Canad. Math. Bull.Vol. **36** (3), 1993 pp. 273–282

BARNES' FIRST LEMMA AND ITS FINITE ANALOGUE

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ABSTRACT. We give a parallel proof of Barnes' first lemma and of its finite analogue. In both cases we use the Mellin transform. In the classical case, the proof avoids the residue theorem. In the finite case the Gamma function is replaced by the Gaussian sum function and the beta function by the Jacobi sum function.

1. **Introduction.** In the study of the solutions of Gauss' hypergeometric differential equation [17] E. W. Barnes discovered in 1910 the following theorem.

THEOREM 1 (BARNES' FIRST LEMMA). *Let a_1, a_2, a_3, a_4 denote four complex numbers such that none of $a_1 + a_2$, $a_2 + a_3$, $a_3 + a_4$, $a_4 + a_1$ is a pole of the Gamma function. Then*

$$\frac{1}{2\pi i}\int_C \Gamma(a_1+s)\Gamma(a_2-s)\Gamma(a_3+s)\Gamma(a_4-s)\,ds = \frac{\Gamma(a_1+a_2)\Gamma(a_2+a_3)\Gamma(a_3+a_4)\Gamma(a_4+a_1)}{\Gamma(a_1+a_2+a_3+a_4)},$$

where C is a contour going from $-i\infty$ to $+i\infty$ leaving the poles of $\Gamma(a_1+s)\Gamma(a_3+s)$ on the left and the poles of $\Gamma(a_2-s)\Gamma(a_4-s)$ on the right.

Note that the hypothesis on a_1, a_2, a_3, a_4 is equivalent to the separation of the poles of $\Gamma(a_2-s)\Gamma(a_4-s)$ from the poles of $\Gamma(a_1+s)\Gamma(a_3+s)$; and that this hypothesis is necessary in view of the righthand side. The standard proofs of Theorem 1 [1, 16, 17] use the theorem of residues and Gauss' summation theorem. Here we present a proof based on the Mellin transform which reduces to computing certain convolution integrals avoiding both theorems.

First, we prove the identity of Theorem 1 when C is a vertical line, which is possible when $\max\big(\Re(a_1),\Re(a_3)\big) < \min\big(\Re(a_2),\Re(a_4)\big)$. Then an analytic continuation argument completes the proof when C is bent.

Our original motivation was to find a new proof of the finite analogue of this theorem (Theorem 2 below) discovered and proved by the first author [7, 8] in her study of the representations of the general linear group over a finite field. In this analogy the Gamma function is replaced by a Gauss sum, and contour integration by a summation on the multiplicative characters of a finite field $\mathbf{F}_q$ with q elements. A representation-theoretic proof is in [7, 13] and an extension to non-archimedean local fields is in [14].

Received by the editors February 4, 1992.
AMS subject classification: Primary: 33A15, 11L05; secondary: 20C99.

THEOREM 2. *Let α_1, α_2, α_3, α_4 denote four multiplicative characters of* $\mathbf{F}_q$. *Then*

$$\frac{1}{q-1}\sum_{\alpha} g(\alpha_1\alpha)g(\alpha_2\alpha^{-1})g(\alpha_3\alpha)g(\alpha_4\alpha^{-1}) = \frac{g(\alpha_1\alpha_2)g(\alpha_2\alpha_3)g(\alpha_3\alpha_4)g(\alpha_4\alpha_1)}{g(\alpha_1\alpha_2\alpha_3\alpha_4)} + q(q-1)\delta(\alpha_1\alpha_2\alpha_3\alpha_4)(\alpha_1\alpha_3)(-1),$$

where $g(\beta)$ is the Gauss sum associated with the character β and $\delta(\beta) = 1$ if β is trivial and zero otherwise.

The well-known analogy between the Gamma function (resp. beta function) and the Gauss sum function (resp. the Jacobi sum function) which dates back to the times of Gauss and Jacobi [10] has enjoyed recently a surge of interest [4, 6, 11, 12].

2. **Classical case.**

2.1 *The beta and Gamma functions.* We recall that the classical Gamma function $\Gamma(s)$ can be defined for $\Re(s) > 0$ as an Eulerian integral of the second kind

$$\Gamma(s) = \int_0^{+\infty} e^{-t}t^{s-1}\,dt, \tag{1}$$

and that it can be analytically continued to the complex plane minus the points $0, -1, -2, \ldots$. In the same fashion, the classical beta function can be defined as an Eulerian integral of the first kind

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \tag{2}$$

for $\Re(x) > 0$ and $\Re(y) > 0$ and we have

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{3}$$

This relation can be used to derive an analytic continuation of $B(x,y)$ to all the points (x,y) of $\mathbf{C}^2$ such that none of $x, y, x+y$ is a pole of Γ.

2.2 *The classical Mellin transform.* This integral transform is best thought of as a Fourier transform on the multiplicative group of the positive reals. If $f(t)$ is a complex-valued function defined for $t \geq 0$, then its Mellin transform $f^*(s)$ is defined on a certain strip $a < \Re(s) < b$ depending on f by the formula

$$f^*(s) = \int_0^{+\infty} f(t)t^{s-1}\,dt. \tag{4}$$

Indeed the change of variable $t = e^{-u}$ yields the Laplace transform of $f(e^{-u})$:

$$f^*(s) = \int_{-\infty}^{+\infty} f(e^{-u})e^{-us}\,du. \tag{5}$$

We note, for future use, that $\Gamma(s)$ is the Mellin transform of e^{-t}, by formula (1).

The function $f(t)$ can be recovered from $f^*(s)$ by contour integration along the vertical line $\Re(s) = \gamma$ for a suitable γ for the following inversion formula

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)t^{-s}\,ds, \tag{6}$$

as proved in, *e.g.* [15, p. 39].

We now define the convolution product $f * h$ of two functions of the real variable t as

$$(f * h)(x) = \int_0^{+\infty} f(t)h(x/t)\frac{dt}{t}.$$

Note that the change of variable $t = \frac{1}{u}$ yields $f * h = h * f$. The motivation for this definition lies in the following lemma, which is Theorem 13 p. 39 of [15].

LEMMA 1. *The Mellin transform of the convolution product of two functions f and h is the pointwise product of the respective Mellin transforms of the two functions.*

$$(f * h)^* = (h * f)^* = f^*h^*.$$

The starting point of our investigations was a variation on Parseval's theorem.

LEMMA 2. *Let f and h denote two functions with Mellin transforms f^* and h^*. Then*

$$\int_0^{+\infty} f(t)h(t)\frac{dt}{t} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)h^*(-s)\,ds.$$

PROOF. Let $k(t) = f(\frac{1}{t})$ so that $k^*(s) = f^*(-s)$. Then by Lemma 1 and the inversion formula (6) applied to $(k * h)(x)$ we get

$$\int_0^{+\infty} h(t)k(x/t)\frac{dt}{t} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} h(s)k(s)x^{-s}\,ds.$$

The result follows on letting $k(s) = f(-s)$ and $x = 1$. ∎

2.3 *Barnes' first lemma for a line.* Let $f_i(t) = t^{a_i}e^{-t}$. Then $f_i^*(s) = \Gamma(a_i + s)$, for $i = 1, 2, 3, 4$ and these four functions can be analytically continued to all the complex plane except at, respectively, $s = -a_i - j, j = 0, 1, \ldots$. Assume, furthermore, that

$$\max\big(\Re(a_1), \Re(a_3)\big) < \min\big(\Re(a_2), \Re(a_4)\big),$$

and let γ denote a real abscissa in the range

$$\Big(\max\big(\Re(a_1), \Re(a_3)\big), \min\big(\Re(a_2), \Re(a_4)\big)\Big).$$

Then Barnes' integral can be expressed as

$$I_B = \frac{1}{2\pi i}\int_D [f_1^*(s)f_3^*(s)][f_2^*(-s)f_4^*(-s)]\,ds$$

where D denotes the vertical line $\Re(z) = \gamma$. By Lemma 1 we see that $f_1^*(s)f_3^*(s) = (f_1*f_3)^*(s)$ and $f_2^*(s)f_4^*(s) = (f_2*f_4)^*(s)$. Applying Lemma 2 with $f = f_1*f_3$ and $h = f_2*f_4$, we obtain

$$I_B = \int_0^{+\infty} (f_1 * f_3)(t)\,(f_2 * f_4)\,(t)\frac{dt}{t}. \tag{7}$$

Using Lemma 1 the convolution products are readily computed and we find

$$(f_1 * f_3)(t) = t^{a_3} \int_0^{+\infty} x^{a_1-a_3} \exp\Big(-x - \frac{t}{x}\Big)\frac{dx}{x},$$

and analogously

$$(f_2 * f_4)(t) = t^{a_4} \int_0^{+\infty} y^{a_2-a_4} \exp\Big(-y - \frac{t}{y}\Big)\frac{dy}{y}.$$

These integrals are closely related to Bessel functions (*cf.* [3, Section 7.3.6, equation (34)]). Substituting these two expressions into formula (7), and summing first on t, we obtain

$$I_B = \int_0^{+\infty}\int_0^{+\infty} x^{a_1-a_3}y^{a_2-a_4} \exp(-x-y)J(x,y)\frac{dx}{x}\frac{dy}{y},$$

where $J(x,y) = \int_0^{+\infty} t^{a_3+a_4} \exp\big(-t\frac{x+y}{xy}\big)\frac{dt}{t}$. After the change of variables $u = t\frac{x+y}{xy}$, a Gamma function appears: $J(x,y) = \Gamma(a_3+a_4)(x+y)^{-a_3-a_4}(xy)^{a_3+a_4}$.

Let $r_B = \frac{I_B}{\Gamma(a_3+a_4)}$. Then, we obtain

$$r_B = \int_0^{+\infty}\int_0^{+\infty} x^{a_1+a_4}y^{a_2+a_3}(x+y)^{-a_3-a_4} \exp(-x-y)\frac{dx}{x}\frac{dy}{y}.$$

The new change of variables $v = x+y$, $dv = dy$ yields

$$r_B = \int_0^{+\infty} v^{-a_3-a_4-1}e^{-v} \int_0^{v} (v-x)^{a_2+a_3-1}x^{a_1+a_4}\frac{dx}{x}\,dv.$$

Letting $x = vu$, $\frac{dx}{x} = \frac{du}{u}$ entails

$$r_B = \int_0^{+\infty} v^{a_1+a_4} \int_0^1 u^{a_1+a_4}v^{a_2+a_3}(1-u)^{a_2+a_3-1}v^{-a_3-a_4}e^{-v}\frac{du}{u}\frac{dv}{v},$$

which factors out into

$$r_B = \int_0^1 u^{a_1+a_4-1}(1-u)^{a_2+a_3-1}\,du \int_0^{+\infty} v^{a_1+a_2}e^{-v}\frac{dv}{v}.$$

By the formulas of Section 2.1 we recognize the beta and Gamma functions in the righthand side.

$$r_B = B(a_1+a_4, a_2+a_3)\Gamma(a_1+a_2).$$

The result follows on expressing the beta function as a ratio of Gamma functions, *i.e.*

$$B(a_1+a_4, a_2+a_3) = \frac{\Gamma(a_1+a_4)\Gamma(a_2+a_3)}{\Gamma(a_1+a_2+a_3+a_4)}.$$

Note that the apparent lack of symmetry of the proof comes from the expression chosen for the convolution products.

2.4 *Barnes' first lemma for a bent contour.* Let a_1, a_3 be fixed once and for all. Let $\pi(a_2, a_4)$ denote the righthand side of Barnes' first lemma, that is

$$\pi(a_2, a_4) = \frac{\Gamma(a_1 + a_2)\Gamma(a_2 + a_3)\Gamma(a_3 + a_4)\Gamma(a_4 + a_1)}{\Gamma(a_1 + a_2 + a_3 + a_4)}.$$

Let $\Gamma[a_2, a_4; s]$ denote the integrand of the lefthand side, namely

$$\Gamma[a_2, a_4; s] = \Gamma(a_1 + s)\Gamma(a_2 - s)\Gamma(a_3 + s)\Gamma(a_4 - s)(2\pi i)^{-1}.$$

Suppose that the condition $\max\big(\Re(a_1), \Re(a_3)\big) < \min\big(\Re(a_2), \Re(a_4)\big)$ is not met. Let C denote the bent contour occuring in Theorem 1. Then consider b_2 and b_4 such that $\max\big(\Re(a_1), \Re(a_3)\big) < \min\big(\Re(b_2), \Re(b_4)\big)$. Then, we know from the preceding section that there exists a vertical line D such that

$$\pi(b_2, b_4) = \int_D \Gamma[b_2, b_4; s]\, ds.$$

Since the integrand is analytic in s for s to the right of C we can distort the line D to fit the contour C and obtain

$$\pi(b_2, b_4) = \int_{C_a} \Gamma[b_2, b_4; s]\, ds.$$

Both sides of the latter equality are analytic functions of (b_2, b_4). Indeed $\pi(b_2, b_4)$ is a product of analytic functions and analycity for the righthand side follows from Theorem 4.3 of [5, p. 39].

As (b_2, b_4) ranges over an open domain of $\mathbf{C}^2$ determined by a_1 and a_3, the *principle of analytic continuation* (*cf.* [2, p. 124] for a bivariate version) entails

$$\pi(a_2, a_4) = \int_{C_a} \Gamma[a_2, a_4; s]\, ds.$$

This completes our proof of Theorem 1.

3. **Finite case.**

3.1 *Gauss sums and Jacobi sums.* Let p denote a prime number, and $\mathbf{F}_q$ a finite field of characteristic p. Let $\mathbf{F}_q^+$ (resp. $\mathbf{F}_q^\times$) denote the additive (resp. multiplicative) group of $\mathbf{F}_q$ and X the character group of $\mathbf{F}_q^\times$. We denote by δ the Dirac function on X, namely $\delta(\alpha) = 1$ if $\alpha = 1$ and zero otherwise. Let ψ denote a *non-trivial* additive character of $\mathbf{F}_q$ fixed once and for all. The Gauss sum function $g\colon X \longmapsto \mathbf{C}$ is defined by

$$\forall \alpha \in X, \quad g(\alpha) = \sum_{a \in \mathbf{F}_q^\times} \psi(a)\, \alpha(a) \tag{8}$$

Formula (8) is a finite analogue of formula (1); note that, in the latter formula, the function $t \longmapsto e^{-t}$ (resp. $t \longmapsto t^{s-1}$) plays the role of the additive character $\psi(a)$, (resp. the multiplicative character $\alpha(a)$) in (8).

https://doi.org/10.4153/CMB-1993-039-1 Published online by Cambridge University Press

The Jacobi sum function $b\colon X \times X \longmapsto \mathbf{C}$, is defined for $\alpha_1, \alpha_2 \in X$ by

$$b(\alpha_1, \alpha_2) = \sum_{a, 1-a \in \mathbf{F}_q^\times} \alpha_1(a)\, \alpha_2(1-a) \tag{9}$$

Formula (9) is a finite analogue of formula (2). The analogy, mentioned in the introduction, between Gauss sums and Jacobi sums on the one hand, and Gamma and beta functions on the other hand is now obvious. In analogy with formula (3) we have the following identity [9, Chapter 8, Section 3, Theorem 1]

$$b(\alpha_1, \alpha_2) = \frac{g(\alpha_1)g(\alpha_2)}{g(\alpha_1\alpha_2)} + (q-1)\alpha_2(-1)\,\delta(\alpha_1\alpha_2). \tag{10}$$

Whereas in formula (3) $x + y \neq 0$, in formula (10) $\alpha_1\alpha_2 = 1$ is allowed, and this accounts for the term in $\delta(\alpha_1\alpha_2)$. We quote for future use one more property of Gauss sums [9, Proposition 8.2.2]:

$$g(\alpha)g(\alpha^{-1}) = q\alpha(-1) - (q-1)\delta(\alpha). \tag{11}$$

We recall the orthogonality of characters:

$$\sum_{a \in \mathbf{F}_q^\times} \alpha(a)\beta(a) = (q-1)\delta(\alpha\beta). \tag{12}$$

3.2 *The finite Mellin transform.* This transform is a Fourier transform on the group $\mathbf{F}_q^\times$. For a detailed study see [11].

With every complex-valued function f defined on $\mathbf{F}_q^\times$, we associate a complex-valued function f^* defined on X by the formula:

$$f^*(\alpha) = \sum_{a \in \mathbf{F}_q^\times} f(a)\,\alpha(a).$$

The following inversion formula can easily be verified.

LEMMA 3. *For all $f\colon \mathbf{F}_q^\times \longmapsto \mathbf{C}$ we have*

$$\forall a \in \mathbf{F}_q^\times, \quad f(a) = \frac{1}{q-1} \sum_{\alpha \in X} \alpha^{-1}(a) f^*(\alpha).$$

Note, for future use, that $g(\alpha)$ is the Mellin transform of $a \longmapsto \psi(a)$, just as $\Gamma(s)$ is the Mellin transform of $t \longmapsto e^{-t}$. The convolution is defined as

$$(f * h)(a) = \sum_{c \in \mathbf{F}_q^\times} f(c)\, h(ac^{-1}). \tag{13}$$

It is easy to check, that, as in Lemma 1, we have

$$(f * h)^*(\alpha) = f^*(\alpha)\, h^*(\alpha). \tag{14}$$

The variation on Parseval's lemma that we shall employ is

(15) $$\frac{1}{q-1}\sum_{\alpha\in X} f^*(\alpha)\, h^*(\alpha^{-1}) = \sum_{a\in \mathbf{F}_q^\times} f(a)\, h(a).$$

3.3 *The finite analogue.* As in Section 2.3, define four auxiliary functions $f_i = \alpha_i\psi$, for $i = 1,2,3,4$. Then their Mellin transforms are $f_i^*(\alpha) = g(\alpha_i\alpha)$. The Barnes sum S_B can be written as

$$S_B = \frac{1}{q-1}\sum_{\alpha\in X} f_1^*(\alpha) f_3^*(\alpha) f_2^*(\alpha^{-1}) f_4^*(\alpha^{-1}),$$

or, equivalently by formula (14)

(16) $$S_B = \frac{1}{q-1}\sum_{\alpha\in X} (f_1 * f_3)^*(\alpha)\,(f_2 * f_4)^*(\alpha^{-1}).$$

Applying formula (15) with $f = f_1 * f_3$, and $h = f_2 * f_4$, we get

(17) $$S_B = \sum_{a\in \mathbf{F}_q^\times} (f_1 * f_3)(a)\,(f_2 * f_4)(a).$$

Define for $a \in \mathbf{F}_q^\times$ and $\beta \in X$ the Kloosterman sum

$$K(\beta; a) = \sum_{x\in \mathbf{F}_q^\times} \beta(x)\,\psi\Big(x + \frac{a}{x}\Big).$$

The convolution products are easy to express with these sums. A direct application of formula (13) yields

$$(f_1 * f_3)(a) = \alpha_3(a)\, K(\alpha_1\alpha_3^{-1}; a)$$

as well as

$$(f_2 * f_4)(a) = \alpha_4(a)\, K(\alpha_2\alpha_4^{-1}; a).$$

Substituting these two expressions into formula (17), we get

$$S_B = \sum_{x,y\in \mathbf{F}_q^\times} (\alpha_1\alpha_3^{-1})(x)\,(\alpha_2\alpha_4^{-1})(y)\,\psi(x+y)\, J_q(x,y),$$

where $J_q(x,y) = \sum_{a\in \mathbf{F}_q^\times} (\alpha_3\alpha_4)(a)\psi(\frac{a(x+y)}{xy})$. The evaluation of $J_q(x,y)$ is somewhat more complicated than the evaluation of its complex counterpart $J(x,y)$ in Section 2.3 as the next lemma shows.

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LEMMA 4. *For $x, y \in \mathbf{F}_q^\times$ we have*

$$J_q(x,y) = (\alpha_3\alpha_4)\Big(\frac{xy}{x+y}\Big) g(\alpha_3\alpha_4)$$

if $x+y \neq 0$, and

$$J_q(x,-x) = (q-1)\,\delta(\alpha_3\alpha_4).$$

PROOF. If $y = -x$ we get

$$J_q(x,y) = \sum_{a\in\mathbf{F}_q^\times} (\alpha_3\alpha_4)(a) = (q-1)\,\delta(\alpha_3\alpha_4),$$

by formula (12). If $x+y \neq 0$, we can make the change of variable $c = a\frac{x+y}{xy}$, and the Gauss sum $g(\alpha_3\alpha_4)$ appears. ∎

Substituting into the Barnes sum, and distinguishing again the cases $x+y = 0$ and $x+y \neq 0$, we get

$$S_B = (q-1)^2(\alpha_2\alpha_4)(-1)\,\delta(\alpha_1\alpha_2)\,\delta(\alpha_3\alpha_4) + g(\alpha_3\alpha_4)m_B,$$

where m_B is the analogue of r_B of Section 2.3 and is given by

$$m_B = \sum_{x,y,x+y\neq 0} (\alpha_1\alpha_4)(x)\,(\alpha_2\alpha_3)(y)\,(\alpha_3^{-1}\alpha_4^{-1})\big(x+y\psi(x+y)\big).$$

Making the same change of variables as in Section 4, we get

$$m_B = g(\alpha_1\alpha_2)\,b(\alpha_1\alpha_4,\alpha_2\alpha_3).$$

Using the connection between Gauss sums and Jacobi sums of Section 2.5, formula (10), we get

$$b(\alpha_1\alpha_4,\alpha_2\alpha_3) = \frac{g(\alpha_1\alpha_4)g(\alpha_2\alpha_3)}{g(\alpha_1\alpha_4\alpha_2\alpha_3)} + (q-1)\,\delta(\alpha_1\alpha_4\alpha_2\alpha_3)\,(\alpha_2\alpha_3)(-1).$$

We now tidy up the correcting terms. Let

$$\Delta_B = S_B - \frac{g(\alpha_1\alpha_2)g(\alpha_2\alpha_3)g(\alpha_3\alpha_4)g(\alpha_4\alpha_1)}{g(\alpha_1\alpha_2\alpha_3\alpha_4)}.$$

From the previous calculations we obtain

$$\begin{aligned}\Delta_B = {}& (q-1)^2\alpha_2\alpha_4(-1)\delta(\alpha_1\alpha_2)\,\delta(\alpha_3\alpha_4)\\ &+ (q-1)\,\delta(\alpha_1\alpha_4\alpha_2\alpha_3)\,g(\alpha_1\alpha_2)\,g(\alpha_3\alpha_4)\,(\alpha_2\alpha_3)(-1).\end{aligned}$$

Note that

$$g(\alpha_1\alpha_2)\,g(\alpha_3\alpha_4)\,\delta(\alpha_1\alpha_4\alpha_2\alpha_3) = g(\alpha_1\alpha_2)\,g\big((\alpha_1\alpha_2)^{-1}\big)\,\delta(\alpha_1\alpha_4\alpha_2\alpha_3).$$

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From formula (11) of Section 2.5 we see that

$$g(\alpha_1\alpha_2)g\big((\alpha_1\alpha_2)^{-1}\big) = q(\alpha_1\alpha_2)(-1) - (q-1)\delta(\alpha_1\alpha_2).$$

Altogether we obtain

$$\begin{aligned}\Delta_B/(q-1) = (q-1)\,\delta(\alpha_1\alpha_2)\,\delta(\alpha_3\alpha_4)\,(\alpha_2\alpha_4)(-1)\\ + q(\alpha_2\alpha_3)(-1)\,\delta(\alpha_1\alpha_4\alpha_2\alpha_3)\,(\alpha_1\alpha_2)(-1)\\ - (q-1)(\alpha_2\alpha_3)(-1)\,\delta(\alpha_1\alpha_4\alpha_2\alpha_3)\,\delta(\alpha_1\alpha_2).\end{aligned}$$

The first and the third terms cancel out and we obtain

$$\Delta_B/(q-1) = q(\alpha_1\alpha_3)(-1)\,\delta(\alpha_1\alpha_4\alpha_2\alpha_3),$$

as promised in Theorem 2.

4. **Conclusion.** We have developed in parallel the proof of Barnes' first lemma and of its finite analogue. However, the symmetry breaks down in two ways.

Firstly, there are some unavoidable analytic topics which appear in the classical case as in Section 2.4. Such considerations are not necessary in the finite case where integrals are replaced by finite sums. The validity conditions on the a_i have no finite equivalent.

Secondly, the correcting terms which appear in the righthand side of the finite field analogue do not occur in the classical case. A recourse to distribution theory might be needed at this point.

Altogether, it would be nice to have a common framework for both situations.

ACKNOWLEDGEMENT. The second author thanks W. Li and J. P. Serre for their interest in this work. Special thanks are due to A. Jean-Marie for advice on Section 2.

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