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On the nine-point conic.

By Professor ALLARDICE.

It is well known that the properties of the orthocentre and of the nine-point circle of a triangle may be most symmetrically stated when the triangle and its orthocentre are looked upon as the vertices of a four-point, the opposite sides of which intersect at right angles. This point of view leads naturally to a generalisation of the nine-point circle, by consideration of any four-point in place of the orthic four-point—a generalisation which was first given in detail by Beltrami in the year 1863; though the theorems involved had been previously stated by T. T. Wilkinson.* A number of papers have since been written on the nine-point conic; but they have for the most part merely given Beltrami's results over again, and have generally been written in ignorance of his work. In this paper I propose giving the properties of the nine-point conic from a different point of view, associating them with the triangle instead of the four-point. There are certain advantages belonging to each point of view. If, for instance, we consider a triangle ABC with its orthocentre H as an orthic four-point, any proof that shows that the nine-point circle touches the inscribed (or an escribed) circle of the triangle ABC, will, in general, also show that it touches the inscribed (and escribed) circles of the triangles HCB, CHA and BAH. On the other hand, as the nine-point conic circumscribes the diagonal triangle of the four-point, if the four-point is given, the nine-point conic is definitely determined; whereas, if the triangle be considered, as the fourth vertex of the four-point may be taken arbitrarily, a number of nine-point conics are obtained, touching the same inscribed conic.

Let H (Fig. 8) be any point in the plane of a triangle ABC; and let AH, BH, CH meet the opposite sides of ABC in L, M, N;

* *The Lady's and Gentleman's Diary*, 1858, p. 87

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then MN, NL, LM meet BC, CA, AB in points lying in a straight line l. I shall speak of l as the polar line* of the point H, and of H as the polar point of the line l. If the coordinates of H are $(1/l, 1/m, 1/n)$, the equation of l is $l\alpha + m\beta + n\gamma = 0$.

The envelope of the polar lines of the points of a given straight line $l\alpha + m\beta + n\gamma = 0$ is the inscribed conic $l^{1/2}\alpha^{1/2} + m^{1/2}\beta^{1/2} + n^{1/2}\gamma^{1/2} = 0$.

If we take another straight line $p\alpha + q\beta + r\gamma = 0$, the corresponding conic is $p^{1/2}\alpha^{1/2} + q^{1/2}\beta^{1/2} + r^{1/2}\gamma^{1/2} = 0$. (Fig. 8.)

These two conics have one common tangent, besides the sides of the triangle, namely the polar line of the point of intersection of the two given straight lines. Hence the equation of the common tangent is $\alpha/(mr - nq) + \beta/(np - lr) + \gamma/(lq - mp) = 0$.

We may find the point of contact corresponding to a given point of $l\alpha + m\beta + n\gamma = 0$, as follows:—

Let (λ, μ, ν), $(\lambda + d\lambda, \mu + d\mu, \nu + d\nu)$ be consecutive points on $l\alpha + m\beta + n\gamma = 0$; so that $l\lambda + m\mu + n\nu = 0$, $ld\lambda + md\mu + nd\nu = 0$.

We find for the coordinates of the intersection of the polar lines of these points the values

$$\{\lambda^2(\nu d\mu - \mu d\nu),\ \mu^2(\lambda d\nu - \nu d\lambda),\ \nu^2(\mu d\lambda - \lambda d\mu)\};$$

and it may easily be shown that these are proportional to

$$(l\lambda^2,\ m\mu^2,\ n\nu^2).$$

We may obtain the envelope by putting $\lambda = (\alpha/l)^{1/2}$, etc., in the relation $l\lambda + m\mu + n\nu = 0$.

The points of contact of the common tangent to the two inscribed conics (Fig. 8) lie on the conic that passes through their six points of contact with the sides.

The equation of this conic is

$$lp\alpha^2 + mq\beta^2 + nr\gamma^2 - (lq + mp)\alpha\beta - (mr + nq)\beta\gamma - (np + lr)\gamma\alpha = 0.$$

To prove this we have merely to verify that the points $\{l(mr - nq)^2, m(np - lr)^2, n(lq - mp)^2\}$ and $\{p(mr - nq)^2, q(np - lr)^2, r(lq - mp)^2\}$ lie on this conic.

We may now show that a conic may be drawn to pass through L, M, N, (Fig. 8), to pass through the intersection of the conic PQR with the straight line l, and to touch the conic PQR at the point just considered, namely, $\{p(mr - nq)^2, q(np - lr)^2, r(lq - mp)^2\}$.

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* When the pole and polar with respect to a conic are referred to, the terms "pole" and "polar," simply, will be employed.

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Consider the equation

$$(l\alpha+m\beta+n\gamma)\{\alpha/(mr-nq)+\beta/(np-lr)+\gamma/(lq-mp)\}$$
$$+k\{p^2\alpha^2+q^2\beta^2+r^2\gamma^2-2pq\alpha\beta-2qr\beta\gamma-2rp\gamma\alpha\}=0. \quad \text{(A)}$$

This represents a conic through the intersection of the conic PQR with the straight line l and the common tangent to the two inscribed conics; and we may determine k so that this conic shall pass through L, the coordinates of which are $(0, 1/m, 1/n)$.

The value of k is easily found to be

$$2lmn/(mr-nq)(np-lr)(lq-mp);$$

and the symmetrical form of k shows that the conic also passes through the points M and N.

The conic thus obtained is a nine-point conic, and hence when an inscribed conic is given, there is a twofold infinity of nine-point conics that touch it, and that pass through its intersections with one of the straight lines that determine the nine-point conic.

Geometrical Method.

I have obtained geometrical proofs of all the theorems proved above analytically, with the exception of the theorem regarding the contact property.

1. The envelope of the polar lines of the points of a given straight line is an inscribed conic. (Fig. 9.)

Let δ be any point in the straight line $\beta\gamma\alpha$, XY the polar line of δ. It is obvious that the envelope of XY is a curve inscribed in the triangle; for the polar line of a point in a side (other than a vertex) is that side itself.

We have the theorem that if p, q, r, s are four points in a straight line, and if r', s' are the harmonic conjugates of r and s with respect to p and q, then the cross-ratios $(pqrs)$ and $(pqr's')$ are equal. This is a property of an involution.

Now $\quad (\gamma\alpha\beta\delta)=(\text{B}\,.\,\gamma\alpha\beta\delta)=(\text{AC}\beta\text{Q})=(\text{ACMY})$,

and $\quad (\gamma\alpha\beta\delta)=(\text{C}\,.\,\gamma\alpha\beta\delta)=(\gamma\text{BAR})=(\text{NBAZ})$;

hence XYZ envelopes a conic, which touches the sides at L, M and N.

Some Properties of the Figure.

Let U be the point of contact with the tangent XY; then, considering the conic as inscribed in the quadrilateral BZYC, we

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see that BY, CZ, LU, NM are concurrent. This gives a simple construction for the point of contact, U, of the polar line of a given point δ in l. Similarly CZ and AX intersect in a point O_1, through which LN and MU pass; and AX and BY intersect in a point O_2, through which LM and NU pass. Thus the four-point $\delta O_1 O O_2$ and the four-side QZYR have ABC for a common diagonal triangle.

If we take another point V on XY as the point of contact of a second conic, then VO meets BC in L′, etc., and thus the points L′, M′, N′ are determined, and then the straight line l', which must also pass through δ.

Thus the triangles LMN, L′M′N′, etc., the vertices of which move along the sides of ABC, have their sides passing through the fixed points O, O_1, O_2.

It is obvious from Carnot's Theorem that L, M, N, L′, M′, N′ lie on a conic. This may also be proved by means of Pascal's Theorem, by showing that the opposite sides of the hexagon NN′LL′MM′ meet in collinear points (by the use of the theorems of Ceva and Menelaus).

Again, by considering the Pascal hexagon M′NLMN′L′, we see that M′N and MN′ intersect on O_1O_2; and similarly N′L and NL′ intersect on O_2O, and L′M and LM′ on OO_1.

By considering the hexagon UMN′VM′N, since UM and VM′ intersect in O_1, MN′ and M′N on O_1O_2 and N′V and NU in O_2, we see that these six points lie on a conic; similarly, U, V, N, N′, L, L′ lie on a conic, as do also U, V, L, L′, M, M′. This does not prove that the eight points lie on a conic, which, however, is the case.

2. The Dual Theorem.

The locus of the polar points of lines passing through a fixed point is a conic circumscribed about the triangle. (Fig. 10.)

Let δ be any point, l a straight line passing through δ; let l meet the sides in the points α, β, γ, let S be the polar point of l, and let δA, δB, δC meet the sides in λ, μ, ν.

We have $(\delta . CA\mu\beta) = (CA\mu\beta) = (CA\mu' M) = (B . CA\mu' M)$;

and $(\delta . CA\mu\beta) = (\nu AB\gamma) = (\nu' ABN) = (C . \nu' ABN)$.

Hence the locus of the intersection of BM and CN, that is, the locus of the point S, is a conic circumscribing the triangle.

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The tangents at the vertices are the straight lines joining the vertices to the points λ', μ', ν', where $\lambda'\mu'\nu'$ is the polar line of δ.

Suppose now that δ (Fig. 9) is the given point, and l the straight line passing throught it. The circum-conic will pass through S, the polar point of l; and the tangent at S will be the straight line $\lambda\mu\nu$, where λ is the intersection of MN and O_1O_2; etc.

[This is the dual of the construction of the point of contact in the previous case.]

This method of treatment may be connected with the method that involves the theory of pole and polar, as follows:—

Let (λ, μ, ν) be a point on the straight line $l\alpha + m\beta + n\gamma = 0$, then (λ, μ, ν) transforms into a circum-conic (the locus of the polar points of lines passing through it). The pole of $l\alpha + m\beta + n\gamma = 0$, with respect to this conic is a point on the inscribed conic, namely, the point where the polar line of (λ, μ, ν) touches the conic.

For (λ, μ, ν) transforms into the conic $\lambda\beta\gamma + \mu\gamma\alpha + \nu\alpha\beta = 0$, where $l\lambda + m\mu + n\nu = 0$. The pole of $l\alpha + m\beta + n\gamma = 0$, with respect to this conic is $(l\lambda^2, m\mu^2, n\nu^2)$, which lies on the straight line $\alpha/\lambda + \beta/\mu + \gamma/\nu = 0$, the polar line of the point (λ, μ, ν).

The Dual Theorem.

If (l, m, n) is any point, $\lambda\alpha + \mu\beta + \nu\gamma = 0$ any straight line through it, then $\lambda\alpha + \mu\beta + \nu\gamma = 0$ transforms into an inscribed conic (the envelope of the polar line of points on it). The envelope of the polar of (l, m, n) with respect to this conic is a circumscribed conic; and the tangent at the point $(1/\lambda, 1/\mu, 1/\nu)$ given by $\lambda\alpha + \mu\beta + \nu\gamma = 0$ is $l\lambda^2\alpha + m\mu^2\beta + n\nu^2\gamma = 0$, which contains the point $(1/\lambda, 1/\mu, 1/\nu)$, the polar point of the line $\lambda\alpha + \mu\beta + \nu\gamma = 0$.

Geometrical Proof.

The former of the last two theorems may be proved geometrically as follows:—

The point δ (Fig. 9) transforms into a circum-conic through S, the polar point of the straight line l; and the tangents at the points A, B, C, and S are the straight lines AX, BY, CZ, and $\lambda\mu\nu$.

The point U is the pole of l with respect to this conic; for O is the pole of BC and hence OL is the polar of α; similarly O_1M is the polar of β, and O_2N the polar of γ; and OL, O_1M, O_2N intersect in U.

Moreover X is the pole of AP, and therefore XYZ is the polar of δ.

A more general case.

If (λ, μ, ν) be a point on $l\alpha + m\beta + n\gamma = 0$, and if we transform (λ, μ, ν) into a circum-conic, then the locus of the pole of another straight line $p\alpha + q\beta + r\gamma = 0$, with respect to this conic, is a conic (a nine-point conic).

The equation of this conic is

$$lp\alpha^2 + mq\beta^2 + nr\gamma^2 - (lq + mp)\alpha\beta - (mr + nq)\beta\gamma - (np + lr)\gamma\alpha = 0;$$

which is symmetrical with respect to (l, m, n) and (p, q, r).

This theorem may be proved analytically; and also geometrically as follows:—

Suppose (Fig. 9) that l' does not pass through δ. When l and l' are given lines, L, M, N, L′, M′, N′, are fixed points; if δ be a point on l (but not on l'), XYZ is its polar line, and the sides of LMN pass through O, O_1, O_2.

The point δ transforms into a conic through A, B, C, and S. The polar of α' with respect to this conic is OL′, the polar of β' is O_1M', and the polar of γ' is O_2N'. Hence the pole of l' is the intersection of OL′, O_1M', and O_2N' (which will not, in general, lie on XYZ).

Now O_1 lies on the fixed straight line LN,

$$\therefore\ M'O_1 \barwedge AO_1, \text{ and } N'O_2 \barwedge AO_2; \text{ but } AO_1 \barwedge AO_2, \ \therefore\ M'O_1 \barwedge N'O_2.$$

Hence the locus of the intersection of $M'O_1$ and $N'O_2$ is a conic passing through L′, M′, and N′.

When O_1 coincides with N, O also coincides with N, and O_2 is at γ; and in this case the point of the conic lies at N. Thus the conic passes through the points L, M, N; and the same conic is obtained whether we take the circum-conics corresponding to the points of l, and then the poles of l', or the circum-conics corresponding to the points of l' and then the poles of l.

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If now δ be the point of intersection of l and l' (Fig. 9), it is obvious that the conic passes through U and V.

If SA meet l' in T′, the conic will meet SA in the point which is harmonically conjugate to T′ with respect to A and S, and will meet BS and CS in the analogous points. The points where the conic meets S′A, S′B, and S′C may be determined with respect to the line l' in a similar manner.

This reasoning must be applied with care; for it might seem that the conic ought to meet SA in the two points that are harmonically conjugate to A and S with respect to T′ and T; or even that each of the points ought to be harmonically conjugate to both T and T′, with respect to S and A, which of course is impossible. The reason of the anomaly is that L, which is a point of the conic lying on SA, is given by the point γ; and for the point γ the circum-conic breaks up into two straight lines, for which case the relation between pole and polar ceases, in general, to exist.

The Contact Property.

Take any straight line (p, q, r) (Fig. 11) and its inscribed conic, touching at the points P, Q, and R; draw any straight line, (l, m, n), to meet this conic in the points S and S′. A conic may be drawn to touch the inscribed conic, to pass through S and S′, and to meet the sides in the points that correspond to the straight line (l, m, n), (not indicated in the figure).

We can draw a circum-conic to touch SS′ at S; let δ be the point that gives this conic, and (λ, μ, ν) the straight line, passing through δ, that gives the point S.

To construct the point δ. Let SS′ meet the sides of the triangle LMN in the points α, β, γ; then Aα, Bβ, Cγ meet the sides of ABC in collinear points X, Y, and Z; and δ is the polar point of the straight line XYZ, while (λ, μ, ν) is the polar line of S. (This construction is a consequence of the properties of Fig. 9.)

The point δ' and the straight line (λ', μ', ν') may be derived in a similar way from the point S′; and the straight line $\delta\delta'$ is the line which, taken along with (l, m, n), gives the conic required.

The common tangent is the polar line of the point of intersection of the straight lines (p, q, r) and (l, m, n).

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Analytical Proof of the Contact Property.

Let the coordinates of S be (a, b, c); of S′, (a', b', c').

We have then the following equations:—

SS′	$(bc'-b'c)\alpha+(ca'-c'a)\beta+(ab'-a'b)\gamma=0.$
MN	$-\alpha/a+\beta/b+\gamma/c=0.$
aA	$\beta/b^2(ca'-c'a)+\gamma/c^2(ab'-a'b)=0.$
XYZ	$\alpha/a^2(b'c-b'c)+\beta/b^2(ca'-c'a)+\gamma/c^2(ab'-a'b)=0.$

Thus δ is $\{a^2(bc'-b'c),\ b^2(ca'-c'a),\ c^2(ab'-ab')\}$;

and δ' is $\{a'^2(bc'-b'c),\ b'^2(ca'-c'a),\ c'^2(ab'-a'b)\}$.

We get for the equation of $\delta\delta'$

$$(bc'+b'c)\alpha+(ca'+c'a)\beta+(ab'+a'b)\gamma=0;$$

or $\quad P\alpha+Q\beta+R\gamma=0$, where $P/Q=(bc'+b'c)/(ca'+c'a)$

$$=(b/c+b'/c')/(a/c+a'/c').$$

We have now to take account of the fact that S and S′ are the intersections of $\qquad l\alpha+m\beta+n\gamma=0$, and

$$p^2\alpha^2+q^2\beta^2+r^2\gamma^2-2pq\alpha\beta-2qr\beta\gamma-2rp\gamma\alpha=0.$$

Eliminating α, we get

$$(pm+ql)^2\beta^2+2(p^2mn+pqnl+prlm-qrl^2)\beta\gamma+(pn+rl)^2\gamma^2=0.$$

whence $\quad \beta/\gamma+\beta'/\gamma'=-2(p^2mn+pqnl+prlm-qrl^2)/(pm+ql)^2$;

similarly, $\alpha/\gamma+\alpha'/\gamma'=-2(qpmn+q^2nl+qrlm-prm^2)/(pm+ql)^2$.

Writing $p=lp'$, $q=mq'$, $r=nr'$, we get

$$\frac{P/l}{Q/m}=\frac{p'^2+p'r'+p'q'-q'r'}{q'p'+q'^2+q'r'-p'r'};$$

which may be shown to be the required relation.

That the nine-point conic passes through S and S′ is obvious geometrically; for this conic may be obtained by taking the circum-conics corresponding to the points of $\delta\delta'$, and taking the poles of (l, m, n) with respect to these conics. Now the circum-conics corresponding to the points δ and δ' touch (l, m, n) at the points S and S′; and hence the poles of (l, m, n), for these conics, are the points S and S′.

It is obvious that the polar line of the point of intersection of (p, q, r) and (l, m, n) touches the inscribed conic; we may verify that the point of contact lies on the nine-point conic.

The coordinates of the point of contact are

$$\{p(qn-rm)^2,\ q(rl-pn)^2,\ r(pm-ql)^2\}.$$

If we put

$$A=pmn+qnl+rlm,\ \ P=pA-qrl^2,\ \ Q=qA-rpm^2, R=rA-pqn^2,$$

we may write the equation of the nine-point conic in the form

$$lP\alpha^2+mQ\beta^2+nR\gamma^2-(lQ+mP)\alpha\beta-(mR+nQ)\beta\gamma-(nP+lR)\gamma\alpha=0,$$

$$\text{(since } \delta\delta' \text{ is } P\alpha+Q\beta+R\gamma=0).$$

This equation may be shown to be identical with equation (A) above.

We have thus to show that $AU-V\equiv 0$ where

$$U\equiv lp\alpha^2+mq\beta^2+nr\gamma^2-(lq+mp)\alpha\beta-(mr+nq)\beta\gamma-(np+lr)\gamma\alpha;$$

$$V\equiv qrl^3\alpha^2+rpm^3\beta^2+pqn^3\gamma^2-pmn(qn+rm)\beta\gamma-qnl(rl+pn)\gamma\alpha$$
$$-rlm(pm+ql)\alpha\beta;$$

and (α, β, γ) is the point of contact.

Writing $l=\lambda p,\ m=\mu q,\ n=\nu r,\ p\alpha=x,\ q\beta=y,\ r\gamma=z,$
we have, neglecting factors which are powers of pqr, and substituting the values of x, y, z,

$$U\equiv\lambda(\mu-\nu)^4+\mu(\nu-\lambda)^4+\nu(\lambda-\mu)^4-(\lambda+\mu)(\nu-\mu)^2(\lambda-\nu)^2$$
$$-(\mu+\nu)(\lambda-\nu)^2(\mu-\lambda)^2-(\nu+\lambda)(\mu-\lambda)^2(\nu-\mu)^2;$$

$$V\equiv\lambda^3(\mu-\nu)^4+\mu^3(\nu-\lambda)^4+\nu^3(\lambda-\mu)^4-\lambda\mu(\lambda+\mu)(\nu-\mu)^2(\lambda-\nu)^2$$
$$-\mu\nu(\mu+\nu)(\lambda-\nu)^2(\mu-\lambda)^2-\nu\lambda(\nu+\lambda)(\mu-\lambda)^2(\nu-\mu)^2.$$

Both U and V may be shown to vanish identically; V is easily multiplied out if it be remarked that no letter occurs to a higher power than the fourth, and that, in every term, one letter at least must occur to as high a power as the third.

The conics $U=0$ and $V=0$, which pass through the point of contact, are nine-point conics; U is given by the straight lines (l, m, n) and (p, q, r), and V by the straight lines (l, m, n) and $qrl^2\alpha+rpm^2\beta+pqn^2\gamma=0$.

It is a consequence of one of the theorems given above that the conic $U=0$, must pass through the point of contact of the nine-point conic, that the common tangent must touch the inscribed conic given by (l, m, n), and that the conic $U=0$ must also pass through the point of contact of this inscribed conic.

The nine-point conic also passes through the points of contact of the common tangent to the inscribed conics given by the straight lines $\delta\delta'$ and (l, m, n).

In the case of the nine-point circle, (l, m, n) is the straight line at infinity, $\delta\delta'$ is $\alpha\cos A + \beta\cos B + \gamma\cos C = 0$, the polar line of the orthocentre, and (p, q, r) is $\alpha\cos^2\frac{1}{2}A + \beta\cos^2\frac{1}{2}B + \gamma\cos^2\frac{1}{2}C = 0$, the polar line of the point of concurrence of the straight lines joining the vertices to the points of contact of the inscribed circle.

It may be remarked further that from the identical equation

$$2[(lq+mp)\alpha\beta + (mr+nq)\beta\gamma + (np+lr)\gamma\alpha]$$
$$= (l\alpha + m\beta + n\gamma)(p\alpha + q\beta + r\gamma) - [lp\alpha^2 + mq\beta^2 + nr\gamma^2 - (lq+mp)\alpha\beta - \text{etc.}],$$

we see that a circum-conic may be drawn to pass through the intersections of any nine-point conic with the two straight lines that determine it.

In the case under consideration we thus get a circum-conic that passes through the points S and S′ (Fig. 11) and through the points where the nine-point conic meets $\delta\delta'$.

The point that gives this conic may be constructed by starting with the straight lines (l, m, n) and (p, q, r), and making the dual of the construction by means of which the straight line $\delta\delta'$ was derived from the points S and S′.

[This is a construction for the point $(lq+mp, mr+nq, np+lr)$ when the straight lines $l\alpha + m\beta + n\gamma = 0$ and $p\alpha + q\beta + r\gamma = 0$ are given.]

It may be proved analytically, and is obvious geometrically that this point is the intersection of (λ, μ, ν) and (λ', μ', ν').

In the case of the nine-point *circle* the theorem is that the radical axis of the circum-circle and the nine-point circle is the straight line $\alpha\cos A + \beta\cos B + \gamma\cos C = 0$, the polar line of the orthocentre.

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