}^{(p_{n-m}-1)} + (x-1)(y-1)\sum_{k=1}^{m}(q_{n-k}+1)G_k\Delta_{n-k}.$$

Since $(x+y)G_m - xyG_{m-1} = G_{m+1}$, we have (5.1).

Now we are in position to prove Theorem 3. We use induction on n. For $n=1$, the theorem is clear. Assume the theorem proved for Δ_k, where $1 \leqslant k \leqslant n-1$. Without loss of generality we may suppose that $q_{n-1} < 0$. By Lemma 5 we only have to prove for the case $p_n = 1$. Then there exists an integer m such that:

(I) $1 \leqslant m \leqslant n-1$, $p_{n-m+1} = p_{n-m+2} = \cdots = p_{n-1} = 1$, $p_{n-m} \neq 1$ and $q_{n-m}, q_{n-m+1}, \ldots, q_{n-1} < 0$,

(II) $1 \leqslant m \leqslant n-2$, $p_{n-m} = p_{n-m+1} = p_{n-m+2} = \cdots = p_{n-1} = 1$, $q_{n-m}, q_{n-m+1}, \ldots, q_{n-1} < 0$ and $q_{n-m-1} > 0$, or

(III) $m = n-1$, $p_1 = p_2 = \cdots = p_{n-1} = 1$, $q_1, q_2, \ldots, q_{n-1} < 0$.

To prove Theorem 3, it suffices to prove that $\Delta_{n-m} \in \Lambda^{\varepsilon}(r,s)$ implies $\Delta_n \in \Lambda^{(-1)^m\varepsilon}(r, s+m)$, where by Lemma 6

$$\Delta_n = G_{m+1}\Delta_{n-m} - xyG_m\Delta_{n-m}^{(p_{n-m}-1)} + (x-1)(y-1)\sum_{k=1}^{m}(q_{n-k}+1)G_k\Delta_{n-k}. \tag{5.2}$$

By Lemma 4, we have

$$G_{m+1}\Delta_{n-m} \in \Lambda^{(-1)^m\varepsilon}(r, s+m). \tag{5.3}$$

By inductive hypothesis, $\Delta_{n-k} \in \Lambda^{(-1)^{m-k}\varepsilon}(r, s+m-k)$ for $1 \leqslant k \leqslant m$. Then by Lemma 4, $G_k\Delta_{n-k} \in \Lambda^{(-1)^{m-1}\varepsilon}(r, s+m-1)$; hence we obtain

$$(x-1)(y-1)\sum_{k=1}^{m}(q_{n-k}+1)G_k\Delta_{n-k}\begin{cases} = 0 & \text{if } q_{n-k} = -1 \text{ for any } k, \\ \in \Lambda^{(-1)^m\varepsilon}(r, s+m) & \text{otherwise.}\end{cases} \tag{5.4}$$

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Case (I). If $p_{n-m} \neq 1$, then by inductive hypothesis,

$$\Delta_{n-m}^{(p_{n-m}-1)} \in \begin{cases} \Lambda^{\varepsilon}(r, s-1) & \text{if } p_{n-m} \geqslant 2, \\ \Lambda^{\varepsilon}(r-1, s) & \text{if } p_{n-m} \leqslant -1. \end{cases}$$

Thus, using Lemma 4, we have

$$(5.5) \qquad -xyG_m\Delta_{n-m}^{(p_{n-m}-1)} \in \begin{cases} \Lambda^{(-1)^m\varepsilon}(r+1, s+m-1) & \text{if } p_{n-m} \geqslant 2, \\ \Lambda^{(-1)^m\varepsilon}(r, s+m) & \text{if } p_{n-m} \leqslant -1. \end{cases}$$

Case (II). If $p_{n-m} = 1$ and $q_{n-m-1} > 0$, then by inductive hypothesis,

$$\Delta_{n-m}^{(p_{n-m}-1)} = \Delta_{n-m-1} \in \Lambda^{\varepsilon}(r, s-1).$$

Thus, using Lemma 4, we have

$$(5.6) \qquad -xyG_m\Delta_{n-m}^{(p_{n-m}-1)} \in \Lambda^{(-1)^m\varepsilon}(r+1, s+m-1).$$

Case (III). Since $m = n-1$ and $p_1 = 1$, we have

$$(5.7) \qquad -xyG_m\Delta_{n-m}^{(p_{n-m}-1)} = 0.$$

From (5.2) $\sim$ (5.7), we have $\Delta_n \in \Lambda^{(-1)^m\varepsilon}(r, s+m)$. This completes the proof of Theorem 3.

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Department of Mathematics
Kobe University
Kobe 657
Japan

Author's current address:
Department of Mathematics
Kyushu University 33
Fukuoka 812
Japan

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