ON REDFIELD'S GROUP REDUCTION FUNCTIONS

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1. Introduction. In 1927, J. H. Redfield **(12)**, discussed some of the links between combinatorial analysis and permutation groups, including such topics as group transitivity, the enumeration of certain geometrical configurations, and the construction of various permutation isomorphs of a given group. Except for a revision of Redfield's treatment of transitivity by D. E. Littlewood **(6)**, this 1927 paper appears to have been overlooked. However, it has recently been described **(5)** as a remarkable pioneering paper which appears to contain or anticipate virtually all of the enumeration results for graphs which have been discovered and developed during the last thirty years.[1] Redfield associated every permutation group with a type of symmetric function which he called a group reduction function, and introduced two connective operations between such functions which we shall denote by $\cap$ and $\cup$.

In this paper I treat the topics considered by Redfield, except those dealt with by Littlewood, in a modern context from the point of view of group representation theory, and discard Redfield's symmetric functions in favour of group characters. The operations $\cap$, $\cup$ are then equivalent to well-known compositions, the scalar product and the inner product, of certain group characters.

This disentanglement of the group characters from the symmetric functions is not, however, sufficient in itself to resolve the ambiguities which beset Redfield, and which he was unable to analyse. These ambiguities are completely removed in § 6 by discarding the group characters in favour of the marks **(1)** of transitive permutation groups.

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[1]The referee suggested the following quotation from **(5)**: "it contains

(1) The exact formula of Read's Superposition Theorem.

(2) Apparently the first published definition of the cycle index of a permutation group under the name of the 'group reduction function.'

(3) Formulas for the cycle index of the symmetric, alternating, cyclic and dihedral groups.

(4) The cycle index of the group of symmetries of a 3-cube. He actually substitutes $1 + x$ into this cycle index, thereby giving the first known example of Pólya's theorem. This also anticipates the enumeration of the symmetry types of boolean functions due to Pólya and Slepian.

(5) A substitution of $1/(1 - x)$ into this cycle index. This is a device . . . for enumerating graphs in which any number of lines are permitted to join the same two points.

(6) The number of graphs with p points and q lines for $p = 5$ and $q = 4$ as a solution of a problem involving the number of types of binary relations."

2. Symmetric functions and induced characters. If ψ is the character of a representation σ of a subgroup H of a group G, it is well known **(4)** that the character ψ' of G induced by ψ is given by

$$\psi'(x) = \frac{g}{hg_\rho} \sum_{z \in C_\rho \cap H} \psi(z), \tag{1}$$

where g, h are the orders of G, H respectively, and C_ρ is the class of G, of order g_ρ, which contains x.

In the special case when σ is the principal unit representation of H, that is, the representation in which every element of H is mapped on $+1$, we have

$$\psi'(x) = gh_\rho/hg_\rho, \tag{2}$$

where h_ρ is the number of elements in $C_\rho \cap H$. This is the character of the transitive permutation representation of G induced by H. The character of any permutation representation will be referred to as a permutation character, whether the representation is transitive or not.

Consider any faithful permutation representation R, of degree m, of a finite group G of order g. Let $x \in G$, and let R_x have a_r cycles of length r, where $r = 1, 2, \dots, m$. With R_x associate a product $S_x = s_1^{a_1} s_2^{a_2} \dots s_m^{a_m}$, where

$$s_p = \sum_{j=1}^{t} \alpha_j^p,$$

the α_j's being indeterminates, and t having no specified value in general. Let ν be any representation of G, not necessarily as a permutation group, with character ϕ. Then we associate with R and ν the symmetric function

$$F(R, \nu) = \frac{1}{g} \sum_{x \in G} \phi(x) S_x.$$

LEMMA 1.

$$F(R, \nu) = \frac{1}{m!} \sum_{\rho} g_\rho \phi_\rho' S_\rho,$$

where ϕ' is the character of $\mathfrak{S}_m$ induced by ϕ, the summation is over all classes C_ρ, of order g_ρ, of $\mathfrak{S}_m$, and $S_\rho = S_z$ for $z \in C_\rho$.[2]

Proof. If R is a faithful representation of G, then G can be considered as a subgroup of $\mathfrak{S}_m$, and ν induces a representation ν' of $\mathfrak{S}_m$ such that, from (1), for any s belonging to the class C_ρ of $\mathfrak{S}_m$

$$\phi'(s) = \frac{m!}{gg_\rho} \sum_{z \in C_\rho \cap G} \phi(z).$$

Hence

$$\phi_\rho' S_\rho = \frac{m!}{gg_\rho} \sum_{z \in C_\rho \cap G} \phi(z) S_z$$

[2] $\mathfrak{S}_m$ is the symmetric group on m symbols.

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and

$$F(R,\nu) = \frac{1}{g}\sum_{x\in G}\phi(x)S_x$$

$$= \frac{1}{g}\sum_{\rho}\Big[\sum_{z\in C_\rho\cap G}\phi(z)S_z\Big]$$

$$= \frac{1}{g}\sum_{\rho}\frac{gg_\rho}{m!}\phi_\rho' S_\rho$$

$$(3)\qquad = \frac{1}{m!}\sum_{\rho} g_\rho \phi_\rho' S_\rho.$$

Thus $F(R,\nu)$ may be defined either in terms of the character ϕ of G, or in terms of the character ϕ' of $\mathfrak{S}_m$. We may write $F(R,\nu) = F(R',\nu')$, where R' is the representation of $\mathfrak{S}_m$ as a permutation group of degree m. If η is any representation of $\mathfrak{S}_m$, we shall say that $F(R',\eta)$ is based on η, or on the character of η.

These symmetric functions include many well-known functions as special cases. Here we are concerned with the case when ν is the principal unit representation of any finite group G, and R is any permutation representation of G on m letters. In this case $F(R,\nu)$ is Redfield's group reduction function of G with respect to R **(12)**. Redfield observed that the same group reduction function may be shared by two or more distinct permutation groups. We return to this ambiguity in § 6.

This special case of $F(R,\nu)$ has also been used by Pólya **(9)**, Riordan **(13)**, Read **(10, 11)**, and others, and has been called the cycle index, or cycle indicator of R. When R is the permutation representation of $\mathfrak{S}_n$ induced by $\mathfrak{S}_{n-2}\dot{\times}\mathfrak{S}_2$, and ν is the principal unit representation of $\mathfrak{S}_n$, then $F(R,\nu)$ is the function G_n used by Riordan **(13)** in enumerating linear graphs.

3. Scalar product of two functions defined on a group. Let ϕ,ψ be any functions defined for every element $x \in G$, and having values in some field F whose characteristic is not a divisor of the order g of G. Then the scalar product of ϕ and ψ is defined as (see **4**, 270; **8**, 95)

$$(\phi,\psi) = \frac{1}{g}\sum_{x\in G}\phi(x)\psi(x^{-1}).$$

The property of the scalar product we require is that if

$$\phi = \sum_i \alpha_i\chi_i, \qquad \psi = \sum_i \beta_i\chi_i,$$

where $\alpha_i, \beta_i \in F$, and χ_i is an absolutely irreducible character of G, then

$$(\phi,\psi) = \sum_i \alpha_i\beta_i.$$

In particular if ϕ is absolutely irreducible, then (ϕ,ψ) is the multiplicity of ϕ in ψ.

The concept of scalar product has appeared in various forms in the literature, and the equivalence of the various forms appears not to have been appreciated. It has frequently been entangled with symmetric functions (**2**, **6**, **10**, **11**, **12**). Thus if D_Θ is a differential operator (**2**) obtained from a symmetric function

$$\Theta = \sum_{\rho \nmid m} k_\rho S_\rho,$$

based on a representation of character θ of $\mathfrak{S}_m$, by replacing s_i^α by $i^\alpha(\partial^\alpha/\partial s_i^\alpha)$, we have the following result:

LEMMA 2. *If ϕ, ψ are any two characters of $\mathfrak{S}_m$, and*

$$\Phi = \frac{1}{m!}\sum_\rho g_\rho \phi_\rho S_\rho, \qquad \Psi = \frac{1}{m!}\sum g_\rho \psi_\rho S_\rho,$$

then

$$D_\Phi \Psi = D_\Psi \Phi = (\phi, \psi).$$

Proof. If $\rho = 1^{a_1}2^{a_2}\ldots m^{a_m}$ is a partition of m, then

$$D_\Phi \Psi = \left[\frac{1}{m!}\sum_\rho g_\rho \phi_\rho 1^{a_1} 2^{a_2} \ldots m^{a_m} \frac{\partial^{a_1+a_2+\ldots+a_m}}{\partial s_1^{a_1}\partial s_2^{a_2}\ldots \partial s_m^{a_m}}\right]\Psi$$

$$= \left[\sum_\rho \frac{1}{a_1!\, a_2! \ldots a_m!}\phi_\rho \frac{\partial^{a_1+a_2+\ldots+a_m}}{\partial s_1^{a_1}\partial s_2^{a_2}\ldots \partial s_m^{a_m}}\right]\frac{1}{m!}\sum_\rho g_\rho \psi_\rho S_\rho$$

$$= \frac{1}{m!}\sum_\rho g_\rho \phi_\rho \psi_\rho = (\phi, \psi),$$

since in $\mathfrak{S}_m$ every class contains the inverse of every element in the class.

Redfield (**12**, 436-438) introduces an operation between two symmetric functions defined by

(i) $$s_{\lambda_1}^{l_1} s_{\lambda_2}^{l_2} \ldots \cap s_{\mu_1}^{m_1} s_{\mu_2}^{m_2} \ldots = \delta_{\lambda_1}^{l_1}\delta_{\lambda_2}^{l_2}\ldots s_{\mu_1}^{m_1} s_{\mu_2}^{m_2}\ldots,$$

where $\delta_w = w(\partial/\partial s_w)$, and

(ii) $$(A + B) \cap C = (A \cap C) + (B \cap C),$$

where A, B, C are symmetric functions. Clearly the following lemma holds.

LEMMA 3. $\Phi \cap \Psi = D_\Phi \Psi = (\phi, \psi)$.

Read (**10**, 422) defines a composition of two symmetric functions

$$\Phi = \sum_\rho A_\rho s_1^{a_1} s_2^{a_2} \ldots s_m^{a_m}, \qquad \Psi = \sum_\rho B_\rho s_1^{a_1} s_2^{a_2} \ldots s_m^{a_m}$$

as

$$N\{\Phi * \Psi\} = \sum_\rho A_\rho B_\rho a_1!\, a_2! \ldots a_m!\, 2^{a_2} 3^{a_3} \ldots m^{a_m}.$$

Hence

$$N\{\Phi * \Psi\} = \sum_{\rho} A_{\rho} \frac{\partial^{a_1}}{\partial s_1^{a_1}} \cdot 2^{a_2} \frac{\partial^{a_2}}{\partial s_2^{a_2}} \cdot 3^{a_3} \ldots m^{a_m} \frac{\partial^{a_m}}{\partial s_m^{a_m}} B_{\rho}\, s_1^{a_1} s_2^{a_2} \ldots s_m^{a_m},$$

and so we have:

LEMMA 4. $N\{\Phi * \Psi\} = D_{\Phi}\Psi = (\phi, \psi)$, *where* Φ, Ψ *are based on the functions* ϕ, ψ *defined over* $\mathfrak{S}_m$.

Littlewood (**6**, 165) defines a function $N(G_1, G_2)$ of two subgroups G_1, G_2 of $\mathfrak{S}_m$ taking non-negative integer values. He calls it the transitive factor of G_1, G_2 and shows that if

$$\phi = \sum_i \alpha_i \chi_i, \qquad \psi = \sum_i \beta_i \chi_i$$

are the permutation characters of $\mathfrak{S}_m$ induced by G_1, G_2 respectively, and χ_i is an irreducible character of $\mathfrak{S}_m$, then

$$N(G_1, G_2) = \sum_i \alpha_i \beta_i.$$

Hence we have:

LEMMA 5. $N(G_1, G_2) = (\phi, \psi)$.

Both Redfield and Read define their respective compositions $\cap$ and $*$ for more than two symmetric functions. This extension is considered in the following section.

4. Kronecker products of representations. If $\phi(x)$, $\psi(x)$ are the characteristics of an element $x \in G$ in two representations R, S of G, the character of the Kronecker product representation $R \times S$ is called the inner product of ϕ and ψ and is denoted by $[\phi, \psi]$. We write $[\phi, \psi, \theta]$ for the common value of $[[\phi, \psi], \theta]$, $[[\phi, \theta], \psi]$, $[[\theta, \psi], \phi]$, and similarly we have a unique definition of $[\phi_1, \phi_2, \ldots, \phi_r]$, where ϕ_i is any character of G.

Redfield (**12**, 438) defines a second composition $A \cup B$ between two symmetric functions A, B of the same weight m. Thus, if ϕ, ψ are characters of $\mathfrak{S}_m$, and the symmetric functions based on ϕ, ψ are Φ, Ψ as in Lemma 2, then $\Phi \cup \Psi$ is defined by means of

(i) $s_1^{a_1} s_2^{a_1} \ldots s_m^{a_m} \cup s_1^{a_1} s_2^{a_2} \ldots s_m^{a_m}$
$$= (s_1^{a_1} s_2^{a_2} \ldots s_m^{a_m} \cap s_1^{a_1} s_2^{a_2} \ldots s_m^{a_m})\, s_1^{a_1} s_2^{a_2} \ldots s_m^{a_m},$$

(ii) $s_1^{a_1} s_2^{a_2} \ldots s_m^{a_m} \cup s_1^{b_1} s_2^{b_2} \ldots s_m^{b_m} = 0$
when $1^{a_1} 2^{a_2} \ldots m^{a_m}$ and $1^{b_1} 2^{b_2} \ldots m^{b_m}$ are different partitions,

(iii) $(A + B) \cup C = A \cup C + B \cup C$.

We have

$$\Phi \cup S_\rho = \frac{1}{m!} g_\rho\, \phi_\rho\, 1^{a_1} 2^{a_2} \ldots m^{a_m} a_1!\, a_2! \ldots a_m!\, s_1^{a_1} s_2^{a_2} \ldots s_m^{a_m} = \phi_\rho\, S_\rho,$$

and

$$\Phi \cup \Psi = \frac{1}{m!} \sum_{\rho} g_{\rho}\, \phi_{\rho}\, \psi_{\rho}\, S_{\rho},$$

which is the symmetric function of weight m based on $[\phi, \psi]$. Hence we have:

LEMMA 6. *If Φ, Ψ, Θ are symmetric functions of weight m based on characters ϕ, ψ, θ of $\mathfrak{S}_m$ respectively, and $\Phi \cup \Psi = \Theta$, then $\theta = [\phi, \psi]$.*

If $\Phi_1 \cup \Phi_2 \cup \ldots \cup \Phi_r = \Theta$, then it is clear that Θ is based on $[\phi_1, \phi_2, \ldots, \phi_r]$, and we are able to extend the concept of a scalar product of two characters of $\mathfrak{S}_m$ to that of a scalar product of several characters of $\mathfrak{S}_m$ as follows.

DEFINITION. *If 1_m is the principal unit character of $\mathfrak{S}_m$, and $\phi_1, \phi_2, \ldots, \phi_r$ are class functions defined over $\mathfrak{S}_m$, then the multiple scalar product $(\phi_1, \phi_2, \ldots, \phi_r)$ is defined as $(1_m, [\phi_1, \phi_2, \ldots, \phi_r])$.*

In particular $(\phi, \psi) = (1_m, [\phi, \psi])$, $\Phi_1 \cap \Phi_2 \cap \ldots \cap \Phi_r = (\phi_1, \phi_2, \ldots, \phi_r)$, and $N\{\Phi_1 * \Phi_2 * \ldots * \Phi_r\} = (\phi_1, \phi_2, \ldots, \phi_r)$.

Since every class function defined over $\mathfrak{S}_m$ is a linear combination of the irreducible characters of $\mathfrak{S}_m$, we have by § 3:

LEMMA 7. *If $\phi_1, \phi_2, \ldots, \phi_r$ are class functions defined over $\mathfrak{S}_m$, then $(\phi_1, \phi_2, \ldots, \phi_r)$ is the multiplicity of the principal unit character of $\mathfrak{S}_m$ in $[\phi_1, \phi_2, \ldots, \phi_r]$.*

5. Application of Frobenius' reciprocity theorem. A formulation of Frobenius' reciprocity theorem in terms of scalar products has been given by P. Hall **(3)** as follows.

If ϕ, ψ are characters of subgroups H, K respectively of a finite group G, and ϕ^G, ψ^G are the corresponding induced characters of G, then

$$(\phi^G, \psi^G) = (\phi_K{}^G, \psi) = (\phi, \psi_H{}^G),$$

where $\phi_K{}^G$, $\psi_H{}^G$ respectively denote ϕ^G restricted to K, and ψ^G restricted to H.

When ϕ, ψ are the principal unit characters $1_H, 1_K$ of H, K and 1_G is the principal unit character of G, we have

$$(\phi^G, \psi^G) = (1_G, [\phi^G, \psi^G]) = (1_K, \phi_K{}^G) = (1_H, \psi_H{}^G)$$

so that we have:

LEMMA 8. *If ϕ^G, ψ^G are the transitive permutation characters of G induced by subgroups H, K respectively, then the multiplicities of the principal unit characters in $[\phi^G, \psi^G]$, $\psi_H{}^G$, $\phi_K{}^G$ are the same.*

If the multiple scalar product $(\phi_a, \ldots, \phi_s)$ of permutation characters $\phi_a, \ldots, \phi_s$ of any finite group G is defined as $(1_G, [\phi_a, \ldots, \phi_s])$, then the following theorem replaces Redfield's main theorem **(12**, 445**)**:

THEOREM 1. *If $\phi_1, \ldots, \phi_r$ are the permutation characters of G induced by a complete set of non-conjugate subgroups, then $[\phi_a, \phi_b, \ldots, \phi_s]$ is in general an intransitive permutation character of G, and $(\phi_a, \phi_b, \ldots, \phi_s)$ is the number of its transitive constituents.*

Proof. The Kronecker product of permutation representations of G is clearly a permutation representation. If the analysis of the corresponding character into transitive constituents is given by

$$[\phi_a, \phi_b, \ldots, \phi_s] = \sum_{i=1}^{r} k_i \phi_i,$$

then

$$\begin{aligned}(\phi_a, \phi_b, \ldots, \phi_s) &= (1_G, [\phi_a, \phi_b, \ldots, \phi_s]) \\ &= \left(1_G, \sum_i k_i \phi_i\right) \\ &= \sum_i k_i(1_G, \phi_i) \\ &= \sum_i k_i(1_{H_i}, 1_{H_i})\end{aligned}$$

by the reciprocity theorem, H_i being the subgroup of G which induces the permutation character ϕ_i. Hence $(\phi_a, \phi_b, \ldots, \phi_s)$ is $\sum k_i$, the number of transitive constituents of $[\phi_a, \phi_b, \ldots, \phi_s]$.

Redfield's theorem is, essentially, the particular case of the above theorem when $G = \mathfrak{S}_m$, and it is this special case with which we are mainly concerned here.

6. The unique resolution of permutation representations into transitive constituents. When $m > 2$ the transitive permutation characters of $\mathfrak{S}_m$ are linearly dependent. The analysis of a permutation character of $\mathfrak{S}_m$ into transitive permutation characters in accordance with Theorem 1 is thus not unique in general, though the number of such transitive characters is precisely determined by the theorem. This was one of the difficulties encountered by Redfield (**12**, 446).

To resolve the ambiguity we recall that two permutation matrix representations of a group are equivalent if and only if they can be transformed into each other by a permutation matrix. With this understanding of permutational equivalence the ultimate irreducibles with which we have to deal are not the irreducible characters of the group but are the marks (**1**, 236) of the distinct subgroups in each of the transitive permutation representations of the group. The mark of a subgroup H of a group G in a permutation representation g, of degree r, of G is the number of the r symbols which are invariant under every permutation of g restricted to H. We shall use the term mark of a representation g as the set of marks of all the non-conjugate subgroups of G in that representation.

We are primarily interested here in the case in which G is a symmetric group. Thus when $G = \mathfrak{S}_4$, the representative subgroups can be taken as[3]

$$
\begin{aligned}
G_1 &= 1,\\
G_2 &= \{1, (ab)\},\\
G_3 &= \{1, (ab)(cd)\},\\
G_4 &= \{1, (abc), (acb)\},\\
G_5 &= \{1, (abcd), (ac)(bd), (adcb)\},\\
G_6 &= \{1, (ab)(cd), (ac)(bd), (ad)(bc)\},\\
G_7 &= \{1, (ab), (cd), (ab)(cd)\},\\
G_8 &= \mathfrak{S}_3,\\
G_9 &= \{1, (ac), (bd), (ac)(bd), (abcd), (adcb), (ad)(bc), (ab)(cd)\},\\
G_{10} &= \mathfrak{A}_4,\\
G_{11} &= \mathfrak{S}_4,
\end{aligned}
$$

where the subgroups are written as permutation groups on four or fewer letters merely for convenience of description. The table of marks for $\mathfrak{S}_4$ is found to be as in Table I.

TABLE I

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}	G_{11}
g_1	24										
g_2	12	2									
g_3	12	0	4								
g_4	8	0	0	2							
g_5	6	0	2	0	2						
g_6	6	0	6	0	0	6					
g_7	6	2	2	0	0	0	2				
g_8	4	2	0	1	0	0	0	1			
g_9	3	1	3	0	1	3	1	0	1		
g_{10}	2	0	2	2	0	2	0	0	0	2	
g_{11}	1	1	1	1	1	1	1	1	1	1	1

If $p(m)$ is the number of partitions of m, then $\mathfrak{S}_m$ will have $p(m)$ non-conjugate cyclic subgroups, each generated by an arbitrary element taken from one of the $p(m)$ conjugate classes of $\mathfrak{S}_m$ in turn. Let these, in non-decreasing order, be taken as $G_1, G_2, \ldots, G_{p(m)}$. The mark of any cyclic subgroup in any representation g_r is the characteristic of the generator in g_r regarded as a matrix representation. Hence if in the first $p(m)$ columns of the table of marks we replace $G_1, G_2, \ldots, G_{p(m)}$ at the head of the table by $C_{1^m}, C_{\rho_2}, \ldots, C_{\rho_{p(m)}}$, where C_{ρ_i} is the class of $\mathfrak{S}_m$ which contains a generator of G_i, we obtain a table of the complete set of transitive permutation characters of $\mathfrak{S}_m$.

[3]The same set of subgroups of $\mathfrak{S}_4$ and the table of the complete set of transitive permutation characters of $\mathfrak{S}_4$ are given by **(14, 5)**.

When $m > 2$, this table of transitive permutation characters has more rows than columns. We note, however, that because of the triangular nature of the table of marks, the characters ϕ_i, $i = 1, 2, \ldots, p(m)$, of the representations g_i form a linear basis for all transitive permutation characters of $\mathfrak{S}_m$, and also for all characters of $\mathfrak{S}_m$.

The inner product of two or more transitive permutation characters gives another permutation character, but this can often be decomposed into a sum of integer multiples of transitive characters in more than one way. In the language of group reduction functions instead of permutation characters we have the difficulty encountered by Redfield, namely that the decomposition of $\mathrm{Grf}(G_1) \bigcup \mathrm{Grf}(G_2)$ is not in general unique.

To restore uniqueness all that is needed is to consider the marks, instead of the characters, of the Kronecker products of the g_i's.

Thus for $\mathfrak{S}_4$ the subgroups G_7, G_9 have group reduction functions

$$\tfrac{1}{4}(s_1^4 + 2s_1^2 s_2 + s_2^2), \qquad \tfrac{1}{8}(s_1^4 + 2s_1^2 s_2 + 3s_2^2 + 2s_4)$$

respectively, giving transitive permutation characters $\phi_7 = 6, 2, 2, 0, 0$ and $\phi_9 = 3, 1, 3, 0, 1$ of $\mathfrak{S}_4$ where the classes of $\mathfrak{S}_4$ are taken as $1^4, 1^2 2, 2^2, 13, 4$. We have

$$\mathrm{Grf}(G_7) \bigcap \mathrm{Grf}(G_9) = 2,$$

which is equal to (ϕ_7, ϕ_9) by Lemma 3, and by Theorem 1 is the number of transitive characters in $[\phi_7, \phi_9]$. Also

$$\mathrm{Grf}(G_7) \bigcup \mathrm{Grf}(G_9) = \tfrac{1}{4}(3s_1^4 + 2s_1^2 s_2 + 3s_2^2),$$

which by Lemma 6 is the symmetric function based on $[\phi_7, \phi_9] = [18, 2, 6, 0, 0]$. This character can be written as $\phi_2 + \phi_6$ or as $\phi_3 + \phi_7$.

If m_i denotes the set of marks corresponding to g_i, and which we regard as the mark of g_i, then $g_7 \times g_9$ has the mark $18, 2, 6, 0, 0, 0, 2, 0, 0, 0, 0$, which decomposes uniquely into $m_3 + m_7$ and the ambiguity is resolved.

Another difficulty encountered by Redfield arises from the fact that the same group reduction function can belong to two or more groups which are not equivalent as permutation groups. Thus the groups

$$Q_1 = \{1, (ab)(cd)(e)(f), (ab)(c)(d)(ef), (a)(b)(cd)(ef)\},$$
$$Q_2 = \{1, (ab)(cd)(e)(f), (ac)(bd)(e)(f), (ad)(bc)(e)(f)\},$$

are not equivalent as permutation groups but have the same group reduction function $\tfrac{1}{4}(s_1^6 + 3s_1^2 s_2^2)$, and give the same induced permutation character

$$\phi = 180, 0, 0, 0, 12, 0, \ldots, 0$$

of $\mathfrak{S}_6$, where the two non-zero characteristics belong to the classes $1^6, 1^2 2^2$ of $\mathfrak{S}_6$ in this order. The subgroup $\mathfrak{S}_5 \dot{\times} \mathfrak{S}_1$ of $\mathfrak{S}_6$ induces the usual permutation representation of $\mathfrak{S}_6$ on 6 symbols. Its group reduction function is the product of the Schur functions $\{5\}$ and $\{1\}$, and the corresponding permutation character of $\mathfrak{S}_6$ is

$$\psi = 6, 4, 3, 2, 2, 1, 1, 0, 0, 0, 0$$

where the classes of $\mathfrak{S}_6$ are taken in the sequence $1^6, 1^4 2, 1^3 3, 1^2 4, 1^2 2^2$, 123, 15, 6, 24, 2^3, 3^2. We have $(\phi, \psi) = 3$ and

$$[\phi, \psi] = 1080, 0, 0, 0, 24, 0, 0, 0, 0, 0, 0.$$

Redfield's difficulty here was, in effect, to separate $[\phi, \psi]$ into three transitive characters when ϕ was taken as belonging to Q_1, and into another three when it was taken as belonging to Q_2.

The difficulty vanishes if we consider the inner product, defined in the obvious way, of appropriate rows of the table of marks of $\mathfrak{S}_6$ instead of rows of the transitive permutation character table. We need consider only the subgroups $G_5 = \mathfrak{S}_5 \dot{\times} \mathfrak{S}_1$, $G_3 = Q_1$, $G_4 = Q_2$, and the common subgroups $G_1 = 1$, $G_2 = \{1, (ab)(cd)\}$ of Q_1 and Q_2. The only part of the table of marks of $\mathfrak{S}_6$ which matters for our purpose is that given in Table II, where the

TABLE II

	G_1	G_2	G_3	G_4	G_5
g_1	720				
g_2	360	8			
g_3	180	12	x		
g_4	180	12	0	y	
g_5	6	2	0	2	1

first two columns correspond to the columns $1^6, 1^2 2^2$ of the transitive permutation character table, and x, y are integers >0 whose values we need not in fact determine. Besides distinguishing g_3, g_4, or in effect Q_1, Q_2, which the permutation characters fail to do, the table of marks gives the inner products $[m_3, m_5] = 1080, 24, 0, 0, 0 = 3m_2$ and $[m_4, m_5] = 1080, 24, 0, 2y, 0 = m_1 + 2m_4$, thus giving the different decompositions of $g_3 \times g_5$ and $g_4 \times g_5$ into three transitive constituents, as sought by Redfield.

Summarizing, we have the following elaboration of Theorem 1, giving uniqueness criteria to the decomposition of Redfield's $\mathbf{U}$-products.

THEOREM 2. *If the transitive permutation representations of $\mathfrak{S}_m$ with respect to a complete set of non-conjugate subgroups $G_1, G_2, \ldots, G_r$ have marks $m_1, m_2, \ldots, m_r$ (each a set of r non-negative integers m_i^j, the sets being linearly independent) and characters $\phi_1, \phi_2, \ldots, \phi_r$ (each a set of $p(m)$ non-negative integers ϕ_i^j, the sets being linearly dependent when $m > 2$), then*

(i) *$\phi_i^j = m_i^j$ if the notation is such that when G_j is cyclic it is generated by an element of the class C_j of $\mathfrak{S}_m$,*

(ii) $[m_a, m_b, \ldots, m_k]$ *has a unique expression as*

$$\sum_{i=1}^{r} \alpha_i m_i,$$

where the α_i*'s are non-negative integers, and* $\alpha_i = 0$ *when* G_i *is not a subgroup of* $G_a \cap G_b \cap \ldots \cap G_k$,

(iii) $\sum_{i=1}^{r} \alpha_i = (\phi_a, \phi_b, \ldots, \phi_k)$,

(iv) $[\phi_a, \phi_b, \ldots, \phi_k] = \sum_{i=1}^{r} \alpha_i \phi_i$.

Redfield noted two special cases in which his **U**-products were capable of unique decomposition into group reduction functions. His special cases are replaced by the two following theorems.

THEOREM 3. *If at least one of* $G_a, G_b, \ldots, G_k$ *is cyclic, then* $[\phi_a, \phi_b, \ldots, \phi_k]$ *can be uniquely expressed as* $\alpha_p \phi_p + \alpha_q \phi_q + \ldots + \alpha_t \phi_t$, *where* $\phi_p, \phi_q, \ldots, \phi_t$ *are permutation characters of* $\mathfrak{S}_m$ *induced by subgroups of the cyclic group* $G_a \cap G_b \cap \ldots \cap G_k$, *and* $\alpha_p, \alpha_q, \ldots, \alpha_t$ *are non-negative integers such that* $\alpha_p + \alpha_q + \ldots + \alpha_t = (\phi_a, \phi_b, \ldots, \phi_k)$.

Proof. Take $G_1, G_2, \ldots, G_{p(m)}$ in non-decreasing order as the complete set of non-conjugate cyclic subgroups of $\mathfrak{S}_m$. Then the complete set of r marks in any of the representations $g_1, g_2, \ldots, g_{p(m)}$ is obtained from the set of $p(m)$ numbers in the permutation character ϕ_i, $(i = 1, 2, \ldots, p(m))$, by attaching $r - p(m)$ zeros. Also the $p(m) \times p(m)$ table of characters induced by $G_1, G_2, \ldots, G_{p(m)}$, arranged suitably, will be triangular in the sense that no term in the main diagonal will be zero, and every entry above the diagonal will be zero. Hence $\phi_1, \phi_2, \ldots, \phi_{p(m)}$ are linearly independent.

Now because at least one of $G_a, G_b, \ldots, G_k$ is cyclic, one at least of $m_a, m_b, \ldots, m_k$ will have zeros in the last $r - p(m)$ places, and so $[\phi_a, \phi_b, \ldots, \phi_k]$ will differ from $[m_a, m_b, \ldots, m_k]$ only in that the latter has a further $r - p(m)$ zeros. By Theorem 2, $[m_a, m_b, \ldots, m_k]$ has a unique expression as $\sum \alpha_i m_i$, where the α_i are non-negative integers, and every m_i corresponds to a subgroup G_i of $G_a \cap G_b \cap \ldots \cap G_k$, which is cyclic since at least one of the groups is cyclic. But $m_i = \phi_i$ for a cyclic subgroup G_i, except for the attached zeros, and so $[\phi_a, \phi_b, \ldots, \phi_k] = \sum \alpha_i \phi_i$, where the summation is over all subgroups of $G_a \cap G_b \cap \ldots \cap G_k$, and

$$\sum \alpha_i = (\phi_a, \phi_b, \ldots, \phi_k)$$

by Theorem 2. Since $\phi_1, \phi_2, \ldots, \phi_{p(m)}$ are linearly independent, this decomposition of $[\phi_a, \phi_b, \ldots, \phi_k]$ is unique.

THEOREM 4. *If the subgroups* $G_a, G_b, \ldots, G_k$ *of* $\mathfrak{S}_m$ *are each the direct product of symmetric groups, then* $[\phi_a, \phi_b, \ldots, \phi_k]$ *can be expressed uniquely as*

$$\alpha_p \phi_p + \alpha_q \phi_q + \ldots + \alpha_t \phi_t,$$

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where $\phi_p, \phi_q, \ldots, \phi_t$ are permutation characters of $\mathfrak{S}_m$ induced by subgroups which are themselves direct products of symmetric groups, and $\alpha_p, \alpha_q, \ldots, \alpha_t$ are non-negative integers such that $\alpha_p + \alpha_q + \ldots + \alpha_t = (\phi_a, \phi_b, \ldots, \phi_k)$.

Proof. Let

$$G_a = \mathfrak{S}_1^{\lambda_1} \dot{\times} \mathfrak{S}_2^{\lambda_2} \dot{\times} \ldots \dot{\times} \mathfrak{S}_m^{\lambda_m},$$

$$G_b = \mathfrak{S}_1^{\nu_1} \dot{\times} \mathfrak{S}_2^{\nu_2} \dot{\times} \ldots \dot{\times} \mathfrak{S}_m^{\nu_m},$$

where $(1^{\lambda_1}2^{\lambda_2}\ldots m^{\lambda_m})$, $(1^{\nu_1}2^{\nu_2}\ldots m^{\nu_m}), \ldots$ are partitions of m. Let the symbols on which g_a operates be $A_1, A_2, \ldots$, and those on which g_b operates be $B_1, B_2, \ldots$, and so on. Then the symbols on which the Kronecker product $g_a \times g_b \times \ldots \times g_k$ operates can be taken as all possible sets $\{A_i, B_j, \ldots, K_v\}$, where if $\theta \in \mathfrak{S}_m$, $\{A_i, B_j, \ldots, K_v\}\theta$ is defined as $\{A_i\theta, B_j\theta, \ldots, K_v\theta\}$. By Theorem 2, these symbols fall into $(\phi_a, \phi_b, \ldots, \phi_k)$ transitive sets. Let

$$\{A_{x1}, B_{y1}, \ldots, K_{u1}\}, \{A_{x2}, B_{y2}, \ldots, K_{u2}\}, \ldots, \{A_{xr}, B_{yr}, \ldots, K_{ur}\}$$

form a transitive set S. If θ belongs to the stabilizer of $\{A_{x1}, B_{y1}, \ldots, K_{u1}\}$, that is to the subgroup of $\mathfrak{S}_m$ which leaves this symbol unaltered, then

$$A_{x1}\theta = A_{x1}, \qquad B_{y1}\theta = B_{y1}, \ldots, \qquad K_{u1}\theta = K_{u1}$$

and θ will belong to the stabilizers of $A_{x1}, B_{y1}, \ldots, K_{u1}$ and conversely. Hence the stabilizer of $\{A_{x1}, B_{y1}, \ldots, K_{u1}\}$ is the intersection of the stabilizers of $A_{x1}, B_{y1}, \ldots, K_{u1}$. But these stabilizers are permutation transforms of $G_a, G_b, \ldots, G_k$ respectively, and so the stabilizer of $\{A_{x1}, B_{y1}, \ldots, K_{u1}\}$ is also a direct product of symmetric groups. Thus the transitive representation of $\mathfrak{S}_m$ on the symbols of S is induced by a subgroup of $\mathfrak{S}_m$ which is a direct product of symmetric groups. This applies to each of the $(\phi_a, \phi_b, \ldots, \phi_k)$ transitive sets into which $g_a \times g_b \times \ldots \times g_k$ decomposes.

Hence $[m_a, m_b, \ldots, m_k]$ has a unique expression as

$$\alpha_p m_p + \alpha_q m_q + \ldots + \alpha_t m_t,$$

where the α_i's are non-negative integers, and $m_p, m_q, \ldots, m_t$ belong to the set of marks of representations of $\mathfrak{S}_m$ induced by the $p(m)$ distinct non-conjugate subgroups of $\mathfrak{S}_m$ which are direct products of symmetric groups.

Considering the first $p(m)$ columns of the table of marks of $\mathfrak{S}_m$ as the columns corresponding to cyclic subgroups, it follows that since $\phi_i^j = m_i^j$ for these columns, then $[\phi_a, \phi_b, \ldots, \phi_k]$ can be expressed as

$$\alpha_p \phi_p + \alpha_q \phi_q + \ldots + \alpha_t \phi_t.$$

It remains to show that this expression is unique. To do this we show that the characters ϕ_i corresponding to the $p(m)$ subgroups $\mathfrak{S}_1^{a_1} \dot{\times} \mathfrak{S}_2^{a_2} \dot{\times} \ldots \dot{\times} \mathfrak{S}_m^{a_m}$, taken over all partitions $1^{a_1} 2^{a_2} \ldots m^{a_m}$ of m, are linearly independent.

Arrange the partitions of m so that (ρ_1) precedes (ρ_2) if, after parts common to (ρ_1) and (ρ_2) are struck out, (ρ_2) has at least one part greater than every part of (ρ_1). This ensures that if $(\rho_1) = 1^{a_1} 2^{a_2} \ldots m^{a_m}$, then $\mathfrak{S}_1^{a_1} \dot{\times} \mathfrak{S}_2^{a_2} \dot{\times} \ldots \dot{\times} \mathfrak{S}_m^{a_m}$ has no element whose permutation cycles are given by (ρ_2), and so has no cyclic subgroup corresponding to (ρ_2). With this ordering of the partitions, the part of the table of marks consisting of the first $p(m)$ columns and the $p(m)$ rows corresponding to representations induced by subgroups of type $\mathfrak{S}_1^{a_1} \dot{\times} \mathfrak{S}_2^{a_2} \dot{\times} \ldots \dot{\times} \mathfrak{S}_m^{a_m}$ will have, by (2), non-zero entries on the main diagonal and nothing but zeros above the diagonal. The $p(m)$ rows are thus independent, and so the $p(m)$ permutation characters of $\mathfrak{S}_m$ induced by subgroups of type $\mathfrak{S}_1^{a_1} \dot{\times} \mathfrak{S}_2^{a_2} \dot{\times} \ldots \dot{\times} \mathfrak{S}_m^{a_m}$ are linearly independent. From Theorem 2, $\alpha_p + \alpha_q + \ldots + \alpha_t = (\phi_a, \phi_b, \ldots, \phi_k)$, and the proof is complete.

Redfield used symmetric function formulae to perform the actual decomposition dealt with in this theorem. In the present treatment, the triangular nature of the restricted character table makes the arithmetical reduction extremely simple. His example (**12**, 448) is from MacMahon (**7**) and in the notation of § 6 is the reduction of $g_2 \times g_2 \times g_7$. We have

$$\begin{aligned}[\phi_2, \phi_2, \phi_7] &= 864, 8, 0, 0, 0\\ &= 34\phi_1 + 4\phi_2.\end{aligned}$$

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