Appendix C

Holographic renormalization, one-point functions, and a two-point function

Here we will illustrate the general prescription of Section 5.3.1 for computing Euclidean correlators. We will begin by giving a derivation of the general expression (5.47) for a one-point function to linear order in the external source, although we note that Eq. (5.47) is in fact valid at the nonlinear level [538], as follows from a generalization of the discussion that we shall present. Then, we will calculate the two-point function of a scalar operator $\mathcal{O}(x)$ in $\mathcal{N} = 4$ SYM at zero temperature. In so doing we will provide a derivation of the more general expression (5.55) for a two-point function and then evaluate it explicitly for this particular case. Although our main interest is in four-dimensional boundary theories, for the sake of generality we will present the formulas for a general dimension d.

Let Φ be the scalar field in AdS dual to $\mathcal{O}$. The Euclidean two-point function of $\mathcal{O}$ is then given by the right-hand side of Eq. (5.43) with $n = 2$. In order to evaluate this, we first need to solve the classical equation of motion for Φ subject to the boundary condition (5.29), and then evaluate the action on that solution. Since in order to obtain the two-point function we only need to take two functional derivatives of the action, it suffices to keep only the terms in the action that are quadratic in Φ, ignoring all interaction terms. At this level, the action is given by Eq. (5.20), except without the minus sign inside $\sqrt{-g}$, as appropriate for Euclidean signature:

$$S = -\frac{1}{2} \int dz\, d^d x \, \sqrt{g} \, \left[g^{MN} \partial_M \Phi \partial_N \Phi + m^2 \Phi^2 \right] + \cdots \, . \tag{C.1}$$

Note that we have adopted an overall sign convention for the Euclidean action appropriate for (5.41). The metric is that of pure Euclidean AdS and takes the form

$$ds^2 = \frac{R^2}{z^2} \left(dz^2 + \delta_{\mu\nu} dx^\mu dx^\nu \right) \, . \tag{C.2}$$

We will work in momentum space along the boundary directions. The equation of motion for $\Phi(z, k)$ then takes the form (5.22), which we reproduce here for convenience:

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$$z^{d+1}\partial_z\left(z^{1-d}\partial_z\phi\right) - k^2z^2\Phi - m^2R^2\Phi = 0\,, \tag{C.3}$$

with $k^2 = \delta_{\mu\nu}k^\mu k^\nu$ as appropriate in Euclidean signature.

Integration by parts shows that, when evaluated on a solution Φ_c, S reduces to the boundary term

$$S[\Phi_c] = -\frac{1}{2}\lim_{\epsilon\to 0}\int_{z=\epsilon}\frac{d^dk}{(2\pi)^d}\,\Pi_c(-k)\Phi_c(k)\,, \tag{C.4}$$

where Π_c is the canonical momentum associated with the z-foliation,

$$\Pi = -\sqrt{-g}\,g^{zz}\partial_z\Phi\,, \tag{C.5}$$

evaluated at the solution Φ_c. Since $z = 0$ is a regular singular point of Eq. (C.3), it is possible to choose a basis for $\Phi_{1,2}$ given by

$$\Phi_1 \to R^{\frac{1-d}{2}}z^{d-\Delta}, \qquad \Phi_2 \to R^{\frac{1-d}{2}}z^{\Delta}, \qquad \text{as } z\to 0\,, \tag{C.6}$$

with the corresponding canonical momenta $\Pi_{1,2}(z,k)$ behaving as

$$\Pi_1 \to -(d-\Delta)\,R^{\frac{d-1}{2}}z^{-\Delta}, \qquad \Pi_2 \to -\Delta R^{\frac{d-1}{2}}z^{-(d-\Delta)}, \quad \text{as } z\to 0\,, \tag{C.7}$$

where

$$\Delta = \frac{d}{2} + \nu, \qquad \nu = \sqrt{\frac{d^2}{4} + m^2R^2}\,. \tag{C.8}$$

Note that in (C.6) and (C.7) we only indicated the leading terms in a power series expansion in kz for each function. For example,

$$\Phi_1(z,k) = R^{\frac{1-d}{2}}z^{d-\Delta}\left(1 + a_2(kz)^2 + a_4(kz)^4 + \cdots\right) \tag{C.9}$$

for some constants $a_{2,4}$. Because all the terms in Eq. (C.3) are analytic in k^2, all the expansions are also analytic in k^2. This will be important in demonstrating that the counterterm action that we shall introduce below is local.

Then Φ_c and its canonical momentum can be expanded as

$$\begin{aligned}\Phi_c(z,k) &= A(k)\Phi_1(z,k) + B(k)\Phi_2(z,k)\,,\\ \Pi_c(z,k) &= A(k)\Pi_1(z,k) + B(k)\Pi_2(z,k)\,,\end{aligned} \tag{C.10}$$

as in (5.23), and the classical on-shell action becomes

$$\begin{aligned}S[\Phi_c] = -\frac{1}{2}\lim_{\epsilon\to 0}\int_{z=\epsilon}\frac{d^dk}{(2\pi)^d}\\ &[(A(-k)A(k)\Pi_1(-k)\Phi_1(k) + B(-k)B(k)\Pi_2(-k)\Phi_2(k)\\ &+ A(-k)B(k)(\Pi_1(-k)\Phi_2(k) + \Phi_1(-k)\Pi_2(k))]\,.\end{aligned} \tag{C.11}$$

Note that because $\nu > 0$, in the $\epsilon \to 0$ limit the first term on the right-hand side of (C.11) contains divergences and thus S requires renormalization. These divergences can be interpreted as dual to UV divergences of the boundary gauge theory. A local counterterm action S_{ct} defined on the cut-off surface $z = \epsilon$ can be introduced to cancel the divergences. From (C.11) we need to choose [1]

$$S_{ct} = \frac{1}{2} \int_{z=\epsilon} \frac{d^d k}{(2\pi)^d} \frac{\Pi_1(-k)}{\Phi_1(-k)} \Phi(-k)\Phi(k) . \tag{C.12}$$

Below we will show that this is a local action for sources in the boundary theory. The renormalized on-shell action is then given by

$$S^{(\text{ren})}[\Phi_c] \equiv S[\Phi_c] + S_{ct}[\Phi_c] = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} 2\nu\, A(-k)B(k) , \tag{C.13}$$

where we have dropped terms which vanish in the $\epsilon \to 0$ limit; the action is now finite.

We now impose the (Euclidean momentum space version of the) boundary condition (5.29) on Φ_c. We can use (C.10) or (C.9) to write this boundary condition as

$$\Phi_c(\epsilon, k) \to \Phi_1(\epsilon, k)\phi(k) \qquad \text{as } \epsilon \to 0 , \tag{C.14}$$

which from equation (C.10) gives

$$A(k) = \phi(k) + \text{terms that vanish as } \epsilon \to 0 . \tag{C.15}$$

Note that the boundary condition as written in (C.14) contains a factor $R^{\frac{1-d}{2}}$ (coming from the definition of Φ_1 in (C.6)) that is not written in (5.29). This factor ensures that $\phi(k)$ has the correct engineering dimension for a source coupled to an operator of dimension Δ. Consequently, there are no R factors in equations below.

We also need to impose the condition that Φ_c be regular everywhere in the interior. This extra condition then fixes the solution of (C.3) completely, which in turn determines the ratio $\chi \equiv B/A$ in terms of which $B = \chi\phi$. The renormalized action can now be written as

$$S^{(\text{ren})}[\Phi_c] = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} 2\nu\, \chi\phi(-k)\phi(k) . \tag{C.16}$$

It follows that the one-point function is given by

$$\langle \mathcal{O}(k) \rangle_\phi = \frac{\delta S^{(\text{ren})}[\Phi_c]}{\delta\phi(-k)} = 2\nu\chi\phi(k) = 2\nu B(k) , \tag{C.17}$$

which is the momentum space version of (5.47). The two-point function is then

$$G_E(k) = \frac{\langle \mathcal{O}(k) \rangle_\phi}{\phi(k)} = 2\nu \frac{B(k)}{A(k)} , \tag{C.18}$$

which is Eq. (5.55).

[1] We assume that 2ν is not an integer. If 2ν were an integer, then extra logarithmic terms would arise. See the discussion in Ref. [742].

Let us now verify explicitly that (C.12) is a local action in terms of the boundary source ϕ. Substituting (C.14) into (C.12) we find that

$$\begin{aligned} S_{ct} &= \frac{1}{2}\int_{z=\epsilon} \frac{d^d k}{(2\pi)^d}\, \Pi_1(-k)\Phi_1(-k)\,\phi(-k)\phi(k) \\ &= \frac{1}{2}\int_{z=\epsilon} \frac{d^d k}{(2\pi)^d}\, \epsilon^{d-2\Delta}\left((\Delta - d) + \cdots\right)\phi(-k)\phi(k), \end{aligned} \tag{C.19}$$

where the $\cdots$ in the second line denotes terms of $\mathcal{O}(\epsilon^2)$ and higher. As we noted below (C.9), all the k-dependence in Φ_1 and Π_1 is analytic in k^2, which implies that all the k-dependence in the terms represented by $\cdots$ is also analytic in k^2. So, after doing the Fourier transform to coordinate space, (C.19) is a local action for the boundary source ϕ.

Note that the entire discussion above only uses the form of Eq. (C.3) near $z = 0$ and thus applies to any geometry that is asymptotically AdS. In the case where the boundary theory is $\mathcal{N} = 4$ SYM theory at zero temperature, the bulk geometry is pure AdS and we can evaluate (C.18) explicitly. We begin by noting that, for pure AdS, Eq. (C.3) can in fact be solved exactly, with $\Phi_{1,2}$ given by

$$\Phi_1 = \Gamma(1-\nu)\left(\frac{k}{2}\right)^{\nu} R^{\frac{1-d}{2}} z^{\frac{d}{2}} I_{-\nu}(kz)\,, \quad \Phi_2 = \Gamma(1+\nu)\left(\frac{k}{2}\right)^{-\nu} R^{\frac{1-d}{2}} z^{\frac{d}{2}} I_{\nu}(kz)\,, \tag{C.20}$$

where $I(x)$ is the modified Bessel function of the first kind. Requiring Φ_c to be regular at $z \to \infty$ determines the solution up to an overall multiplicative constant:

$$\Phi_c \propto z^{\frac{d}{2}} K_\nu(kz)\,, \tag{C.21}$$

where $K_\nu(x)$ is the modified Bessel function of the second kind. From (C.10), (C.20) and (C.21) we then find that

$$\frac{B}{A} = \frac{\Gamma(-\nu)}{\Gamma(\nu)}\left(\frac{k}{2}\right)^{2\nu} \tag{C.22}$$

and thus

$$G_E(k) = 2\nu\frac{\Gamma(-\nu)}{\Gamma(\nu)}\left(\frac{k}{2}\right)^{2\nu}. \tag{C.23}$$