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Rigid local systems and motives of type G_2

Michael Dettweiler and Stefan Reiter

With an appendix by Michael Dettweiler and Nicholas M. Katz

Abstract

Using the middle convolution functor MC_χ introduced by N. Katz, we prove the existence of rigid local systems whose monodromy is dense in the simple algebraic group G_2. We derive the existence of motives for motivated cycles which have a motivic Galois group of type G_2. Granting Grothendieck's standard conjectures, the existence of motives with motivic Galois group of type G_2 can be deduced, giving a partial answer to a question of Serre.

Introduction

The method of rigidity was first used by Riemann [Rie57] in his study of Gauß's hypergeometric differential equations ${}_2F_1 = {}_2F_1(a, b, c)$. Consider the monodromy representation

$$\rho : \pi_1^{\mathrm{top}}(\mathbb{P}^1 \setminus \{0, 1, \infty\}, s) \to \mathrm{GL}(V_s)$$

that arises from analytic continuation of the vector space $V_s \simeq \mathbb{C}^2$ of local solutions of ${}_2F_1$ at s, along paths in $\mathbb{P}^1\setminus\{0, 1, \infty\}$ which are based at s. Let γ_i, $i \in \{0, 1, \infty\}$, be simple loops around the points $0, 1, \infty$ (respectively) which are based at s. Then the monodromy representation ρ is *rigid* in the sense that it is determined up to isomorphism by the Jordan canonical forms of $\rho(\gamma_i)$, for $i = 0, 1, \infty$.

One can translate the notion of rigidity into the language of local systems by saying that the local system $\mathcal{L}({}_2F_1)$ on $\mathbb{P}^1\setminus\{0, 1, \infty\}$ which is given by the holomorphic solutions of ${}_2F_1$ is *rigid* in the following sense: the monodromy representation of $\mathcal{L}({}_2F_1)$ (as defined in [Del70]) is determined up to isomorphism by the local monodromy representations at the missing points. This definition of rigidity extends in the obvious way to other local systems. Since Riemann's work, the concept of a rigid local system has proven to be very fruitful and has appeared in many different branches of mathematics and physics (see, e.g., [BH89, Inc56]).

A key observation turned out to be the following: the local sections of the rank-two local system $\mathcal{L}({}_2F_1)$ can be written as linear combinations of convolutions $f * g$, where f and g are solutions of two related Fuchsian systems of *rank one* (see [Kat96, Introduction]). By interpreting the convolution as higher direct image and using a transition to étale sheaves, Katz proved in [Kat96] a vast generalization of the above observation: let $\mathcal{F}$ be any irreducible étale rigid local system on the punctured affine line in the sense specified below; then $\mathcal{F}$ can be transformed to a rank-one sheaf by a suitable iterative application of *middle convolutions* MC_χ and tensor products with rank-one objects to it (see [Kat96, ch. 5]). (The definition of the middle convolution MC_χ and its main properties are recalled in § 1.) This yields the *Katz existence algorithm* for

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irreducible rigid local systems, which tests whether a given set of local representations comes from an irreducible and rigid local system (see [Kat96, ch. 6]). This algorithm works simultaneously in the tame étale case and in the classical case of rigid local systems mentioned above [Kat96, §§ 6.2 and 6.3].

Let $\mathcal{F}$ be a lisse constructible $\bar{\mathbb{Q}}_\ell$-sheaf on a non-empty Zariski open subset $j : U \to \mathbb{P}^1_k$ which is tamely ramified at the missing points $\mathbb{P}^1_k \backslash U$ (cf. [Gro77]). We say that $\mathcal{F}$ is *rigid* if the monodromy representation

$$\rho_{\mathcal{F}} : \pi_1^{\text{tame}}(U, \bar{\eta}) \longrightarrow \mathrm{GL}(\mathcal{F}_{\bar{\eta}})$$

of $\mathcal{F}$ is determined up to isomorphism by the conjugacy classes of the induced representations of tame inertia groups I_s^{tame}, where $s \in D := \mathbb{P}^1 \backslash U$. Sometimes we will call such a sheaf an *étale rigid local system*. If $\mathcal{F}$ is irreducible, then $\mathcal{F}$ is rigid if and only if the following formula holds:

$$\chi(\mathbb{P}^1, j_* \underline{\mathrm{End}}(\mathcal{F})) = (2 - \mathrm{Card}(D)) \mathrm{rk}(\mathcal{F})^2 + \sum_{s \in D} \dim(\mathrm{Centralizer}_{\mathrm{GL}(\mathcal{F}_{\bar{\eta}})}(I_s^{\text{tame}})) = 2;$$

see [Kat96, chs 2 and 6].

In preparation for Theorem 1 below, recall that there exist only finitely many exceptional simple linear algebraic groups over an algebraically closed field which are not isomorphic to a classical group; see [Bor91]. The smallest of these is the group G_2, which admits an embedding into the group GL_7. Let us also fix some notation: let $\mathbf{1}$, $-\mathbf{1}$ and $\mathbf{U}(n)$ denote, respectively, the trivial $\bar{\mathbb{Q}}_\ell$-valued representation, the unique quadratic $\bar{\mathbb{Q}}_\ell$-valued character and the standard indecomposable unipotent $\bar{\mathbb{Q}}_\ell$-valued representation of degree n of the tame fundamental group $\pi_1^{\text{tame}}(\mathbb{G}_{m,k})$, where k is an algebraically closed field of characteristic not equal to 2 or ℓ. The group $\pi_1^{\text{tame}}(\mathbb{G}_{m,k})$ is isomorphic to the tame inertia group I_s^{tame}. This can be used to view representations of I_s^{tame} as representations of $\pi_1^{\text{tame}}(\mathbb{G}_{m,k})$. We prove the following result.

Theorem 1. *Let ℓ be a prime number and let k be an algebraically closed field of characteristic not equal to 2 or ℓ. Let $\varphi, \eta : \pi_1^{\text{tame}}(\mathbb{G}_{m,k}) \to \bar{\mathbb{Q}}_\ell^\times$ be continuous characters such that*

$$\varphi, \eta, \varphi\eta, \varphi\eta^2, \eta\varphi^2, \varphi\overline{\eta} \neq -\mathbf{1}.$$

Then there exists an étale rigid local system $\mathcal{H}(\varphi, \eta)$ of rank 7 on $\mathbb{P}^1_k \backslash \{0, 1, \infty\}$ whose monodromy group is Zariski dense in $G_2(\bar{\mathbb{Q}}_\ell)$ and whose local monodromy is as follows.

- *The local monodromy at 0 is of type*

$$-\mathbf{1} \oplus -\mathbf{1} \oplus -\mathbf{1} \oplus -\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}.$$

- *The local monodromy at 1 is of type*

$$\mathbf{U}(2) \oplus \mathbf{U}(2) \oplus \mathbf{U}(3).$$

- *The local monodromy at ∞ is of the form summarized in the following table.*

Local monodromy at ∞	*Conditions on φ and η*
$\mathbf{U}(7)$	$\varphi = \eta = \mathbf{1}$
$\mathbf{U}(3, \varphi) \oplus \mathbf{U}(3, \overline{\varphi}) \oplus \mathbf{1}$	$\varphi = \eta \neq \mathbf{1}, \varphi^3 = \mathbf{1}$
$\mathbf{U}(2, \varphi) \oplus \mathbf{U}(2, \overline{\varphi}) \oplus \mathbf{U}(1, \varphi^2) \oplus \mathbf{U}(1, \overline{\varphi}^2) \oplus \mathbf{1}$	$\varphi = \eta, \varphi^4 \neq \mathbf{1} \neq \varphi^6$
$\mathbf{U}(2, \varphi) \oplus \mathbf{U}(2, \overline{\varphi}) \oplus \mathbf{U}(3)$	$\varphi = \overline{\eta}, \varphi^4 \neq \mathbf{1}$
$\varphi \oplus \eta \oplus \varphi\eta \oplus \overline{\varphi\eta} \oplus \overline{\eta} \oplus \overline{\varphi} \oplus \mathbf{1}$	$\varphi, \eta, \varphi\eta, \overline{\varphi\eta}, \overline{\eta}, \overline{\varphi}, \mathbf{1}$ *pairwise distinct*

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Theorem 1 is proved in a slightly more general form as Theorem 1.3.1 below, where it is also proved that these are the only étale rigid local systems of rank 7 whose monodromy is dense in G_2. The proof of Theorem 1.3.1 relies heavily on Katz's existence algorithm. Using the canonical homomorphism

$$\pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}) \to \pi_1^{\text{ét}}(\mathbb{P}^1_{\mathbb{C}}\backslash\{0,1,\infty\}),$$

the existence of a rigid local system in the classical sense (corresponding to a representation of the topological fundamental group $\pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\})$) whose monodromy group is Zariski dense in G_2 can easily be derived.

Suppose that one is given several local representations

$$I_s^{\mathrm{tame}} \longrightarrow \mathrm{GL}(V) \quad \text{with } s \in D \cup \{\infty\}$$

which are assumed to come from an irreducible rigid local system $\mathcal{F}$ on $\mathbb{P}^1\backslash D\cup\{\infty\}$. It has been observed empirically that the rigidity condition $\chi(\mathbb{P}^1, j_*\underline{\mathrm{End}}(\mathcal{F})) = 2$ and the (necessary) *irreducibility condition*

$$\chi(\mathbb{P}^1, j_*\mathcal{F}) = (1 - \mathrm{Card}(D))\mathrm{rk}(\mathcal{F}) + \sum_{s\in D\cup\{\infty\}} \dim(\mathcal{F}_s^{I_s^{\mathrm{tame}}}) \leqslant 0 \qquad (0.0.1)$$

contradict each other in many cases; this occurs especially often when the Zariski closure of the monodromy group of $\mathcal{F}$ is supposed to be small in the underlying general linear group. It is thus astonishing that the aforementioned irreducible and rigid G_2-sheaves exist at all. In fact, the local systems given by Theorems 1 and 1.3.1 are the first, and perhaps the only, examples of tamely ramified rigid sheaves such that the Zariski closure of the monodromy group is an exceptional simple algebraic group.

We remark that in positive characteristic, wildly ramified lisse sheaves on $\mathbb{G}_m$ with G_2-monodromy were previously found by Katz (see [Kat88, Kat90]). Also, the conjugacy classes in $G_2(\mathbb{F}_\ell)$ which correspond to the local monodromy of the above rigid local system $\mathcal{H}(\mathbf{1},\mathbf{1})$ have already come up in the work of Feit, Fong and Thompson on the inverse Galois problem (see [FF85, Tho85]); however, only the situation in $G_2(\mathbb{F}_\ell)$ was considered, and the transition to rigid local systems was not made.

We apply the above results to give a partial answer to a question posed by Serre on the existence of motives with exceptional motivic Galois groups. Recall that a *motive* in the Grothendieck sense is a triple $M = (X, P, n)$, $n \in \mathbb{Z}$, where X is a smooth projective variety over a field K and P is an idempotent correspondence; see, e.g., [Saa72]. Motives appear in many branches of mathematics (see [JKS94]) and play a central role in the Langlands program [Lan79]. Granting Grothendieck's standard conjectures, the category of Grothendieck motives has the structure of a Tannakian category. Thus, by the Tannakian formalism, every Grothendieck motive M has conjecturally an algebraic group attached to it, called the *motivic Galois group* of M (see [Del90, Saa72]).

An unconditional theory of *motives for motivated cycles* was developed by André [And96], who formally adjoins a certain homological cycle (the Lefschetz involution) to the algebraic cycles in order to obtain the Tannakian category of motives for motivated cycles. Let us also mention the Tannakian category of *motives for absolute Hodge cycles*, introduced by Deligne [DMOS82] (for a definition of an absolute Hodge cycle, take a homological cycle that satisfies the most visible properties of an algebraic cycle). In both categories, one has the notion of a motivic Galois group, given by the Tannakian formalism. It can be shown that any motivated cycle is an absolute Hodge cycle, so every motive for motivated cycles is also a motive for absolute Hodge cycles; see [And96]. Since the category of motives for motivated cycles is the minimal extension of Grothendieck's category that is unconditionally Tannakian, we will work and state our results mainly in this category.

The motivic Galois group is expected to encode essential properties of a motive. Many open conjectures on motivic Galois groups and related Galois representations are considered in the article [Ser94]. Under the general assumption of Grothendieck's standard conjectures, Serre [Ser94, 8.8] asks the following question 'plus hasardeuse': *do there exist motives whose motivic Galois group is an exceptional simple algebraic group of type G_2 (or E_8)?* It follows from Deligne's work on Shimura varieties that such motives cannot be submotives of abelian varieties or the motives parametrized by Shimura varieties; see [Del79]. Thus, motives with motivic Galois group of type G_2 or E_8 are presumably hard to construct.

There is the notion of a family of motives for motivated cycles; see [And96] and § 3.2. Using this notion, we prove the following result (see also Theorem 3.3.1).

THEOREM 2. *There is a family of motives M_s, parametrized by $S = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, such that for any $s \in S(\mathbb{Q})$ outside a thin set, the motive M_s has a motivic Galois group of type G_2.*

Since the complement of a thin subset of $\mathbb{Q}$ is infinite (see [Ser89]), Theorem 2 implies the existence of infinitely many motives for motivated cycles whose motivic Galois group is of type G_2. A proof of Theorem 2 will be given in § 3. It can be shown that under the assumption of the standard conjectures, the motives M_s are Grothendieck motives with motivic Galois group of type G_2 (see Remark 3.3.2). In this sense, we obtain a positive answer to Serre's question in the G_2 case.

The method of construction of the motives M_s is based on the motivic interpretation of rigid local systems with quasi-unipotent local monodromy, introduced by Katz in [Kat96, ch. 8]. It follows from Katz's work that the sheaf $\mathcal{H}(\mathbf{1}, \mathbf{1})$ in Theorem 1 comes from the cohomology of a smooth affine morphism $\pi : \mathrm{Hyp} \to \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ which arises during the convolution process (see Theorem 2.4.1 and Corollary 2.4.2). Then a desingularization of the relative projective closure of Hyp and the work of André [And96] on families of motives imply that a suitable compactification and specialization of π gives motives over $\mathbb{Q}$ whose motivic Galois groups are of type G_2.

In the appendix to this paper, written jointly with Katz, the Galois representations associated with the above motives M_s are studied. It follows from Theorem A.1 of the appendix that for two coprime integers a and b which each have at least one odd prime divisor, the motive M_s, with $s = 1 + a/b$, gives rise to ℓ-adic Galois representations whose image is Zariski dense in the group G_2. This implies that the motivic Galois group of M_s is of type G_2. By letting a and b vary among the squarefree coprime odd integers greater than 2, one obtains infinitely many non-isomorphic motives $M_{1+a/b}$ with motivic Galois group of type G_2; see Corollary A.2(ii) in the appendix.

We remark that Gross and Savin [GS98] proposed a completely different way of constructing motives with motivic Galois group G_2, which involves looking at the cohomology of Shimura varieties of type G_2 with non-trivial coefficients. The connection between these approaches has yet to be explored. Owing to an observation of Serre, at least the underlying Hodge types coincide.

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1. Middle convolution and G_2-local systems

Throughout this section we fix an algebraically closed field k and a prime number $\ell \neq \mathrm{char}(k)$.

1.1 The middle convolution

Let G be an algebraic group over k and let $\pi : G \times G \to G$ be the multiplication map. Let $D^b_c(G, \bar{\mathbb{Q}}_\ell)$ denote the bounded derived category of constructible $\bar{\mathbb{Q}}_\ell$-sheaves on G (cf. [Del80, § 1] and [Kat96, § 2.2]). Given two objects $K, L \in D^b_c(G, \bar{\mathbb{Q}}_\ell)$, define their !-*convolution* as

$$K *_! L := R\pi_!(K \boxtimes L) \in D^b_c(G, \bar{\mathbb{Q}}_\ell)$$

and their $*$-*convolution* as

$$K *_* L := R\pi_*(K \boxtimes L) \in D^b_c(G, \bar{\mathbb{Q}}_\ell).$$

An element $K \in D^b_c(G, \bar{\mathbb{Q}}_\ell)$ is called a *perverse sheaf* (cf. [BBD82]) if K and its dual $D(K)$ satisfy, respectively,

$$\dim(\mathrm{Supp}(H^i(K))) \leqslant -i \quad \text{and} \quad \dim(\mathrm{Supp}(H^i(D(K)))) \leqslant -i.$$

Suppose that K is a perverse sheaf with the property that for any other perverse sheaf L on G, the sheaves $K *_! L$ and $K *_* L$ are again perverse. Then one can define the *middle convolution* $K *_{\mathrm{mid}} L$ of K and L as the image of $L *_! K$ in $L *_* K$ under the 'forget supports map' in the abelian category of perverse sheaves.

Let us now consider the situation where $G = \mathbb{A}^1_k$. For any non-trivial continuous character

$$\chi : \pi_1^{\mathrm{tame}}(\mathbb{G}_{m,k}) \longrightarrow \bar{\mathbb{Q}}_\ell^\times,$$

let $\mathcal{L}_\chi$ denote the corresponding lisse sheaf of rank one on $\mathbb{G}_{m,k}$. Let $j : \mathbb{G}_m \to \mathbb{A}^1$ denote the inclusion. From $j_*\mathcal{L}_\chi$ one obtains a perverse sheaf $j_*\mathcal{L}_\chi[1]$ on $\mathbb{A}^1$ by placing the sheaf in degree -1. Since !-convolution (and $*$-convolution) with $j_*\mathcal{L}_\chi[1]$ preserves perversity (see [Kat96, ch. 2]), the middle convolution $K *_{\mathrm{mid}} j_*\mathcal{L}_\chi[1]$ is defined for any perverse sheaf K on $\mathbb{A}^1$.

The following notation will be used: for any scheme W and any map $f : W \to \mathbb{G}_m$, define

$$\mathcal{L}_{\chi(f)} := f^*\mathcal{L}_\chi. \tag{1.1.1}$$

The identity character will be denoted by $\mathbf{1}$, and $-\mathbf{1}$ will denote the unique quadratic character of $\pi_1^{\mathrm{tame}}(\mathbb{G}_m)$. The *inverse character* of χ will be denoted by $\overline{\chi}$ (by definition, $\chi \otimes \overline{\chi} = \mathbf{1}$). The following category will be of importance.

Definition 1.1.1. Let $\mathcal{T}_\ell = \mathcal{T}_\ell(k)$ denote the full subcategory of constructible $\bar{\mathbb{Q}}_\ell$-sheaves $\mathcal{F}$ on $\mathbb{A}^1_k$ which satisfy the following conditions.

- There exists a dense open subset $j : U \to \mathbb{A}^1$ such that $j^*\mathcal{F}$ is lisse and irreducible on U and $\mathcal{F} \simeq j_*j^*\mathcal{F}$.
- The lisse sheaf $j^*\mathcal{F}$ is tamely ramified at every point of $\mathbb{P}^1 \backslash U$.
- There are at least two distinct points of $\mathbb{A}^1$ at which $\mathcal{F}$ fails to be lisse.

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The properties of $\mathcal{T}_\ell$ imply that $\mathcal{F}[1] *_{\mathrm{mid}} \mathcal{L}_\chi[1]$ is a single sheaf placed in degree -1 (see [Kat96, ch. 5]), leading to the *middle convolution functor*

$$\mathrm{MC}_\chi : \mathcal{T}_\ell \to \mathcal{T}_\ell, \quad \mathcal{F} \mapsto (\mathcal{F}[1] *_{\mathrm{mid}} \mathcal{L}_\chi[1])[-1];$$

see [Kat96, 5.1.5]. (Note that, by the definition of $\mathcal{T}_\ell$, the sheaf $\mathrm{MC}_\chi(\mathcal{F})$ is again irreducible; cf. [Kat96, Theorem 3.3.3].)

An important property of MC_χ is that

$$\mathrm{MC}_\chi \circ \mathrm{MC}_\rho = \mathrm{MC}_{\chi\rho} \quad \text{if } \chi\rho \neq \mathbf{1}, \quad \mathrm{MC}_\chi \circ \mathrm{MC}_{\overline{\chi}} = \mathrm{Id}. \tag{1.1.2}$$

Let $U \subseteq \mathbb{A}^1$ be an open subset of $\mathbb{A}^1$ such that $\mathcal{F}|_U$ is lisse, and let $\iota : U \to \mathbb{P}^1$ be the canonical inclusion. The sheaf $\mathcal{F} \in \mathcal{T}_\ell$ is called *cohomologically rigid* if the *index of rigidity*

$$\mathrm{rig}(\mathcal{F}) = \chi(\mathbb{P}^1, \iota_*(\underline{\mathrm{End}}(\mathcal{F}|_U)))$$

is equal to 2. Then MC_χ carries rigid elements in $\mathcal{T}_\ell$ to rigid elements in $\mathcal{T}_\ell$ by virtue of the following relation (see [Kat96, 6.0.17]):

$$\mathrm{rig}(\mathcal{F}) = \mathrm{rig}(\mathrm{MC}_\chi(\mathcal{F})). \tag{1.1.3}$$

1.2 The numerology of the middle convolution

We recall the effect of the middle convolution on the Jordan canonical forms of the local monodromy, given by Katz in [Kat96, ch. 6]. Let $\mathcal{F} \in \mathcal{T}_\ell$ and let $j : U \mapsto \mathbb{A}^1_x$ denote an open subset such that $j^*\mathcal{F}$ is lisse. Let $D := \mathbb{A}^1 \backslash U$. Then, for any point $s \in D \cup \{\infty\} = \mathbb{P}^1 \backslash U$, the sheaf $\mathcal{F}$ gives rise to the *local monodromy representation* $\mathcal{F}(s)$ of the tame inertia subgroup $I(s)^{\mathrm{tame}}$ (of the absolute Galois group of the generic point of $\mathbb{A}^1$) at s. The representation $\mathcal{F}(s)$ decomposes as a direct sum of (character) $\otimes$ (unipotent representation), where the sum is over the set of continuous $\bar{\mathbb{Q}}_\ell$-characters ρ of $\pi_1^{\mathrm{tame}}(\mathbb{G}_{m,k}) \simeq I(s)^{\mathrm{tame}}$:

$$\mathcal{F}(s) = \bigoplus_\rho \mathcal{L}_{\rho(x-s)} \otimes \mathrm{Unip}(s, \rho, \mathcal{F}) \quad \text{for all } s \in D$$

and

$$\mathcal{F}(\infty) = \bigoplus_\rho \mathcal{L}_{\rho(x)} \otimes \mathrm{Unip}(\infty, \rho, \mathcal{F}).$$

Here, the following convention is used: if one starts with a rank-one object $\mathcal{F}$, which at $s \in D$ gives rise locally to a character χ_s of $\pi_1^{\mathrm{tame}}(\mathbb{G}_m)$, then

$$\chi_\infty = \prod_{s \in D} \chi_s.$$

For $s \in D \cup \{\infty\}$, write $\mathrm{Unip}(s, \rho, \mathcal{F})$ as a direct sum of Jordan blocks of lengths $\{n_i(s, \rho, \mathcal{F})\}_i$. This leads to a decreasing sequence of non-negative integers

$$e_1(s, \rho, \mathcal{F}) \geqslant e_2(s, \rho, \mathcal{F}) \geqslant \cdots \geqslant e_k(s, \rho, \mathcal{F}) = 0 \quad \text{for } k \gg 0,$$

where the integer $e_j(s, \rho, \mathcal{F})$ is defined to be the number of Jordan blocks in $\mathrm{Unip}(s, \rho, \mathcal{F})$ whose length is at least j.

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Proposition 1.2.1. *Let $\mathcal{F} \in \mathcal{T}_\ell$ be of generic rank n. Then the following hold.*

(i)
$$\operatorname{rk}(\mathrm{MC}_\chi(\mathcal{F})) = \sum_{s\in D} \operatorname{rk}(\mathcal{F}(s)/(\mathcal{F}(s)^{I(s)})) - \operatorname{rk}((\mathcal{F}(\infty)\otimes\mathcal{L}_\chi)^{I(\infty)})$$
$$= \sum_{s\in D}(n - e_1(s,\mathbf{1},\mathcal{F})) - e_1(\infty,\overline{\chi},\mathcal{F}).$$

(ii) *For $s \in D$ and $i \geqslant 1$,*

$$e_i(s,\rho\chi,\mathrm{MC}_\chi(\mathcal{F})) = e_i(s,\rho,\mathcal{F}) \quad \text{if } \rho\neq\mathbf{1} \text{ and } \rho\chi\neq\mathbf{1},$$
$$e_{i+1}(s,\mathbf{1},\mathrm{MC}_\chi(\mathcal{F})) = e_i(s,\bar{\chi},\mathcal{F}),$$
$$e_i(s,\chi,\mathrm{MC}_\chi(\mathcal{F})) = e_{i+1}(s,\mathbf{1},\mathcal{F}).$$

Moreover,

$$e_1(s,\mathbf{1},\mathrm{MC}_\chi(\mathcal{F})) = \operatorname{rk}(\mathrm{MC}_\chi(\mathcal{F})) - n + e_1(s,\mathbf{1},\mathcal{F}).$$

(iii) *For $s=\infty$ and $i \geqslant 1$,*

$$e_i(\infty,\rho\chi,\mathrm{MC}_\chi(\mathcal{F})) = e_i(\infty,\rho,\mathcal{F}) \quad \text{if } \rho\neq\mathbf{1} \text{ and } \rho\chi\neq\mathbf{1},$$
$$e_{i+1}(\infty,\chi,\mathrm{MC}_\chi(\mathcal{F})) = e_i(\infty,\mathbf{1},\mathcal{F}),$$
$$e_i(\infty,\mathbf{1},\mathrm{MC}_\chi(\mathcal{F})) = e_{i+1}(\infty,\overline{\chi},\mathcal{F}).$$

Moreover,

$$e_1(\infty,\chi,\mathrm{MC}_\chi(\mathcal{F})) = \sum_{s\in D}(\operatorname{rk}(\mathcal{F}) - e_1(s,\mathbf{1},\mathcal{F})) - \operatorname{rk}(\mathcal{F}).$$

Proof. Assertion (i) is [Kat96, Corollary 3.3.7]. The first three equalities in (ii) are [Kat96, 6.0.13], and the last equality follows from [Kat96, 6.0.14]. To deduce (iii), we argue as follows. From [Kat96, 3.3.6 and 6.0.5], for any $\mathcal{F}\in\mathcal{T}_\ell$ there exists an $I(\infty)^{\mathrm{tame}}$-representation $M(\infty,\mathcal{F})$ of rank $\sum_{s\in D}(n-e_1(s,\mathbf{1},\mathcal{F}))$ that has the following properties:

$$E_i(\infty,\rho,\mathcal{F}) = e_i(\infty,\rho,\mathcal{F}) \quad \text{if } \rho\neq\mathbf{1},$$
$$E_{i+1}(\infty,\mathbf{1},\mathcal{F}) = e_i(\infty,\mathbf{1},\mathcal{F}) \quad \text{for } i\geqslant 1,$$
$$E_1(\infty,\mathbf{1},\mathcal{F}) = \operatorname{rk}(M(\infty,\mathcal{F})) - \operatorname{rk}(\mathcal{F}),$$

where the numbers $E_i(\infty,\rho,\mathcal{F})$ denote the invariants associated to $M(\infty,\mathcal{F})$, defined in an analogous way to the invariants $e_i(s,\rho,\mathcal{F})$ for $\mathcal{F}(s)$. Moreover, by [Kat96, 6.0.11], we have that

$$E_i(\infty,\rho\chi,\mathrm{MC}_\chi(\mathcal{F})) = E_i(\infty,\rho,\mathcal{F}) \quad \text{for all } i\geqslant 1 \text{ and } \rho.$$

By combining the preceding equations, it follows that if $\rho\chi\neq\mathbf{1}$ and $\rho\neq\mathbf{1}$, then

$$e_i(\infty,\rho\chi,\mathrm{MC}_\chi(\mathcal{F})) = E_i(\infty,\rho\chi,\mathrm{MC}_\chi(\mathcal{F})) = E_i(\infty,\rho,\mathcal{F}) = e_i(\infty,\rho,\mathcal{F}).$$

If $\rho=\mathbf{1}$, then since χ is non-trivial, the following holds:

$$e_{i+1}(\infty,\chi,\mathrm{MC}_\chi(\mathcal{F})) = E_{i+1}(\infty,\chi,\mathrm{MC}_\chi(\mathcal{F})) = E_{i+1}(\infty,\mathbf{1},\mathcal{F}) = e_i(\infty,\rho,\mathcal{F}).$$

Moreover,

$$e_i(\infty,\mathbf{1},\mathrm{MC}_\chi(\mathcal{F})) = E_{i+1}(\infty,\mathbf{1},\mathrm{MC}_\chi(\mathcal{F})) = E_{i+1}(\infty,\overline{\chi},\mathcal{F}) = e_{i+1}(\infty,\overline{\chi},\mathcal{F}),$$

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since χ and hence $\overline{\chi}$ are non-trivial. Finally,

$$\begin{aligned}e_1(\infty,\chi,\mathrm{MC}_\chi(\mathcal{F})) &= E_1(\infty,\chi,\mathrm{MC}_\chi(\mathcal{F}))\\ &= E_1(\infty,\mathbf{1},\mathcal{F})\\ &= \mathrm{rk}(M(\infty,\mathcal{F}))-\mathrm{rk}(\mathcal{F})\\ &= \sum_{s\in D}(\mathrm{rk}(\mathcal{F})-e_1(s,\mathbf{1},\mathcal{F}))-\mathrm{rk}(\mathcal{F}),\end{aligned}$$

where the last equality follows from [Kat96, 6.0.6]. □

Let $\mathcal{F}\in\mathcal{T}_\ell$, and let $\mathcal{L}$ be a middle extension sheaf on $\mathbb{A}^1$ (i.e. there exists an open subset $j:U\to\mathbb{A}^1$ such that $j^*\mathcal{L}$ is lisse and $\mathcal{L}\simeq j_*j^*\mathcal{L}$). Assume that $\mathcal{F}|_U$ is also lisse. Then the *middle tensor product* of $\mathcal{F}$ and $\mathcal{L}$ is defined as

$$\mathrm{MT}_\mathcal{L}(\mathcal{F})=j_*(\mathcal{F}|_U\otimes\mathcal{L}|_U);$$

cf. [Kat96, 5.1.9]. Obviously, the generic rank of $\mathrm{MT}_\mathcal{L}(\mathcal{F})$ is the same as the generic rank of $\mathcal{F}$. For any $s\in D\cup\{\infty\}$, denote by $\chi_{s,\mathcal{L}}$ the unique character ρ with $e_1(s,\rho,\mathcal{L})=1$. Then the following holds (see [Kat96, 6.0.10]):

$$e_i(s,\rho\chi_{s,\mathcal{L}},\mathrm{MT}_\mathcal{L}(\mathcal{F}))=e_i(s,\rho,\mathcal{F}). \tag{1.2.1}$$

1.3 Classification of irreducible rigid local systems with G_2-monodromy

In this section, we give a complete classification of rank-7 rigid sheaves $\mathcal{H}\in\mathcal{T}_\ell$ whose associated monodromy group is Zariski dense in the exceptional simple algebraic group $G_2(\bar{\mathbb{Q}}_\ell)$ whose minimal representation has dimension 7 (we refer to [Bor91] for basic results on the group G_2).

Let us first collect some information on the conjugacy classes of the simple algebraic group G_2 which will be needed in what follows. In Table 1 (on page 937), we list the possible Jordan canonical forms of elements of the group $G_2(\bar{\mathbb{Q}}_\ell)\leqslant\mathrm{GL}_7(\bar{\mathbb{Q}}_\ell)$, together with the dimensions of the centralizers in $G_2(\bar{\mathbb{Q}}_\ell)$ and in $\mathrm{GL}_7(\bar{\mathbb{Q}}_\ell)$.

We use the following conventions: $E_n\in\bar{\mathbb{Q}}_\ell^{n\times n}$ denotes the identity matrix, $\mathbf{J}(n)$ denotes a unipotent Jordan block of length n, $\omega\in\bar{\mathbb{Q}}_\ell^\times$ denotes a primitive third root of unity, and $i\in\bar{\mathbb{Q}}_\ell^\times$ denotes a primitive fourth root of unity. Moreover, an expression of the form $(x\mathbf{J}(2),x^{-1}\mathbf{J}(2),x^2,x^{-2},1)$ denotes a matrix in Jordan canonical form in $\mathrm{GL}_7(\bar{\mathbb{Q}}_\ell)$ with one Jordan block of length 2 having eigenvalue x, one Jordan block of length 2 having eigenvalue x^{-1}, and three Jordan blocks of length 1 having eigenvalues x^2,x^{-2} and 1.

Table 1 can be derived as follows. By Jordan decomposition, any element $g\in G_2(\bar{\mathbb{Q}}_\ell)$ can be written as $g=su$ where $s\in G_2(\bar{\mathbb{Q}}_\ell)$ is a semi-simple element and u is unipotent. By conjugating inside $\mathrm{GL}_7(\bar{\mathbb{Q}}_\ell)$, we may assume that s is in the maximal G_2-torus

$$T=\{\mathrm{diag}(x,y,xy,1,(xy)^{-1},y^{-1},x^{-1})\mid x,y,\in\bar{\mathbb{Q}}_\ell^\times\}.$$

Since the group $G_2(\bar{\mathbb{Q}}_\ell)$ is simply connected, the centralizer $C(s)$ is a connected reductive group of Lie rank two (see [Car93, Theorems 3.5.4 and 3.5.6]). The possibilities for the type of the centralizer $C(s)$ and its Weyl group

$$W(C(s))\subseteq W(G_2)=D_6=\langle(1,6,5,7,2,3),(1,2)(6,7)\rangle,$$

Table 1. The GL_7 conjugacy classes of G_2.

Jordan form	Centralizer dimension in G_2	Centralizer dimension in GL_7	Conditions
E_7	14	49	
$(\mathbf{J}(2), \mathbf{J}(2), E_3)$	8	29	
$(\mathbf{J}(3), \mathbf{J}(2), \mathbf{J}(2))$	6	19	
$(\mathbf{J}(3), \mathbf{J}(3), \mathbf{1})$	4	17	
$\mathbf{J}(7)$	2	7	
$(-E_4, E_3)$	6	25	
$(-\mathbf{J}(2), -\mathbf{J}(2), E_3)$	4	17	
$(-\mathbf{J}(2), -\mathbf{J}(2), \mathbf{J}(3))$	4	11	
$(-\mathbf{J}(3), -\mathbf{1}, \mathbf{J}(3))$	2	9	
$(\omega E_3, 1, \omega^{-1} E_3)$	8	19	
$(\omega\mathbf{J}(2), \omega^{-1}\mathbf{J}(2), \omega, \omega^{-1}, 1)$	4	11	
$(\omega\mathbf{J}(3), \omega^{-1}\mathbf{J}(3), 1)$	2	7	
$(i, i, -1, 1, i^{-1}, i^{-1}, -1)$	4	13	
$(i\mathbf{J}(2), i^{-1}\mathbf{J}(2), -1, -1, 1)$	2	9	
$(x, x, x^{-1}, x^{-1}, 1, 1, 1)$	4	17	$x^2 \neq 1$
$(x, x, x^2, 1, x^{-1}, x^{-1}, x^{-2})$	4	11	$x^4 \neq 1 \neq x^3$
$(x, -1, -x, 1, -x^{-1}, -1, x^{-1})$	2	9	$x^4 \neq 1$
$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), x^2, x^{-2}, 1)$	2	7	$x^4 \neq 1$
$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), \mathbf{J}(3))$	2	7	$x^2 \neq 1$
$(x, y, xy, 1, (xy)^{-1}, y^{-1}, x^{-1})$	2	7	$x, y, xy, 1, (xy)^{-1}, y^{-1}, x^{-1}$ pairwise distinct

which is embedded into $G_2(\bar{\mathbb{Q}}_\ell) \subseteq \mathrm{GL}_7(\bar{\mathbb{Q}}_\ell)$ by the permutation representation of the underlying symmetric group S_7, are summarized in the following table.

$C(s)$	T	$\bar{\mathbb{Q}}_\ell^\times \cdot A_1$	$A_1 \cdot A_1$	A_2	G_2
$W(C(s))$	1	Z_2	$Z_2 \times Z_2$	S_3	D_6

Here, Z_n denotes the cyclic group of order n, S_n denotes the symmetric group on n letters, and D_n denotes the dihedral group of order $2n$. This also implies the centralizer dimensions in G_2 of the semi-simple elements occurring in Table 1.

The $W(G_2)$-conjugacy classes of the subgroups $W(C(s)) \leqslant W(G_2)$ are as follows.

Z_2	$\langle(1,2)(6,7)\rangle$	$\langle(1,6)(2,7)(3,5)\rangle$	$\langle(1,7)(2,6)(3,5)\rangle$
$Z_2 \times Z_2$	$\langle(1,7)(2,6)(3,5), (1,2)(6,7)\rangle$		
S_3	$\langle(1,5,2)(6,7,3), (1,2)(6,7)\rangle$	$\langle(1,5,2)(6,7,3), (1,6)(2,7)(3,5)\rangle$	

The conditions on the eigenvalues of an element $s \in T$, imposed by the condition that s be centralized by one of the above conjugacy classes of subgroups of $W(G_2)$, lead to the following possibilities for the Jordan forms and their centralizers inside $W(G_2)$ (note that if $(1,7)(2,6)(3,5) \in C(s)$, then s has order less than or equal to 2).

s	$W(C(s))$	Conditions
$\mathrm{diag}(x, y, xy, 1, (xy)^{-1}, y^{-1}, x^{-1})$	1	$1, x, y, xy$ pairwise distinct
$\mathrm{diag}(x, x, x^2, 1, x^{-2}, x^{-1}, x^{-1})$	$\langle(1,2)(6,7)\rangle$	$x^2 \neq 1, x^3 \neq 1$
$\mathrm{diag}(x, x^{-1}, 1, 1, 1, x, x^{-1})$	$\langle(1,6)(2,7)(3,5)\rangle$	$x^2 \neq 1$
$\mathrm{diag}(-1, -1, 1, 1, 1, -1, -1)$	$Z_2 \times Z_2$	
$\mathrm{diag}(\omega, \omega, \omega^2, 1, \omega^{-2}, \omega^{-1}, \omega^{-1})$	S_3	$\omega^3 = 1$

The Jordan forms of the unipotent elements u can be found in [Law95] and their centralizers $C(u) = C_{G_2}(u)$ in [Car93, § 13.1]. We note that for any Jordan form, there is only one class in $G_2(\bar{\mathbb{Q}}_\ell)$; cf. [Car93]. The connected component $C(u)^0$ can be written as $C(u)^0 = C \cdot R$, $C \cap R = 1$, where R denotes the unipotent radical of $C(u)$ and C is reductive. These results are summarized as follows.

Jordan form	$\dim(R)$	Type of C	$C(u)/C(u)^0$
$(\mathbf{J}(2), \mathbf{J}(2), E_3)$	5	A_1	1
$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	3	A_1	1
$(\mathbf{J}(3), \mathbf{J}(3), 1)$	4	1	S_3
$(\mathbf{J}(7))$	2	1	1

To obtain the information in Table 1 about a mixed (i.e. neither semi-simple nor unipotent) element $g \in G_2(\bar{\mathbb{Q}}_\ell)$, we make use of the uniqueness of the Jordan decomposition of $g = g_s \cdot g_u = g_u \cdot g_s$ within the group $G_2(\bar{\mathbb{Q}}_\ell)$, where $g_s \in G_2(\bar{\mathbb{Q}}_\ell)$ is semi-simple and $g_u \in G_2(\bar{\mathbb{Q}}_\ell)$ unipotent; cf. [Car93]. The uniqueness implies that

$$C_{G_2}(g) = C_{G_2}(g_s) \cap C_{G_2}(g_u). \tag{1.3.1}$$

Note that the centralizer dimensions of elements in GL_n can be derived from their Jordan form by using the arguments of [Car93, p. 398].

By the structure of the centralizers of unipotent elements listed above, every centralizer of a unipotent element contains an involution, except for $\mathbf{J}(7)$, which explains the occurrence of the mixed elements in the seventh to ninth rows of Table 1. To verify the claim on the centralizer dimensions in G_2, note that the dimension of the centralizer of a non-trivial unipotent element u in a group of type $A_1 \cdot A_1$ is either 2 or 4, depending on whether or not a subgroup of type A_1 is contained in the centralizer of u. The claim concerning the centralizer dimension in G_2 for the seventh to ninth rows of Table 1 therefore follows from the structure of the centralizers of unipotent elements shown above.

Since the centralizer of an element s of order 3 is of type A_2, one has two classes of non-trivial unipotent elements in $C_{G_2}(s)$. Then, the centralizer dimensions in a group of type A_2 may again be derived using [Car93, p. 398], implying by (1.3.1) the centralizer dimensions in G_2 given n the 11th and 12th rows of Table 1.

Let $s = \mathrm{diag}(y, y, y^2, 1, y^{-2}, y^{-1}, y^{-1})$. Since $C_{G_2}(s)$ is of type $\bar{\mathbb{Q}}_\ell^\times \cdot A_1$, the distribution of the eigenvalues of s implies that the non-trivial unipotent elements in $C(s)$ have Jordan form $(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$. Thus there exists an element h with Jordan form $(y\mathbf{J}(2), y^2, 1, y^{-2}, y^{-1}\mathbf{J}(2))$, which explains the occurrence of the mixed elements in rows 14 and 18 of Table 1.

The computation of the centralizer dimension is similar to what was done in the above cases. A similar argument applies to row 19, using the fact that the centralizer of a unipotent element with Jordan form $(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$ contains a group of type A_1.

The only case left is to exclude an element $g = g_s g_u$ with possible Jordan form $(x\mathbf{J}(2), 1, 1, 1, x^{-1}\mathbf{J}(2))$, $x^2 \neq 1$. Since $C_{G_2}(g_s)$ is of type $T_1 \cdot A_1$ (with $T_1 \simeq \bar{\mathbb{Q}}_\ell^\times$), one has $g_s \in T_1$. A similar argument applied to the element $h = g'_s g_u$ occurring in rows 14 and 18 of Table 1 (with Jordan form $(y\mathbf{J}(2), y^2, 1, y^{-2}, y^{-1}\mathbf{J}(2))$) implies the existence of a non-conjugate torus $T_2 \simeq \bar{\mathbb{Q}}_\ell^\times$ with $T_2 \subseteq C_{G_2}(g_u)$, contradicting $C_{G_2}(g_u) = A_1 \cdot R$ where R is the unipotent radical. Summarizing these results leads to Table 1.

We will use the following notation in our next result, Theorem 1.3.1. Let $\mathbf{U}(i)$ denote the $\bar{\mathbb{Q}}_\ell$-valued representation of $\pi_1^{\mathrm{tame}}(\mathbb{G}_m)$ which sends a generator of $\pi_1^{\mathrm{tame}}(\mathbb{G}_m)$ to the Jordan block $\mathbf{J}(i)$. For any character χ of $\pi_1^{\mathrm{tame}}(\mathbb{G}_m)$, let

$$\mathbf{U}(i, \chi) := \chi \otimes \mathbf{U}(i),$$

let

$$-\mathbf{U}(i) := -\mathbf{1} \otimes \mathbf{U}(i)$$

(with $-\mathbf{1}$ denoting the unique quadratic character of $\pi_1^{\mathrm{tame}}(\mathbb{G}_m)$), and let $\mathbf{U}(i, \chi)^j$ denote the j-fold direct sum of the representation $\mathbf{U}(i, \chi)$.

THEOREM 1.3.1. *Let ℓ be a prime number and let k be an algebraically closed field with $\mathrm{char}(k) \neq 2, \ell$. Then the following hold.*

(i) *Let $\alpha_1, \alpha_2 \in \mathbb{A}^1(k)$ be two distinct points and let $\varphi, \eta : \pi_1^{\mathrm{tame}}(\mathbb{G}_{m,k}) \to \bar{\mathbb{Q}}_\ell^\times$ be continuous characters such that*

$$\varphi, \eta, \varphi\eta, \varphi\eta^2, \eta\varphi^2, \varphi\overline{\eta} \neq -\mathbf{1}. \tag{1.3.2}$$

Then there exists an irreducible cohomologically rigid sheaf $\mathcal{H} = \mathcal{H}(\varphi, \eta) \in \mathcal{T}_\ell(k)$ of generic rank 7 whose local monodromy is as follows.

- *The local monodromy at α_1 is $-\mathbf{1}^4 \oplus \mathbf{1}^3$.*
- *The local monodromy at α_2 is $\mathbf{U}(2)^2 \oplus \mathbf{U}(3)$.*
- *The local monodromy at ∞ is of the following form.*

Local monodromy at ∞	*Conditions on φ and η*
$\mathbf{U}(7)$	$\varphi = \eta = \mathbf{1}$
$\mathbf{U}(3, \varphi) \oplus \mathbf{U}(3, \overline{\varphi}) \oplus \mathbf{1}$	$\varphi = \eta \neq \mathbf{1}, \varphi^3 = \mathbf{1}$
$\mathbf{U}(2, \varphi) \oplus \mathbf{U}(2, \overline{\varphi}) \oplus \mathbf{U}(1, \varphi^2) \oplus \mathbf{U}(1, \overline{\varphi}^2) \oplus \mathbf{1}$	$\varphi = \eta, \varphi^4 \neq \mathbf{1} \neq \varphi^6$
$\mathbf{U}(2, \varphi) \oplus \mathbf{U}(2, \overline{\varphi}) \oplus \mathbf{U}(3)$	$\varphi = \overline{\eta}, \varphi^4 \neq \mathbf{1}$
$\varphi \oplus \eta \oplus \varphi\eta \oplus \overline{\varphi\eta} \oplus \overline{\eta} \oplus \overline{\varphi} \oplus \mathbf{1}$	$\varphi, \eta, \varphi\eta, \overline{\varphi\eta}, \overline{\eta}, \overline{\varphi}, \mathbf{1}$ *pairwise distinct*

Moreover, the restriction $\mathcal{H}|_{\mathbb{A}^1_k \setminus \{\alpha_1, \alpha_2\}}$ is lisse and the monodromy group associated to $\mathcal{H}$ is a Zariski dense subgroup of the simple exceptional algebraic group $G_2(\bar{\mathbb{Q}}_\ell)$.

(ii) *Assume that $\mathcal{H} \in \mathcal{T}_\ell$ is a cohomologically rigid $\bar{\mathbb{Q}}_\ell$-sheaf of generic rank 7 which fails to be lisse at ∞ and is such that the monodromy group associated to $\mathcal{H}$ is Zariski dense in the group $G_2(\bar{\mathbb{Q}}_\ell)$. Then $\mathcal{H}$ fails to be lisse at exactly two distinct points $\alpha_1, \alpha_2 \in \mathbb{A}^1(k)$ and,*

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up to a permutation of the points α_1, α_2 and ∞, the above list exhausts all the possible local monodromies of $\mathcal{H}$.

Proof. We introduce the following notation. Let $j : U := \mathbb{A}^1_k \backslash \{\alpha_1, \alpha_2\} \to \mathbb{P}^1$ denote the tautological inclusion. Let $x - \alpha_i$, with $i = 1, 2$, denote the morphism $U \to \mathbb{G}_m$ induced by sending $x \in U$ to $x - \alpha_i$. For any pair of continuous characters $\chi_1, \chi_2 : \pi_1^{\mathrm{tame}}(\mathbb{G}_m) \to \bar{\mathbb{Q}}_\ell^\times$, set

$$\mathcal{L}(\chi_1, \chi_2) := j_*(\mathcal{L}_{\chi_1(x-\alpha_1)} \otimes \mathcal{L}_{\chi_2(x-\alpha_2)}),$$

using the notation of (1.1.1). Let

$$\mathcal{F}_1 = \mathcal{L}(-\mathbf{1}, -\varphi\eta) \in \mathcal{T}_\ell.$$

Define inductively a sequence of sheaves $\mathcal{H}_0, \ldots, \mathcal{H}_6$ in $\mathcal{T}_\ell$ by setting

$$\mathcal{H}_0 := \mathcal{F}_1 \quad \text{and} \quad \mathcal{H}_i := \mathrm{MT}_{\mathcal{F}_{i+1}}(\mathrm{MC}_{\rho_i}(\mathcal{H}_{i-1})) \quad \text{for } i = 1, \ldots, 6,$$

where the $\mathcal{F}_i$ and ρ_i are defined as follows:

$$\mathcal{F}_3 = \mathcal{F}_5 = \mathcal{F}_7 = \mathcal{L}(-\mathbf{1}, \mathbf{1}), \quad \mathcal{F}_2 = \mathcal{F}_6 = \mathcal{L}(\mathbf{1}, -\overline{\varphi}), \quad \mathcal{F}_4 = \mathcal{L}(\mathbf{1}, -\varphi\overline{\eta})$$

and

$$\rho_1 := -\overline{\varphi\eta^2}, \quad \rho_2 := -\varphi\eta^2, \quad \rho_3 := -\overline{\varphi\eta}, \quad \rho_4 := -\varphi\eta, \quad \rho_5 := -\overline{\varphi}, \quad \rho_6 := -\varphi.$$

We now distinguish five cases, which correspond to the different types of local monodromy at ∞ listed above.

Case 1. Let $\varphi = \eta = \mathbf{1}$. The following table lists the local monodromies of the sheaves $\mathcal{H}_0, \ldots, \mathcal{H}_6 = \mathcal{H}$ at the points α_1, α_2 and ∞; the proof is a direct computation, using Proposition 1.2.1 and (1.2.1).

	at α_1	at α_2	at ∞
$\mathcal{H}_0$	$-\mathbf{1}$	$-\mathbf{1}$	$\mathbf{1}$
$\mathcal{H}_1$	$\mathbf{U}(2)$	$-\mathbf{U}(2)$	$\mathbf{U}(2)$
$\mathcal{H}_2$	$-\mathbf{1}^2 \oplus \mathbf{1}$	$\mathbf{U}(3)$	$\mathbf{U}(3)$
$\mathcal{H}_3$	$\mathbf{U}(2)^2$	$\mathbf{U}(2) \oplus -\mathbf{1}^2$	$\mathbf{U}(4)$
$\mathcal{H}_4$	$\mathbf{1}^2 \oplus -\mathbf{1}^3$	$\mathbf{U}(2)^2 \oplus -\mathbf{1}$	$\mathbf{U}(5)$
$\mathcal{H}_5$	$\mathbf{U}(2)^3$	$-\mathbf{U}(2) \oplus -\mathbf{1}^2 \oplus \mathbf{1}^2$	$\mathbf{U}(6)$
$\mathcal{H}_6$	$-\mathbf{1}^4 \oplus \mathbf{1}^3$	$\mathbf{U}(2)^2 \oplus \mathbf{U}(3)$	$\mathbf{U}(7)$

By Proposition 1.2.1 and the results of § 1.1, the sheaf $\mathcal{H} = \mathcal{H}_6$ is a cohomologically rigid irreducible sheaf of rank 7 in $\mathcal{T}_\ell$ which is lisse on the open subset $U = \mathbb{A}^1_k \backslash \{\alpha_1, \alpha_2\} \subseteq \mathbb{A}^1_k$. The lisse sheaf $\mathcal{H}|_U$ corresponds to a representation

$$\rho : \pi_1^{\mathrm{tame}}(\mathbb{A}^1 \backslash \{\alpha_1, \alpha_2\}) \to \mathrm{GL}(V),$$

where V is a $\bar{\mathbb{Q}}_\ell$-vector space of dimension 7. Let G be the image of ρ. Note that G is an irreducible subgroup of $\mathrm{GL}(V)$ since $\mathcal{H}$ is irreducible. In the following, we fix an isomorphism $V \simeq \bar{\mathbb{Q}}_\ell^7$. This induces an isomorphism $\mathrm{GL}(V) \simeq \mathrm{GL}_7(\bar{\mathbb{Q}}_\ell)$, so we can view G as a subgroup of $\mathrm{GL}_7(\bar{\mathbb{Q}}_\ell)$.

We want to show that G is contained in a conjugate of the group $G_2(\bar{\mathbb{Q}}_\ell) \leqslant \mathrm{GL}_7(\bar{\mathbb{Q}}_\ell)$. To do this, we argue as in [Kat90, § 4.1]. First, note that the local monodromy at $s \in \{\alpha_1, \alpha_2, \infty\}$ can be (locally) conjugated in $\mathrm{GL}_7(\bar{\mathbb{Q}}_\ell)$ into the orthogonal group $O_7(\bar{\mathbb{Q}}_\ell)$. It then follows from the rigidity of the representation ρ that there exists an element $x \in \mathrm{GL}(V)$ such that

$$\mathrm{Transpose}(\rho(g)^{-1}) = \rho(g)^x \quad \forall g \in \pi_1^{\mathrm{tame}}(\mathbb{A}^1 \backslash \{\alpha_1, \alpha_2\}).$$

In other words, the group G respects the non-degenerate bilinear form given by the element x^{-1}. Since G is irreducible and the dimension of V is 7, this form has to be symmetric. Thus we can assume that G is contained in the orthogonal group $O_7(\bar{\mathbb{Q}}_\ell)$. By the results of Aschbacher [Asc87, Theorem 5 parts (2) and (5)], an irreducible subgroup G of $\mathrm{O}_7(K)$ (where K denotes an algebraically closed field or a finite field) lies inside an $\mathrm{O}_7(K)$-conjugate of $G_2(K)$ if and only if G has a non-zero invariant in the third exterior power $\Lambda^3(V)$ of $V = K^7$. In our case, this is equivalent to

$$H^0(U, \Lambda^3(\tilde{\mathcal{H}})) \simeq H^0(\mathbb{P}^1, j_*\Lambda^3(\tilde{\mathcal{H}})) \neq \{0\}, \tag{1.3.3}$$

where $\tilde{\mathcal{H}} = \mathcal{H}|_U$. Poincaré duality implies that the previous formula is equivalent to

$$H_c^2(U, \Lambda^3(\tilde{\mathcal{H}})) \simeq H^2(\mathbb{P}^1, j_*\Lambda^3(\tilde{\mathcal{H}})) \neq \{0\}. \tag{1.3.4}$$

The Euler–Poincaré formula implies that

$$\begin{aligned}
\chi(\mathbb{P}^1, j_*\Lambda^3(\tilde{\mathcal{H}})) &= h^0(\mathbb{P}^1, j_*\Lambda^3(\tilde{\mathcal{H}})) - h^1(\mathbb{P}^1, j_*\Lambda^3(\tilde{\mathcal{H}})) + h^2(\mathbb{P}^1, j_*\Lambda^3(\tilde{\mathcal{H}})) \\
&= \chi(U) \cdot \mathrm{rk}(\Lambda^3(\tilde{\mathcal{H}})) + \sum_{s \in \{\alpha_1, \alpha_2, \infty\}} \dim(\Lambda^3(\tilde{\mathcal{H}})^{I(s)}) \\
&= -35 + 19 + 13 + 5 \\
&= 2.
\end{aligned} \tag{1.3.5}$$

Note that

$$\chi(U) = h^0(U) - h^1(U) + h^2(U) = -1, \quad \mathrm{rk}(\Lambda^3(\tilde{\mathcal{H}})) = 35$$

and, for $s = \alpha_1, \alpha_2$ and ∞, the dimensions of the local invariants $\dim(\Lambda^3(\tilde{\mathcal{H}})^{I(s)})$ equal 19, 13 and 5, respectively.

The latter claim can be verified as follows. The group G_2 leaves a line fixed in the third exterior power $\Lambda^3(V)$ of its minimal representation $V = \bar{\mathbb{Q}}_\ell^7$; cf. [Asc87]. Hence [OV90, Table 5] implies (for dimension reasons) that the $G_2(\bar{\mathbb{Q}}_\ell)$-module $\Lambda^3(V)$ decomposes as

$$\Lambda^3(V) = V \oplus S^2(V), \tag{1.3.6}$$

where $S^2(V)$ is the second symmetric power of V. For $s = \alpha_1$, we have to determine the dimension of the fixed space of an involution in $G_2(\bar{\mathbb{Q}}_\ell)$ under Λ^3, which can easily be seen to have dimension 19, using the decomposition in (1.3.6). For $s = \infty$, we have to determine the dimension of the fixed space of a regular unipotent element u, which can be assumed to be contained in the image of the sixth symmetric power $S^6(\rho)$ of the standard representation ρ of $\mathrm{SL}_2(\bar{\mathbb{Q}}_\ell)$. Since

$$S^2(S^p(\rho)) = \sum_{i \geqslant 0} S^{2p-4i}(\rho), \tag{1.3.7}$$

where $S^0(\rho) = 1$ and $S^k(\rho) = 0$ for k negative (cf. [OV90, Table 5]), this implies that $\dim(\Lambda^3(\tilde{\mathcal{H}})^{I(\infty)}) = 5$, because $\Lambda^3(S^6(\rho)) = S^6(\rho) \oplus (S^{12}(\rho) \oplus S^8(\rho) \oplus S^4(\rho) \oplus 1)$ (by (1.3.6)) and each summand adds a 1 to the dimension of the fixed space. The claim for $\dim(\Lambda^3(\tilde{\mathcal{H}})^{I(\alpha_2)})$ can be verified by analogous reasoning, using

$$S^2(V_1 \oplus V_2) = S^2(V_1) \oplus S^2(V_2) \oplus (V_1 \otimes V_2) \quad \text{and} \quad S^p(\rho) \otimes S^q(\rho) = \sum_{i \geqslant 0} S^{p+q-2i}(\rho)$$

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together with formula (1.3.7), by embedding the element $(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$ into the image of $\rho \oplus \rho \oplus S^2(\rho)$.

It follows from (1.3.5) and the equivalence of (1.3.3) and (1.3.4) that $h^0(U, \Lambda^3(\tilde{\mathcal{H}})) \geqslant 1$. Therefore, the monodromy group G can be assumed to be contained in $G_2(\bar{\mathbb{Q}}_\ell)$. Let $\overline{G}$ denote the Zariski closure of G in $G_2(\bar{\mathbb{Q}}_\ell)$. By [Asc87, Corollary 12], either a Zariski closed proper maximal subgroup of $G_2(\bar{\mathbb{Q}}_\ell)$ is reducible or G is isomorphic to the group $\mathrm{SL}_2(\bar{\mathbb{Q}}_\ell)$ acting on the vector space of homogeneous polynomials of degree 6. In the latter case, the unipotent elements of the image of $\mathrm{SL}_2(\bar{\mathbb{Q}}_\ell)$ are either equal to the identity matrix or conjugate in $\mathrm{GL}_7(\bar{\mathbb{Q}}_\ell)$ to a Jordan block of length 7. Since the local monodromy of $\mathcal{H}$ at α_2 is not of this form, G must coincide with $\mathrm{G}_2(\bar{\mathbb{Q}}_\ell)$. This finishes the proof of assertion (i) in Case 1.

In Cases 2–5 that follow, we shall list only the local monodromy of the sheaves

$$\mathcal{H}_0, \ldots, \mathcal{H}_6 = \mathcal{H}.$$

In each case, rigidity implies that the image of $\pi_1(U)$ is contained in an orthogonal group, and one can compute that an analogue of (1.3.5) holds. Thus the image of $\pi_1(U)$ is Zariski dense in G_2 by the same arguments as in Case 1.

Case 2. Suppose $\varphi = \eta$ and that φ is non-trivial of order 3. Then the local monodromies of $\mathcal{H}_0, \ldots, \mathcal{H}_6 = \mathcal{H}$ at α_1, α_2 and ∞ are as follows.

	at α_1	at α_2	at ∞
$\mathcal{H}_0$	$-\mathbf{1}$	$-\overline{\varphi}$	$\overline{\varphi}$
$\mathcal{H}_1$	$\mathbf{U}(2)$	$-\varphi \oplus -\overline{\varphi}$	$\varphi \oplus \overline{\varphi}$
$\mathcal{H}_2$	$-\mathbf{1}^2 \oplus \mathbf{1}$	$\varphi \oplus \overline{\varphi} \oplus \mathbf{1}$	$\varphi \oplus \overline{\varphi} \oplus \mathbf{1}$
$\mathcal{H}_3$	$\mathbf{U}(1, \varphi)^2 \oplus \mathbf{1}^2$	$\overline{\varphi} \oplus \mathbf{1} \oplus \mathbf{U}(1, -\mathbf{1})^2$	$\mathbf{U}(2, \varphi) \oplus \mathbf{1} \oplus \overline{\varphi}$
$\mathcal{H}_4$	$\mathbf{1}^2 \oplus -\mathbf{1}^3$	$-\varphi \oplus \mathbf{U}(1, \overline{\varphi})^2 \oplus \mathbf{1}^2$	$\mathbf{U}(2) \oplus \mathbf{U}(2, \overline{\varphi}) \oplus \varphi$
$\mathcal{H}_5$	$\mathbf{U}(1, \overline{\varphi})^3 \oplus \mathbf{1}^3$	$\mathbf{U}(2, -\overline{\varphi}) \oplus \mathbf{1}^2 \oplus \mathbf{U}(1, -\overline{\varphi})^2$	$\mathbf{U}(3, \varphi) \oplus \mathbf{U}(2) \oplus \overline{\varphi}$
$\mathcal{H}_6$	$-\mathbf{1}^4 \oplus \mathbf{1}^3$	$\mathbf{U}(2)^2 \oplus \mathbf{U}(3)$	$\mathbf{U}(3, \varphi) \oplus \mathbf{U}(3, \overline{\varphi}) \oplus \mathbf{1}$

Case 3. Suppose $\varphi = \eta$ and $\varphi^4 \neq \mathbf{1} \neq \varphi^6$. Then the local monodromies are as given in the following table.

	at α_1	at α_2	at ∞
$\mathcal{H}_0$	$-\mathbf{1}$	$-\varphi^2$	φ^2
$\mathcal{H}_1$	$\overline{\varphi}^3 \oplus \mathbf{1}$	$-\overline{\varphi}^2 \oplus -\overline{\varphi}$	$\overline{\varphi}^4 \oplus \overline{\varphi}^2$
$\mathcal{H}_2$	$-\mathbf{1}^2 \oplus \mathbf{1}$	$\varphi \oplus \varphi^2 \oplus \mathbf{1}$	$\varphi^3 \oplus \overline{\varphi} \oplus \varphi$
$\mathcal{H}_3$	$\mathbf{U}(1, \overline{\varphi}^2)^2 \oplus \mathbf{1}^2$	$\overline{\varphi} \oplus \mathbf{1} \oplus \mathbf{U}(1, -\mathbf{1})^2$	$\overline{\varphi}^2 \oplus \varphi \oplus \overline{\varphi}^3 \oplus \overline{\varphi}$
$\mathcal{H}_4$	$\mathbf{1}^2 \oplus -\mathbf{1}^3$	$-\varphi \oplus \mathbf{U}(1, \varphi^2)^2 \oplus \mathbf{1}^2$	$\varphi^2 \oplus \mathbf{1} \oplus \varphi^3 \oplus \overline{\varphi} \oplus \varphi$
$\mathcal{H}_5$	$\mathbf{U}(1, \overline{\varphi})^3 \oplus \mathbf{1}^3$	$\mathbf{U}(2, -\overline{\varphi}) \oplus \mathbf{1}^2 \oplus \mathbf{U}(1, -\overline{\varphi})^2$	$\mathbf{1} \oplus \mathbf{U}(2, \overline{\varphi}^2) \oplus \varphi \oplus \overline{\varphi}^3 \oplus \overline{\varphi}$
$\mathcal{H}_6$	$-\mathbf{1}^4 \oplus \mathbf{1}^3$	$\mathbf{U}(2)^2 \oplus \mathbf{U}(3)$	$\mathbf{U}(2, \varphi) \oplus \mathbf{U}(2, \overline{\varphi}) \oplus \varphi^2 \oplus \overline{\varphi}^2 \oplus \mathbf{1}$

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Case 4. Suppose $\varphi = \overline{\eta}$ and $\varphi^4 \neq \mathbf{1}$. Then we have the following.

	at α_1	at α_2	at ∞
$\mathcal{H}_0$	$-\mathbf{1}$	$-\mathbf{1}$	$\mathbf{1}$
$\mathcal{H}_1$	$\varphi \oplus \mathbf{1}$	$-\mathbf{1} \oplus -\overline{\varphi}$	$\mathbf{U}(2)$
$\mathcal{H}_2$	$-\mathbf{1}^2 \oplus \mathbf{1}$	$\overline{\varphi} \oplus \overline{\varphi}^2 \oplus \mathbf{1}$	$\mathbf{U}(3, \overline{\varphi})$
$\mathcal{H}_3$	$\mathbf{U}(2)^2$	$\varphi \oplus \mathbf{1} \oplus \mathbf{U}(1, -\varphi^2)^2$	$\varphi^2 \oplus \mathbf{U}(3, \varphi)$
$\mathcal{H}_4$	$\mathbf{1}^2 \oplus -\mathbf{1}^3$	$-\varphi \oplus \mathbf{U}(1, \varphi^2)^2 \oplus \mathbf{1}^2$	$\varphi^2 \oplus \mathbf{1} \oplus \mathbf{U}(3, \varphi)$
$\mathcal{H}_5$	$\mathbf{U}(1, \overline{\varphi})^3 \oplus \mathbf{1}^3$	$\mathbf{U}(2, -\overline{\varphi}) \oplus \mathbf{1}^2 \oplus \mathbf{U}(1, -\overline{\varphi})^2$	$\mathbf{1} \oplus \mathbf{U}(2, \overline{\varphi}^2) \oplus \mathbf{U}(3, \overline{\varphi})$
$\mathcal{H}_6$	$-\mathbf{1}^4 \oplus \mathbf{1}^3$	$\mathbf{U}(2)^2 \oplus \mathbf{U}(3)$	$\mathbf{U}(2, \varphi) \oplus \mathbf{U}(2, \overline{\varphi}) \oplus \mathbf{U}(3)$

Case 5. The characters

$$\varphi, \eta, \varphi\eta, \overline{\varphi\eta}, \overline{\eta}, \overline{\varphi}, \mathbf{1}$$

are pairwise disjoint and

$$\varphi\overline{\eta} \neq -\mathbf{1} \neq \varphi\eta^2, \varphi^2\eta.$$

Then the local monodromies are as follows.

	at α_1	at α_2	at ∞
$\mathcal{H}_0$	$-\mathbf{1}$	$-\varphi\eta$	$\varphi\eta$
$\mathcal{H}_1$	$\overline{\varphi\eta^2} \oplus \mathbf{1}$	$-\overline{\varphi\eta} \oplus -\overline{\varphi}$	$\overline{\varphi\eta} \oplus \overline{\varphi^2\eta^2}$
$\mathcal{H}_2$	$\mathbf{1} \oplus -\mathbf{1}^2$	$\eta \oplus \eta^2 \oplus \mathbf{1}$	$\eta \oplus \overline{\varphi} \oplus \varphi\eta^2$
$\mathcal{H}_3$	$\mathbf{U}(1, \overline{\varphi\eta})^2 \oplus \mathbf{1}^2$	$\overline{\eta} \oplus \mathbf{1} \oplus \mathbf{U}(1, -\varphi\overline{\eta})^2$	$\overline{\eta} \oplus \overline{\varphi\eta^2} \oplus \varphi \oplus \overline{\eta}^2$
$\mathcal{H}_4$	$\mathbf{U}(1, -\mathbf{1})^3 \oplus \mathbf{1}^2$	$-\varphi \oplus \mathbf{U}(1, \varphi^2)^2 \oplus \mathbf{1}^2$	$\varphi \oplus \overline{\eta} \oplus \varphi^2\eta \oplus \varphi\overline{\eta} \oplus \varphi\eta$
$\mathcal{H}_5$	$\mathbf{U}(1, \overline{\varphi})^3 \oplus \mathbf{1}^3$	$\mathbf{U}(2, -\overline{\varphi}) \oplus \mathbf{U}(1, -\overline{\varphi})^2 \oplus \mathbf{1}^2$	$\overline{\varphi} \oplus \overline{\eta\varphi^2} \oplus \eta \oplus \overline{\varphi\eta} \oplus \overline{\varphi}\eta \oplus \overline{\varphi}^2$
$\mathcal{H}_6$	$-\mathbf{1}^4 \oplus \mathbf{1}^3$	$\mathbf{U}(2)^2 \oplus \mathbf{U}(3)$	$\mathbf{1} \oplus \overline{\eta\varphi} \oplus \eta\varphi \oplus \overline{\eta} \oplus \eta \oplus \overline{\varphi} \oplus \varphi$

This finishes the proof of assertion (i).

Now let us prove assertion (ii) of the theorem. Let D be a reduced effective divisor on $\mathbb{A}^1$, let $U := \mathbb{A}^1 \setminus D$, and let $j : U \to \mathbb{P}^1$ denote the obvious inclusion. A sheaf $\mathcal{H} \in \mathcal{T}_\ell(k)$ which is lisse on U is cohomologically rigid if and only if

$$\operatorname{rig}(\mathcal{H}) = (1 - \operatorname{Card}(D))\operatorname{rk}(\mathcal{H}|_U)^2 + \sum_{s \in D \cup \infty} \sum_{i,\chi} e_i(s, \chi, \mathcal{H})^2 = 2 \tag{1.3.8}$$

(cf. [Kat96, 6.0.15]). (Note that the sum $\sum_{i,\chi} e_i(s, \chi, \mathcal{H})^2$ gives the dimension of the centralizer of the local monodromy in the group $\mathrm{GL}_{\operatorname{rk}(\mathcal{H}|_U)}(\overline{\mathbb{Q}}_\ell)$; see [Kat96, 3.1.15].) Another necessary condition for $\mathcal{H}$ to be contained in $\mathcal{T}_\ell$ is that $\mathcal{H}$ be irreducible. This implies that

$$\chi(\mathbb{P}^1, j_*(\mathcal{H}|_U)) = (1 - \operatorname{Card}(D))\operatorname{rk}(\mathcal{H}) + \sum_{s \in D \cup \{\infty\}} e_1(s, \mathbf{1}, \mathcal{H}) \leqslant 0, \tag{1.3.9}$$

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Table 2. The possible centralizer dimensions.

Case	$\sum_{i,\chi} e_i(\alpha_1,\chi,\mathcal{H})^2$	$\sum_{i,\chi} e_i(\alpha_2,\chi,\mathcal{H})^2$	$\sum_{i,\chi} e_i(\infty,\chi,\mathcal{H})^2$
P_1	29	13	9
P_2	29	11	11
P_3	25	19	7
P_4	25	17	9
P_5	25	13	13
P_6	19	19	13
P_7	17	17	17

since the same arguments as for (1.3.5) apply here. Assume first that $\mathrm{Card}(D)>2$ and that $\mathcal{H}$ fails to be lisse at all points of D. Then, by (1.3.8),

$$\sum_{s\in D\cup\infty}\sum_{i,\chi} e_i(s,\chi,\mathcal{H})^2 = 2+(\mathrm{Card}(D)-1)7^2.$$

Since $\sum_{i,\chi} e_i(s,\chi,\mathcal{H})^2 \leqslant 29$ by Table 1, one immediately concludes that the cardinality of D is less than or equal to 3. If $\mathrm{Card}(D)=3$, then, by Table 1, the following combinations of the centralizer dimensions can occur:

$$(25,25,25,25),\quad (29,29,29,13),\quad (29,29,25,17).$$

In each case, one obtains a contradiction to (1.3.9) (using a quadratic twist at each local monodromy in the $(25,25,25,25)$ case).

Thus $D=\{\alpha_1,\alpha_2\}$, where α_1 and α_2 are two distinct points of $\mathbb{A}^1(k)$, and

$$\sum_{s\in\{\alpha_1,\alpha_2,\infty\}}\sum_{i,\chi} e_i(s,\chi,\mathcal{H})^2 = 7^2+2=51.$$

This leaves us with seven possible cases $P_1,\ldots,P_7$, which are listed in Table 2.

Using Table 1 and the inequality of (1.3.9), one can exclude P_1, P_4 and P_7 by possibly twisting the local monodromy at α_1,α_2 and ∞ by three suitable (at most quadratic) characters whose product is $\mathbf{1}$. The case P_5 can be excluded by means of the inequality in (1.3.9) and a twist by suitable characters of order at most 4 whose product is $\mathbf{1}$.

Since the monodromy representation of $\mathcal{H}$ is dense in the group $G_2(\bar{\mathbb{Q}}_\ell)$, one obtains an associated sheaf $\mathrm{Ad}(\mathcal{H})\in\mathcal{T}_\ell$ of generic rank 14, given by the adjoint representation of G_2. This is again irreducible, which implies that

$$\chi(\mathbb{P}^1, j_*(\mathrm{Ad}(\mathcal{H})|_U)) = (1-\mathrm{Card}(D))\cdot 14+\sum_{s\in D\cup\infty}\dim(C_{G_2}(\mathcal{H}(s)))\leqslant 0.$$

This can be used to exclude the cases P_2 and P_6, because in these cases one has, respectively,

$$\chi(\mathbb{P}^1, j_*\underline{\mathrm{End}}_{G_2}(\mathcal{H}|_U)) = -14+8+4+4>0$$

and

$$\chi(\mathbb{P}^1, j_*\underline{\mathrm{End}}_{G_2}(\mathcal{H}|_U)) = -14+6+6+4>0.$$

By the same argument, in case P_3 one can exclude the possibility that the centralizer dimension 19 comes from a non-unipotent character.

Thus, up to a permutation of the points α_1, α_2 and ∞, we are left with the following possibility for the local monodromy: the local monodromy at α_1 is an involution, the local monodromy at α_2 is unipotent of the form $\mathbf{U}(2)^2 \oplus \mathbf{U}(3)$, and the local monodromy at ∞ is regular, i.e. the dimension of the centralizer is 7. Let us explain the conditions in formula (1.3.2) of Theorem 1.3.1. By Table 1, the regularity of the local monodromy at ∞ implies that $\varphi, \eta, \varphi\eta \neq -\mathbf{1}$. It follows from the properties of Katz's algorithm and the regularity at ∞ that $\mathcal{F}_1, \ldots, \mathcal{F}_6$ also must not equal $\mathbf{1}$ (see Remark 1.3.2(i)), implying the further condition $\varphi\bar{\eta} \neq -\mathbf{1}$ by the definition of $\mathcal{F}_4$. In addition, each ρ_i has to not equal $\mathbf{1}$ in order for MC_{ρ_i} to be defined, which implies $\varphi\eta^2 \neq -\mathbf{1}$. The condition $\eta\varphi^2 \neq -\mathbf{1}$ is non-redundant only in Case 5. In this case, by Proposition 1.2.1(i), the equality $\eta\varphi^2 = -\mathbf{1}$ implies that the rank of $\mathrm{MC}_{-\overline{\varphi\eta}}(\mathcal{H}_2)$ is 3 instead of 4, producing a contradiction. (We remark that the conditions $\varphi^4 \neq \mathbf{1} \neq \varphi^6$ in Case 3 and the condition $\varphi^4 \neq \mathbf{1}$ in Case 4 also follow from (1.3.2) and are stated only for completeness in the table of Theorem 1.3.1(i). Also, the conditions of (1.3.2) can alternatively be derived from similar considerations as for $\eta\varphi^2 = -\mathbf{1}$.) □

Remark 1.3.2. Let us give a few remarks on the general construction of the sheaves in the proof of Theorem 1.3.1.

(i) The construction of the various sheaves $\mathcal{H} = \mathcal{H}_6$ follows Katz's existence algorithm for rigid local systems [Kat96, ch. 6] in reverse order, using the invertibility of MC_χ stated in (1.1.2). Since the local monodromy of $\mathcal{H}$ at ∞ is always regular by the discussion following Table 2, it follows from Proposition 1.2.1(iii) that the local monodromy at ∞ also has to be regular in each intermediate step (making the tables of local monodromies plausible, at least for the local monodromy at ∞). Together with the multiplicativity of MC stated in (1.1.2) and Proposition 1.2.1, this also implies the conditions $\mathcal{F}_2, \mathcal{F}_4, \mathcal{F}_6 \neq \mathbf{1}$ (as well as $\mathcal{F}_1, \mathcal{F}_3, \mathcal{F}_5 \neq \mathbf{1}$, which is automatically fulfilled). Proposition 1.2.1 further implies that the local monodromy at α_1 has to be involutive in each second intermediate step. The monodromy groups of $\mathcal{H}_2, \mathcal{H}_4, \mathcal{H}_6$ are contained in a general orthogonal group. In addition, the monodromy groups of $\mathcal{H}_1, \mathcal{H}_3, \mathcal{H}_5$ in Case 1 are symplectic groups. (The latter two claims can be derived from rigidity and the properties of the local monodromy.)

(ii) We stress that while Theorem 1.3.1 gives all possible local monodromies of rigid local systems of rank 7 with monodromy group Zariski dense in G_2, it does not list all possible pairs of characters φ, η for which one can apply a sequence of middle tensor products and middle convolutions in order to obtain such a local system. There is some freedom in choosing the sequence of intermediate sheaves $\mathcal{H}_1, \ldots, \mathcal{H}_5$ while fixing $\mathcal{H}_6$. Thus one can, for example, interchange φ with η, corresponding to the action of the Weyl group $W(G_2)$ on the maximal torus.

(iii) In some special cases, the monodromy group of $\mathcal{H}_2$ is even a finite group. This phenomenon arises in Case 2, where the monodromy group of $\mathcal{H}_2$ is the imprimitive reflection group $Z_2^2.Z_3$. In Case 3 with φ being a fifth root of unity, we obtain $A_5 \times Z_5$ as the monodromy group of $\mathcal{H}_2$. Also, in Case 5 with η being a third (respectively, a fifth) root of unity and $\overline{\varphi\eta}$ being a fifth (respectively, a third) root of unity, one obtains a finite monodromy group. In this case the monodromy group of $\mathcal{H}_2$ is $A_5 \times Z_3$ (respectively, $A_5 \times Z_5$).

Let us assume that $\alpha_1 = 0, \alpha_2 = 1$ and $k = \bar{\mathbb{Q}}$. Let $\iota : \mathbb{A}^1_{\bar{\mathbb{Q}}} \to \mathbb{A}^1_{\mathbb{Q}}$ be the base-change map. We say that a sheaf $\mathcal{H}$ as in Theorem 1.3.1 is *defined over* $\mathbb{Q}$ if $\mathcal{H}|_{\mathbb{A}_{\bar{\mathbb{Q}}}\setminus\{0,1\}}$ is of the form $\iota^*(\mathfrak{H})$ where $\mathfrak{H}$ is lisse on $\mathbb{A}^1_{\mathbb{Q}}\setminus\{0, 1\}$.

Theorem 1.3.3. *If a sheaf $\mathcal{H}$ as in Theorem 1.3.1(i) is defined over $\mathbb{Q}$, then the trace of the local monodromy at ∞ is contained in $\mathbb{Q}$. This is the case if and only if the local monodromy at ∞ takes one of the following forms.*

$\mathbf{U}(7)$	
$\mathbf{U}(3,\varphi)\oplus\mathbf{U}(3,\overline{\varphi})\oplus\mathbf{1}$	φ *of order* 3
$\mathbf{U}(2,\varphi)\oplus\mathbf{U}(2,\overline{\varphi})\oplus\mathbf{U}(3)$	φ *of order* 3 *or* 6
$\varphi\oplus\eta\oplus\varphi\eta\oplus\overline{\varphi\eta}\oplus\overline{\eta}\oplus\overline{\varphi}\oplus\mathbf{1}$	φ *of order* 7 *or* 14 *and* $\eta=\varphi^2$ φ *of order* 12 *and* $\eta=-\varphi$

Proof. The first assertion follows from the structure of the local fundamental group at ∞. Using the local monodromy of the sheaves $\mathcal{H}$ listed in Theorem 1.3.1(i), one obtains the result via an explicit computation. □

2. The motivic interpretation of the rigid G_2-sheaves

In this section we recall the motivic interpretation of the middle convolution in the universal setup of [Kat96, ch. 8]. This leads to an explicit geometric construction of the rigid G_2-sheaves found in the previous section.

2.1 Basic definitions

Let us recall the setup of [Kat96, ch. 8]. Let k denote an algebraically closed field and ℓ a prime number which is invertible in k. Further, let $\alpha_1,\ldots,\alpha_n$ be pairwise disjoint points of $\mathbb{A}^1(k)$ and ζ a primitive root of unity in k. Fix an integer $N\geqslant 1$ such that $\mathrm{char}(k)$ does not divide N and let

$$R:=R_{N,\ell}:=\mathbb{Z}\left[\zeta_N,\frac{1}{N\ell}\right],$$

where ζ_N denotes a primitive Nth root of unity. Set

$$S_{N,n,\ell}:=R_{N,\ell}[T_1,\ldots,T_n][1/\Delta],\quad \Delta:=\prod_{i\neq j}(T_i-T_j).$$

Fix an embedding $R\to\overline{\mathbb{Q}}_\ell$ and let E denote the fraction field of R. For a place λ of E, let E_λ denote the λ-adic completion of E. Let $\phi: S_{N,n,\ell}\to k$ denote the unique ring homomorphism for which $\phi(\zeta_N)=\zeta$ and such that

$$\phi(T_i)=\alpha_i\quad\text{for } i=1,\ldots,n.$$

Let $\mathbb{A}^1_{S_{N,n,\ell}}\setminus\{T_1,\ldots,T_n\}$ denote the affine line over S with the n sections $T_1,\ldots,T_n$ deleted. Consider, more generally, the spaces

$$\mathbb{A}(n,r+1)_R:=\mathrm{Spec}\left(R[T_1,\ldots,T_n,X_1,\ldots,X_{r+1}]\left[\frac{1}{\Delta_{n,r}}\right]\right)$$

where

$$\Delta_{n,r}:=\left(\prod_{i\neq j}(T_i-T_j)\right)\left(\prod_{a,j}(X_a-T_j)\right)\left(\prod_k(X_{k+1}-X_k)\right)$$

(here the indices i,j run through $\{1,\ldots,n\}$, the index a runs through $\{1,\ldots,r+1\}$, and the index k runs through $\{1,\ldots,r\}$; when $r=0$ the empty product $\prod_k(X_{k+1}-X_k)$ is understood to be 1).

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Define

$$\mathrm{pr}_i : \mathbb{A}(n, r+1)_R \to \mathbb{A}^1_{S_{N,n,\ell}} \backslash \{T_1, \ldots, T_n\},$$
$$(T_1, \ldots, T_n, X_1, \ldots, X_{r+1}) \mapsto (T_1, \ldots, T_n, X_i).$$

On $(\mathbb{G}_m)_R$ with coordinate Z, one has the Kummer covering of degree N of equation $Y^N = Z$. This is a connected $\mu_N(R)$-torsor whose existence defines a surjective homomorphism $\pi_1((\mathbb{G}_m)_R) \to \mu_N(R)$. The chosen embedding $R \to \overline{\mathbb{Q}}_\ell$ defines a faithful character

$$\chi_N : \mu_N(R) \to \overline{\mathbb{Q}}_\ell^\times,$$

and the composite homomorphism

$$\pi_1((\mathbb{G}_m)_R) \to \mu_N(R) \to \overline{\mathbb{Q}}_\ell^\times$$

defines the Kummer sheaf $\mathcal{L}_{\chi_N}$ on $(\mathbb{G}_m)_R$. For any scheme W and any map $f : W \to (\mathbb{G}_m)_R$, define

$$\mathcal{L}_{\chi(f)} := f^* \mathcal{L}_\chi.$$

2.2 The middle convolution of local systems

Denote by $\mathrm{Lisse}(N, n, \ell)$ the category of lisse $\bar{\mathbb{Q}}_\ell$-sheaves on

$$\mathbb{A}(n, 1)_R = (\mathbb{A}^1 - (T_1, \ldots, T_n))_{S_{N,n,\ell}}.$$

For each non-trivial $\overline{\mathbb{Q}}_\ell$-valued character χ of the group $\mu_N(R)$, Katz defined in [Kat96] a left exact *middle convolution functor*

$$\mathrm{MC}_\chi : \mathrm{Lisse}(N, n, \ell) \longrightarrow \mathrm{Lisse}(N, n, \ell)$$

as follows.

Definition 2.2.1. View the space $\mathbb{A}(n, 2)_R$ with its second projection pr_2 to $\mathbb{A}(n, 1)_R$ as a relative $\mathbb{A}^1$ with coordinate X_1, minus the $n+1$ sections $T_1, \ldots, T_n$ and X_2. Compactify the morphism pr_2 into the relative $\mathbb{P}^1$,

$$\overline{\mathrm{pr}}_2 : \mathbb{P}^1 \times \mathbb{A}(n, 1)_R \to \mathbb{A}(n, 1)_R,$$

by filling in the sections $T_1, \ldots, T_n, X_2, \infty$. Moreover, let $j : \mathbb{A}(n, 2)_R \longrightarrow \mathbb{P}^1 \times \mathbb{A}(n, 1)_R$ denote the natural inclusion. The *middle convolution of* $\mathcal{F} \in \mathrm{Lisse}(N, n, \ell)$ *and* $\mathcal{L}_\chi$ is defined as

$$\mathrm{MC}_\chi(\mathcal{F}) := R^1(\overline{\mathrm{pr}}_2)_!(j_*(\mathrm{pr}_1^*(\mathcal{F}) \otimes \mathcal{L}_{\chi(X_2 - X_1)})) \in \mathrm{Lisse}(N, n, \ell)$$

(see [Kat96, § 8.3]).

For any $\mathcal{F} \in \mathrm{Lisse}(N, n, \ell)$ and any non-trivial character χ as above, let $\mathcal{F}_k$ denote the restriction of $\mathcal{F}$ to the geometric fibre $U_k = \mathbb{A}^1_k \backslash \{\alpha_1, \ldots, \alpha_n\}$ of $(\mathbb{A}^1 - (T_1, \ldots, T_n))_{S_{N,n,\ell}}$ defined by the homomorphism $\phi : S \to k$. Define χ_k to be the restriction of χ to $\mathbb{G}_{m,k}$ and let $j : U_k \to \mathbb{P}^1_k$ denote the inclusion. Then the following holds:

$$\mathrm{MC}_{\chi_k}(j_* \mathcal{F}_k)|_{U_k} = \mathrm{MC}_\chi(\mathcal{F})_k, \tag{2.2.1}$$

where on the left-hand side the middle convolution $\mathrm{MC}_{\chi_k}(\mathcal{F}_k)$ is defined as in § 1.1, and on the right-hand side the middle convolution is defined as in Definition 2.2.1 above (see [Kat96, Lemma 8.3.2]).

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2.3 The motivic interpretation of the middle convolution

In [Kat96, Theorems 8.3.5 and 8.4.1] the following result is proved.

Theorem 2.3.1. *Fix an integer $r \geqslant 0$. For a choice of $n(r+1)$ characters*

$$\chi_{a,i} : \mu_N(R) \to \bar{\mathbb{Q}}_\ell^\times, \quad i = 1, \ldots, n, \ a = 1, \ldots, r+1,$$

and a choice of r non-trivial characters

$$\rho_k : \mu_N(R) \to \bar{\mathbb{Q}}_\ell^\times, \quad k = 1, \ldots, r,$$

define a rank-one sheaf $\mathcal{L}$ on $\mathbb{A}(n, r+1)_R$ by setting

$$\mathcal{L} := \bigotimes_{a,i} \mathcal{L}_{\chi_{a,i}(X_a - T_i)} \bigotimes_k \mathcal{L}_{\rho_k(X_{k+1} - X_k)}.$$

Then the following hold.

(i) *The sheaf $\mathcal{K} := R^r(\mathrm{pr}_{r+1})_!(\mathcal{L})$ is mixed of integral weights in $[0, r]$. There exists a short exact sequence of lisse sheaves on $\mathbb{A}^1_{S_{N,n,\ell}} \setminus \{T_1, \ldots, T_n\}$,*

$$0 \to \mathcal{K}_{\leqslant r-1} \to \mathcal{K} \to \mathcal{K}_{=r} \to 0,$$

such that $\mathcal{K}_{\leqslant r-1}$ is mixed of integral weights less than or equal to $r-1$ and where $\mathcal{K}_{=r}$ is punctually pure of weight r.

(ii) *Let $\chi = \chi_N : \mu_N(R) \to \bar{\mathbb{Q}}_\ell^\times$ be the faithful character defined in the previous section and let $e(a,i)$, with $i = 1, \ldots, n$, $a = 1, \ldots, r+1$, and $f(k)$, with $k = 1, \ldots, r$, be integers such that*

$$\chi_{a,i} = \chi^{e(a,i)} \quad \text{and} \quad \rho_k = \chi^{f(k)}.$$

In the product space $\mathbb{G}_{m,R} \times \mathbb{A}(n, r+1)_R$, consider the hypersurface Hyp given by the equation

$$Y^N = \left(\prod_{a,i} (X_a - T_i)^{e(a,i)}\right)\left(\prod_{k=1,\ldots,r} (X_{k+1} - X_k)^{f(k)}\right),$$

and let

$$\pi : \mathrm{Hyp} \to \mathbb{A}^1_{S_{N,n,\ell}} \setminus \{T_1, \ldots, T_n\},$$
$$(Y, T_1, \ldots, T_n, X_1, \ldots, X_{r+1}) \mapsto (T_1, \ldots, T_n, X_{r+1}).$$

The group $\mu_N(R)$ acts on Hyp by permuting Y alone, inducing an action of $\mu_N(R)$ on $R^r\pi_!(\bar{\mathbb{Q}}_\ell)$. Then the sheaf $\mathcal{K}$ is isomorphic to the χ-component $(R^r\pi_!\bar{\mathbb{Q}}_\ell)^\chi$ of $R^r\pi_!(\bar{\mathbb{Q}}_\ell)$.

(iii) *For $a = 1, \ldots, r+1$, let*

$$\mathcal{F}_a = \mathcal{F}_a(X_a) := \bigotimes_{i=1,\ldots,n} \mathcal{L}_{\chi_{a,i}(X_a - T_i)} \in \mathrm{Lisse}(N, n, \ell).$$

Let

$$\begin{aligned} \mathcal{H}_0 &:= \mathcal{F}_1, \\ \mathcal{H}_1 &:= \mathcal{F}_2 \otimes \mathrm{MC}_{\rho_1}(\mathcal{H}_0), \\ &\ \vdots \\ \mathcal{H}_r &:= \mathcal{F}_{r+1} \otimes \mathrm{MC}_{\rho_r}(\mathcal{H}_{r-1}). \end{aligned}$$

Then $\mathcal{K}_{=r} = \mathcal{H}_r$.

2.4 Motivic interpretation for rigid G_2-sheaves

Let $\epsilon : \pi_1(\mathbb{G}_{m,R}) \to \mu_N(R)$ be the surjective homomorphism of § 2.1. By composition with ϵ, every character $\chi : \mu_N(R) \to \bar{\mathbb{Q}}_\ell^\times$ gives rise to a character of $\pi_1(\mathbb{G}_{m,R})$, again denoted by χ. For a sheaf $\mathcal{K} \in \mathrm{Lisse}(N, n, \ell)$ which is mixed of integral weights in $[0, r]$, let $W^r(\mathcal{K})$ denote the weight-r quotient of $\mathcal{K}$.

Theorem 2.4.1. *Let φ, η and $\mathcal{H}(\varphi, \eta)$ be as in Theorem 1.3.1. Let N denote the least common multiple of 2 and the orders of φ, η. Further, let $\chi = \chi_N : \mu_N(R) \to \bar{\mathbb{Q}}_\ell^\times$ be the character of order N which is defined in § 2.1, and let n_1 and n_2 be integers such that*

$$\varphi = \chi_k^{n_1} \quad \text{and} \quad \eta = \chi_k^{n_2} \text{ for } n_1, n_2 \in \mathbb{Z},$$

where χ_k is the restriction of χ to $\mathbb{G}_{m,k}$. Let $\mathrm{Hyp} = \mathrm{Hyp}(n_1, n_2)$ denote the hypersurface in $\mathbb{G}_{m,R} \times \mathbb{A}(2, 6+1)_R$ defined by the equation

$$Y^N = \left(\prod_{1 \leqslant a \leqslant 7; 1 \leqslant i \leqslant 2} (X_a - T_i)^{e(a,i)}\right)\left(\prod_{1 \leqslant k \leqslant 6} (X_{k+1} - X_k)^{f(k)}\right),$$

where the numbers $e(a, i)$ and the $f(k)$ are as given in the following two tables.

$e(1,1)$	$e(2,1)$	$e(3,1)$	$e(4,1)$	$e(5,1)$	$e(6,1)$	$e(7,1)$
$\frac{N}{2}$	0	$\frac{N}{2}$	0	$\frac{N}{2}$	0	$\frac{N}{2}$
$e(1,2)$	$e(2,2)$	$e(3,2)$	$e(4,2)$	$e(5,2)$	$e(6,2)$	$e(7,2)$
$\frac{N}{2} + n_1 + n_2$	$\frac{N}{2} - n_1$	0	$\frac{N}{2} + n_1 - n_2$	0	$\frac{N}{2} - n_1$	0

$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$
$\frac{N}{2} - n_1 - 2n_2$	$\frac{N}{2} + n_1 + 2n_2$	$\frac{N}{2} - n_1 - n_2$	$\frac{N}{2} + n_1 + n_2$	$\frac{N}{2} - n_1$	$\frac{N}{2} + n_1$

Let

$$\pi = \pi(n_1, n_2) : \mathrm{Hyp}(n_1, n_2) \to \mathbb{A}^1_{S_{N,n,\ell}} \setminus \{T_1, T_2\}$$

be given by $(Y, T_1, T_2, X_1, \ldots, X_7) \mapsto (T_1, T_2, X_7)$. Then the higher direct image sheaf $W^6[(R^6\pi_!\bar{\mathbb{Q}}_\ell)^\chi]$ is contained in $\mathrm{Lisse}(N, n, \ell)$. Moreover, for any algebraically closed field k whose characteristic does not divide ℓN, one has an isomorphism

$$\mathcal{H}(\varphi, \eta)|_{\mathbb{A}^1_k \setminus \{\alpha_1, \alpha_2\}} = (W^6[(R^6\pi_!\bar{\mathbb{Q}}_\ell)^\chi])|_{\mathbb{A}^1_k \setminus \{\alpha_1, \alpha_2\}}.$$

Proof. This is just a restatement of Theorem 2.3.1 in the setting of Theorem 1.3.1. The last formula follows from (2.2.1) and Theorem 2.3.1(iii). □

We now turn to the special case where $n_1 = n_2 = 0$, $N = 2$, $\alpha_1 = 0$ and $\alpha_2 = 1$. In this case, the higher direct image sheaves which occur in Theorem 2.4.1 can be expressed in terms of the cohomology of a smooth *and proper* map of schemes over $\mathbb{Q}$. This will be crucial in the next section.

Corollary 2.4.2. *Let $N = 2$ and $n_1 = n_2 = 0$, and let*

$$\mathrm{Hyp} = \mathrm{Hyp}(0, 0) \subseteq \mathbb{G}_{m,R} \times \mathbb{A}(2, 6+1)_R$$

be the associated hypersurface equipped with the structural morphism $\pi = \pi(0, 0) : \mathrm{Hyp} \to \mathbb{A}^1_S \setminus \{T_1, T_2\}$. Let $\pi_\mathbb{Q} : \mathrm{Hyp}_\mathbb{Q} \to \mathbb{A}^1_\mathbb{Q} \setminus \{0, 1\}$ denote the base-change of π induced by $T_1 \mapsto 0$ and $T_2 \mapsto 1$. Then the following hold.

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(i) *There exist a smooth and projective scheme* X *over* $\mathbb{A}^1_{\mathbb{Q}}\backslash\{0,1\}$ *and an open embedding of* $j:\mathrm{Hyp}_{\mathbb{Q}}\to X$ *such that*

$$D = X\backslash \mathrm{Hyp}_{\mathbb{Q}} = \bigcup_{i\in I} D_i$$

is a strict normal crossings divisor over $\mathbb{A}^1\backslash\{0,1\}$. *The involutory automorphism* σ *of* Hyp *(given by* $Y\mapsto -Y$*) extends to an automorphism* σ *of* X.

(ii) *Let* $\coprod_{i\in I} D_i$ *denote the disjoint union of the components of* D, *and let*

$$\pi_X : X\to \mathbb{A}^1_{\mathbb{Q}}\backslash\{0,1\} \quad \text{and} \quad \pi_{\coprod D_i} : \coprod_{i\in I} D_i \to \mathbb{A}^1_{\mathbb{Q}}\backslash\{0,1\}$$

denote the structural morphisms. Let $\mathcal{G} := W^6[(R^6\pi_!\bar{\mathbb{Q}}_\ell)^\chi]|_{\mathbb{A}^1_{\mathbb{Q}}\backslash\{0,1\}}$. *Then*

$$\mathcal{G} \simeq \Pi[\ker(R^6(\pi_X)_*(\bar{\mathbb{Q}}_\ell)\to R^6(\pi_{\coprod D_i})_*(\bar{\mathbb{Q}}_\ell))],$$

where Π *denotes the formal sum* $(\sigma-1)/2$.

Proof. Let $\Delta\subseteq \mathbb{A}^7_{X_1,\dots,X_7}$ be the divisor defined by the vanishing of

$$\prod_{i=1}^{6}(X_{i+1}-X_i)\prod_{i=1}^{7}X_i\prod_{i=1}^{7}(X_i-1). \tag{2.4.1}$$

Let $S=\mathbb{A}^1_{X_7}\backslash\{0,1\}$ and let $\mathbb{P}^6_S := \mathbb{P}^6_{X_0,\dots,X_6}\times S$. Consider the embedding

$$\mathbb{A}^7 \longrightarrow \mathbb{P}^6_S, \quad (x_1,\dots,x_7)\longmapsto ([1:x_1:\cdots:x_6],x_7).$$

Let

$$L := \mathbb{P}^6_S\backslash(\mathbb{A}^7\backslash\Delta) = \bigcup_i L_i,$$

where the L_i are the irreducible components of L. By Theorem 2.4.1, the hypersurface $\mathrm{Hyp}_{\mathbb{Q}}\subseteq \mathbb{G}_m\times(\mathbb{A}^7\backslash\Delta)$ is an unramified double cover of $\mathbb{A}^7\backslash\Delta$ defined by

$$Y^2=\prod_{i=1}^{6}(X_{i+1}-X_i)\prod_{i=1,3,5,7}X_i\prod_{i=1,2,4,6}(X_i-1). \tag{2.4.2}$$

Consider the embedding

$$\mathbb{G}_m\times(\mathbb{A}^7\backslash\Delta)\longrightarrow \mathbb{P}^1\times\mathbb{P}^6_S, \quad (y,x_1,\dots,x_7)\longmapsto([1,y],[1:x_1:\cdots:x_6],x_7)$$

and view Hyp as a subscheme of $\mathbb{P}^1\times\mathbb{P}^6_S$ via this embedding. Let $\overline{X}\subseteq\mathbb{P}^1\times\mathbb{P}^6_S$ be the Zariski closure of Hyp. By projecting onto $\mathbb{P}^6_S$, we obtain a ramified double cover $\alpha:\overline{X}\to\mathbb{P}^6_S$. The singularities of $\overline{X}$ are situated over the singularities of the ramification locus R of $\alpha:\overline{X}\to\mathbb{P}^6_S$, which is a subdivisor of L by (2.4.2).

There is a standard resolution of any linear hyperplane arrangement $L=\bigcup_i L_i\subseteq\mathbb{P}^n$ that is given in [ESV92, § 2]. By this we mean a birational map $\tau:\tilde{\mathbb{P}}^n\to\mathbb{P}^n$ which factors into several blow-ups and which has the following properties: the inverse image of L under τ is a strict normal crossings divisor in $\tilde{\mathbb{P}}^n$, and the strict transform of L is non-singular (see [ESV92, Claim in § 2]). The standard resolution depends only on the combinatorial intersection behaviour of the irreducible components L_i of L; therefore it can be defined for locally trivial families of hyperplane arrangements.

In our case, we obtain a birational map $\tau:\tilde{\mathbb{P}}^6_S\to\mathbb{P}^6_S$ such that $\tilde{L}:=\tau^{-1}(L)$ is a relative strict normal crossings divisor over S and such that the strict transform of L is smooth over S.

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Let $\tilde{\alpha}: \tilde{X} \to \tilde{\mathbb{P}}^6_S$ denote the pullback of the double cover α along τ, and let $\tilde{R}$ be the ramification divisor of $\tilde{\alpha}$. Then $\tilde{R}$ is a relative strict normal crossings divisor since it is contained in $\tilde{L}$. Write $\tilde{R}$ as a union $\bigcup_k \tilde{R}_k$ of irreducible components. By successively blowing up the (strict transforms of the) intersection loci $\tilde{R}_{k_1} \cap \tilde{R}_{k_2}$ with $k_1 < k_2$, one ends up with a birational map $\hat{f}: \hat{\mathbb{P}}^6_S \to \tilde{\mathbb{P}}^6_S$. Let $\hat{\alpha}: X \to \hat{\mathbb{P}}^6_S$ denote the pullback of the double cover $\tilde{\alpha}$ along $\hat{f}$. Then the strict transform of $\tilde{R}$ in $\hat{\mathbb{P}}^6_S$ is a disjoint union of smooth components. Moreover, since $\tilde{R}$ is a normal crossings divisor, the exceptional divisor of the map $\hat{f}$ has no components in common with the ramification locus of $\hat{\alpha}$. It follows that the double cover $\hat{\alpha}: X \to \hat{\mathbb{P}}^6_S$ is smooth over S and that $D = X \backslash \mathrm{Hyp}$ is a strict normal crossings divisor over S. This desingularization is obviously equivariant with respect to σ, which finishes the proof of assertion (i).

Let $\pi_X: X \to S$ denote the structural map (the composition of $\hat{\alpha}: X \to \hat{\mathbb{P}}^6_S$ with the natural map $\hat{\mathbb{P}}^6_S \to S$). There exists an $n \in \mathbb{N}$ such that the morphism π_X extends to a morphism $X_A \to \mathbb{A}^1_A \backslash \{0,1\}$ of schemes over $A := \mathbb{Z}[1/(2n)]$. We assume that n is big enough that $D_A := X_A \backslash \mathrm{Hyp}_A$ is a normal crossings divisor over $\mathbb{A}^1_A \backslash \{0,1\}$. In the following, we shall mostly omit the subscript $_A$ but will tacitly work in the category of schemes over A (making use of the fact that A is finitely generated over $\mathbb{Z}$, so as to be able to apply Deligne's results on the Weil conjectures). Let $\pi_D: D \to \mathbb{A}^1 \backslash \{0,1\}$ and $\pi_{\mathrm{Hyp}}: \mathrm{Hyp} \to \mathbb{A}^1 \backslash \{0,1\}$ be the structural morphisms. The excision sequence gives an exact sequence of sheaves

$$R^5(\pi_D)_*(\bar{\mathbb{Q}}_\ell) \to R^6(\pi_{\mathrm{Hyp}})_!(\bar{\mathbb{Q}}_\ell) \to R^6(\pi_X)_*(\bar{\mathbb{Q}}_\ell) \to R^6(\pi_D)_*(\bar{\mathbb{Q}}_\ell) \to R^7(\pi_{\mathrm{Hyp}})_!(\bar{\mathbb{Q}}_\ell). \tag{2.4.3}$$

By exactness and the work of Deligne (see [Del80]), the kernel of the map $R^6(\pi_{\mathrm{Hyp}})_!(\bar{\mathbb{Q}}_\ell) \to R^6(\pi_X)_*(\bar{\mathbb{Q}}_\ell)$ is an integral constructible sheaf which is mixed of weights less than or equal to 5. Since the mixed weights of $R^n\pi_!\bar{\mathbb{Q}}_\ell$ (respectively, $R^n\pi_*\bar{\mathbb{Q}}_\ell$) are at most (respectively, at least) n by [Del80], the exact sequence in (2.4.3) implies an isomorphism

$$W^6(R^6(\pi_{\mathrm{Hyp}})_!(\bar{\mathbb{Q}}_\ell)) \longrightarrow \mathrm{im}(R^6(\pi_{\mathrm{Hyp}})_!(\bar{\mathbb{Q}}_\ell) \to R^6(\pi_X)_*(\bar{\mathbb{Q}}_\ell)). \tag{2.4.4}$$

By the exactness of (2.4.3) and functoriality, one thus obtains the following chain of isomorphisms:

$$\begin{aligned} W^6(R^6(\pi_{\mathrm{Hyp}})_!(\bar{\mathbb{Q}}_\ell))^\chi &\simeq \mathrm{im}(R^6(\pi_{\mathrm{Hyp}})_!(\bar{\mathbb{Q}}_\ell) \to R^6(\pi_X)_*(\bar{\mathbb{Q}}_\ell))^\chi \\ &\simeq \ker(R^6(\pi_X)_*(\bar{\mathbb{Q}}_\ell) \to R^6(\pi_D)_*(\bar{\mathbb{Q}}_\ell))^\chi, \end{aligned} \tag{2.4.5}$$

where the superscript $^\chi$ stands for the χ-component of the higher direct image in the sense of Theorem 2.3.1 (the notion extends in an obvious way to X and to D).

We claim that the natural map

$$\ker(R^6(\pi_X)_*(\bar{\mathbb{Q}}_\ell) \to R^6(\pi_D)_*(\bar{\mathbb{Q}}_\ell))^\chi \longrightarrow \ker(R^6(\pi_X)_*(\bar{\mathbb{Q}}_\ell) \to R^6(\pi_{\coprod_i D_{A,i}})_*(\bar{\mathbb{Q}}_\ell))^\chi \tag{2.4.6}$$

is an isomorphism. To show this, we argue as follows. Since the sheaf $W^6(R^6(\pi_{\mathrm{Hyp}})_!(\bar{\mathbb{Q}}_\ell))^\chi$ is lisse (see Theorem 2.4.1), the isomorphisms given in (2.4.5) imply that

$$\ker(R^6(\pi_X)_*(\bar{\mathbb{Q}}_\ell) \to R^6(\pi_D)_*(\bar{\mathbb{Q}}_\ell))^\chi$$

is lisse. It follows from proper base change that

$$\ker(R^6(\pi_X)_*(\bar{\mathbb{Q}}_\ell) \to R^6(\pi_{\coprod_i D_{A,i}})_*(\bar{\mathbb{Q}}_\ell))^\chi$$

is lisse. Thus, by the specialization theorem (see [Kat90, 8.18.2]), in order to prove that the map in (2.4.6) is an isomorphism it suffices to verify this for any *closed* geometric point $\bar{s}$ of Hyp.

In view of (2.4.5), we thus have to show that

$$W^6(H^6_c(\mathrm{Hyp}_{\bar{s}}, \bar{\mathbb{Q}}_\ell))^\chi \simeq \ker\biggl(H^6(X_{\bar{s}}, \bar{\mathbb{Q}}_\ell) \to H^6\biggl(\coprod_i D_{\bar{s},i}, \bar{\mathbb{Q}}_\ell\biggr)\biggr)^\chi. \qquad (2.4.7)$$

It is proved in [Kat96, § 9.4.3] that the Leray spectral sequence

$$E_2^{p,q} = H^p(X_{\bar{s}}, R^q j_* \bar{\mathbb{Q}}_\ell) \Rightarrow H^{p+q}(\mathrm{Hyp}_{\bar{s}}, \bar{\mathbb{Q}}_\ell)$$

for the inclusion map $j_* : \mathrm{Hyp}_{\bar{s}} \to X_{\bar{s}}$ degenerates at 3. Together with the fact that d_2 has bidegree $(2, -1)$, this implies that

$$\begin{aligned} W^6(H^6(\mathrm{Hyp}_{\bar{s}}, \bar{\mathbb{Q}}_\ell)) &= E_3^{6,0} = E_2^{6,0}/E_2^{4,1} \\ &= H^6(X_{\bar{s}}, \bar{\mathbb{Q}}_\ell)/\mathrm{im}(H^4(X_{\bar{s}}, R^1 j_* \bar{\mathbb{Q}}_\ell)) \\ &= H^6(X_{\bar{s}}, \bar{\mathbb{Q}}_\ell)/\mathrm{im}\biggl(H^4\biggl(\coprod_i D_{\bar{s},i}, \bar{\mathbb{Q}}_\ell(-1)\biggr)\biggr), \end{aligned}$$

where in the last equality we have used $R^1 j_* \bar{\mathbb{Q}}_\ell = \bigoplus_i \bar{\mathbb{Q}}_\ell(-1)|_{(D_i)_{\bar{s}}}$ by purity (cf. [Gro72/73] and [Kat96, § 9.4.3]). Taking duals under Poincaré duality, we obtain

$$W^6(H^6_c(\mathrm{Hyp}_{\bar{s}}, \bar{\mathbb{Q}}_\ell)) \simeq \ker\biggl(H^6(X_{\bar{s}}, \bar{\mathbb{Q}}_\ell) \to H^6\biggl(\coprod_i D_{\bar{s},i}, \bar{\mathbb{Q}}_\ell\biggr)\biggr),$$

which implies (2.4.7) upon taking χ-parts. Hence the map in (2.4.6) is an isomorphism as claimed. So,

$$\begin{aligned} W^6(R^6\pi_{\mathrm{Hyp}_A *}(\bar{\mathbb{Q}}_\ell))^\chi &\simeq \ker(R^6(\pi_{X_A})_*(\bar{\mathbb{Q}}_\ell) \to R^6\pi_{\coprod_{i\in I} D_{A,i}*}(\bar{\mathbb{Q}}_\ell))^\chi \\ &= \Pi(\ker(R^6(\pi_{X_A})_*(\bar{\mathbb{Q}}_\ell) \to R^6(\pi_{\coprod_{i\in I} D_{A,i}})_*(\bar{\mathbb{Q}}_\ell))), \end{aligned}$$

where the last equality can be seen to be a tautology using the representation theory of finite (cyclic) groups. It follows that

$$\mathcal{G} = W^6(R^6(\pi_{\mathrm{Hyp}})_!(\bar{\mathbb{Q}}_\ell))^\chi|_{\mathbb{A}^1_{\mathbb{Q}}\setminus\{0,1\}} \simeq \Pi(\ker(R^6(\pi_X)_*(\bar{\mathbb{Q}}_\ell) \to R^6(\pi_{\coprod D_i})_*(\bar{\mathbb{Q}}_\ell))),$$

as claimed. □

3. Relative motives with motivic Galois group G_2

3.1 Preliminaries on motives

For an introduction to the theory of motives, including the basic definitions and properties, we refer the reader to the book of André [And04]. Let K and E denote fields of characteristic zero. Let $\mathcal{V}_K$ denote the category of smooth and projective varieties over K. If $X \in \mathcal{V}_K$ is purely d-dimensional, we denote by $\mathrm{Corr}^0(X, X)_E$ the E-algebra of codimension-d cycles in $X \times X$, modulo homological equivalence (the multiplication is given by the usual composition of correspondences). This notion extends by additivity to an arbitrary object $X \in \mathcal{V}_K$. A *Grothendieck motive with values in* E is then a triple $M = (X, p, m)$ where $X \in \mathcal{V}_K$, $m \in \mathbb{Z}$, and $p \in \mathrm{Corr}^0(X, X)_E$ is idempotent. To any $X \in \mathcal{V}_K$ one can associate a motive $h(X) = (X, \Delta(X), 0)$, called the *motive of* X, where $\Delta(X) \subseteq X \times X$ denotes the diagonal.

One also has the theory of *motives for motivated cycles* due to André [And96], where the ring of correspondences $\mathrm{Corr}^0(X, X)_E$ is replaced by a larger ring $\mathrm{Corr}^0_{\mathrm{mot}}(X, X)_E$ of *motivated cycles* by adjoining a certain homological cycle (the Lefschetz involution) to $\mathrm{Corr}^0(X, X)_E$ (see [And96, And04]).

The formal definition of a motivated cycle is as follows. For $X, Y \in \mathcal{V}_K$, let pr_X^{XY} denote the projection $X \times Y \to X$. Then a *motivated cycle* is an element $(\mathrm{pr}_X^{XY})_*(\alpha \cup *_{XY}(\beta)) \in H^*(X)$ where α and β are E-linear combinations of algebraic cycles on $X \times Y$ and $*_{XY}$ is the Lefschetz involution on $H^*(X \times Y)$ relative to the class of line bundles $\eta_{X\times Y} = [X] \otimes \eta_Y + \eta_X \otimes [Y]$ (with η_X and η_Y being classes of arbitrary ample line bundles L_X and L_Y in $H^2(X)$ and $H^2(Y)$, respectively). Define $\mathrm{Corr}^0_{\mathrm{mot}}(X, X)_E$ to be the ring of the motivated codimension-d cycles, in analogy to $\mathrm{Corr}^0(X, X)_E$. A motive for motivated cycles with values in E is then a triple $M = (X, p, m)$ where $X \in \mathcal{V}_K$, $m \in \mathbb{Z}$, and $p \in \mathrm{Corr}^0_{\mathrm{mot}}(X, X)_E$ is idempotent with respect to the composition of motivated cycles.

The category of motivated cycles is a neutral Tannakian category [And96, § 4]. Thus, by the Tannakian formalism (see [Del90]), every motive for motivated cycles M with values in E has attached to it an algebraic group G_M over E, called the *motivic Galois group* of M. Similarly, granting Grothendieck's standard conjectures, the category of motives has the structure of a Tannakian category. Thus, by the Tannakian formalism and by assuming the standard conjectures, every motive in the Grothendieck sense M has attached to it an algebraic group $\tilde{G}_M$, called the *motivic Galois group* of M. The following lemma and subsequent remark were communicated to the authors by André.

LEMMA 3.1.1. *Let $M = (X, p, n)$ be a motive for motivated cycles with motivic Galois group G_M. Assume that Grothendieck's standard conjectures hold. Then the motive M is defined by algebraic cycles, and the motivic Galois group $\tilde{G}_M$ in the Grothendieck sense coincides with the motivic Galois group G_M of motives for motivated cycles.*

Proof. The first assertion follows from the fact that the standard conjectures predict the algebraicity of the Lefschetz involution in the auxiliary spaces $X \times X \times Y$ that are used to define the projector p (see [Saa72]). The second assertion is a consequence of the following interpretation of G_M (and of $\tilde{G}_M$). The motivic Galois group for motivated cycles G_M is the stabilizer of all motivated cycles which appear in the realizations of all submotives of the mixed tensors $M^{\otimes n} \otimes (M^*)^{\otimes n}$, where M^* denotes the dual of M (this can be seen using the arguments in [And04, § 6.3]). Similarly, under the assumption of the standard conjectures, the motivic Galois group $\tilde{G}_M$ is the stabilizer of all algebraic cycles which appear in the realizations of submotives of the mixed tensors of M; see [And04, § 6.3]. Under the standard conjectures these spaces coincide, so $G_M = \tilde{G}_M$. □

Remark 3.1.2. The above lemma can be strengthened or expanded as follows. It is possible to define unconditionally and purely in terms of algebraic cycles a group which, under the standard conjectures, will indeed be the motivic Galois group of the motive $X = (X, \mathrm{Id}, 0)$, where X is a smooth projective variety. Specifically, let G_X^{alg} be the closed subgroup of $\prod_i \mathrm{GL}(H^i(X)) \times \mathbb{G}_m$ that fixes the classes of algebraic cycles on powers of X (viewed as elements of $H(X)^{\otimes n} \otimes \mathbb{Q}(r)$, with the factor $\mathbb{G}_m$ acting on $\mathbb{Q}(1)$ by homotheties). Then the motivic Galois group G_X is related to G_X^{alg} by (cf. [And04, 9.1.3]) $G_X = \mathrm{im}(G_{X\times Y}^{\mathrm{alg}} \to G_X^{\mathrm{alg}})$ for a suitable projective smooth variety Y. Under the standard conjectures, one may take Y to be a point.

3.2 Results on families of motives

It is often useful to consider variations of motives over a base. Suppose that one is given the following data:

(i) a smooth and geometrically connected variety S over a field $K \subseteq \mathbb{C}$;

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(ii) smooth and projective S-schemes X and Y of relative dimensions d_X and d_Y, respectively, equipped with invertible ample line bundles L_X and L_Y, with $*$ denoting the Lefschetz involution relative to $[(L_{X/S})_s] \otimes [Y_s] + [X_s] \otimes [(L_{Y/S})_s]$;

(iii) two $\mathbb{Q}$-linear combinations Z_1 and Z_2 of integral codimension-$(d_X + d_Y)$ subvarieties in $X \times_S X \times_S Y$ which are flat over S and such that for one (and thus for all) $s \in S(\mathbb{C})$, the class

$$p_s := (\mathrm{pr}^{X_s \times X_s \times Y_s}_{X_s \times X_s})_*([(Z_1)_s] \cup *[(Z_2)_s]) \in H^{2d_X}(X_s \times X_s)(d_x) \subseteq \mathrm{End}(H^*(X_s))$$

satisfies $p_s \circ p_s = p_s$.

(iv) an integer j.

Then the assignment $s \mapsto (X_s, p_s, j)$, $s \in S(\mathbb{C})$, defines a *family of motives* in the sense of [And96, § 5.2]. The following result is due to André (see [And96, Theorem 5.2 and § 5.3]).

Theorem 3.2.1. *Let $s \mapsto (X_s, p_s, j)$, $s \in S(\mathbb{C})$, be a family of motives with coefficients in E, and let $H_E(M_s) := p_s(H^*_B(X_s, E)(j))$ be the E-realization of M_s, where $H^*_B(X_s, E)$ denotes the singular cohomology ring of $X_s(\mathbb{C})$. Then there exists a meagre subset $\mathrm{Exc} \subseteq S(\mathbb{C})$ and a local system of algebraic groups $G_s \leqslant \mathrm{Aut}(H_E(M_s))$ on $S(\mathbb{C})$ such that the following hold.*

(i) *$G_{M_s} \subseteq G_s$ for all $s \in S(\mathbb{C})$.*

(ii) *$G_{M_s} = G_s$ if and only if $s \notin \mathrm{Exc}$.*

(iii) *G_s contains the image of a subgroup of finite index of $\pi_1^{\mathrm{top}}(S(\mathbb{C}), s)$.*

(iv) *Let $\mathfrak{g}'_s$ denote the Lie algebra of G_s, and let $\mathfrak{h}_s$ denote the Lie algebra of the Zariski closure of the image of $\pi_1^{\mathrm{top}}(S(\mathbb{C}), s)$. Then the Lie algebra $\mathfrak{h}_s$ is an ideal in $\mathfrak{g}'_s$.*

Moreover, if S is an open subscheme of $\mathbb{P}^n$ which is defined over a number field K, then $\mathrm{Exc} \cap \mathbb{P}^n(K)$ is a thin subset of $\mathbb{P}^n(K)$ (thin in the sense of [Ser89]).

3.3 Motives with motivic Galois group G_2

An algebraic group G defined over a subfield of $\bar{\mathbb{Q}}_\ell$ is said to be *of type* G_2 if the group of $\bar{\mathbb{Q}}_\ell$-points $G(\bar{\mathbb{Q}}_\ell)$ is isomorphic to the simple exceptional algebraic group $G_2(\bar{\mathbb{Q}}_\ell)$ (see [Bor91] for the definition of the algebraic group G_2). It is the aim of this section to prove the existence of motives for motivated cycles having a motivic Galois group of type G_2.

We start with the situation of Corollary 2.4.2. Let $S := \mathbb{A}^1_{\mathbb{Q}} \setminus \{0, 1\}$, let $\pi_{\mathbb{Q}} : \mathrm{Hyp}_{\mathbb{Q}} \to S$ be as in Corollary 2.4.2, and let $\pi_X : X \to S$ be the strict normal crossings compactification of Hyp given by Corollary 2.4.2. Let

$$\mathcal{G} \simeq \Pi[\ker(R^6(\pi_X)_*(\bar{\mathbb{Q}}_\ell) \to R^6(\pi_{\coprod D_i})_*(\bar{\mathbb{Q}}_\ell))] \qquad (3.3.1)$$

be as in Corollary 2.4.2, where the D_i are the components of the normal crossings divisor $D = X \setminus \mathrm{Hyp}$ over S. We want to use the right-hand side of this isomorphism to define a family of motives $(N_s)_{s \in S(\mathbb{C})}$ for motivated cycles such that the $\bar{\mathbb{Q}}_\ell$-realization of N_s coincides naturally with the stalk $\mathcal{G}_s$ of $\mathcal{G}$. This is done in three steps.

- Let

$$\psi_s^* : H^*(X_s, \bar{\mathbb{Q}}_\ell) \longrightarrow H^*\Big(\coprod_i D_{s,i}, \bar{\mathbb{Q}}_\ell\Big)$$

be the map induced by the tautological map $\psi_s := \coprod_i D_{s,i} \to X_s$. Let $\Gamma_{\psi_s} \in \mathrm{Corr}^0_{\mathrm{mot}}(X_s, \coprod_i D_{s,i})_{\mathbb{Q}}$ be the graph of ψ_s. Note that Γ_{ψ_s} can be viewed as a morphism of motives

$$\Gamma_{\psi_s} = \psi_s^* : h(X_s) \longrightarrow h\bigg(\coprod_i D_{s,i}\bigg).$$

Since the category of motives for motivated cycles is abelian (see [And96, § 4]), there exists a kernel motive

$$K_s = (X_s, p_s, 0), \quad p_s \in \mathrm{Corr}^0_{\mathrm{mot}}(X_s, X_s)_{\mathbb{Q}},$$

of the morphism ψ_s^* such that

$$p_s(H^*(X_s, \bar{\mathbb{Q}}_\ell)) = \ker\bigg(H^*(X_s, \bar{\mathbb{Q}}_\ell) \to H^*\bigg(\coprod_i D_{s,i}, \bar{\mathbb{Q}}_\ell\bigg)\bigg).$$

- The Künneth projector $\pi^6_{X_s} : H^*(X_s) \to H^i(X_s)$ is also contained in $\mathrm{Corr}^0_{\mathrm{mot}}(X_s, X_s)_{\mathbb{Q}}$ (see [And96, Proposition 2.2]).
- Let Π_s denote the following projector in $\mathrm{Corr}^0_{\mathrm{mot}}(X_s, X_s)_{\mathbb{Q}}$:

$$\Pi_s := \tfrac{1}{2}(\Delta(X_s) - \Gamma_{\sigma_s}),$$

where $\Delta(X_s) \subseteq X_s \times X_s$ denotes the diagonal of X_s and $\Gamma_{\sigma_s} \leqslant X_s \times X_s$ denotes the graph of σ_s. By construction, the action of Π_s on $H^6(X_s)$ is the same as the action induced by the idempotent $\Pi = (1-\sigma)/2$ which occurs in Corollary 2.4.2.

By (3.3.1) one has

$$\mathcal{G}_s = \Pi\bigg[\ker\bigg(H^6(X_s, \bar{\mathbb{Q}}_\ell) \to H^6\bigg(\coprod_i D_{s,i}, \bar{\mathbb{Q}}_\ell\bigg)\bigg)\bigg] \quad \forall s \in S(\mathbb{C}).$$

Thus, by combining the above arguments, one sees that the stalk $\mathcal{G}_s$ is the $\bar{\mathbb{Q}}_\ell$-realization $H_{\bar{\mathbb{Q}}_\ell}(N_s)$ of the motives

$$N_s := (X_s, \Pi_s \cdot p_s \cdot \pi^6_{X_s}, 0) \quad \text{with } \Pi_s \cdot p_s \cdot \pi^6_{X_s} \in \mathrm{Corr}^0_{\mathrm{mot}}(X_s, X_s)_{\mathbb{Q}}.$$

We set

$$M_s := N_s(3) = (X_s, \Pi_s \cdot p_s \cdot \pi^6_{X_s}, 3).$$

Theorem 3.3.1. *The motives M_s, $s \in S(\mathbb{C})$, form a family of motives such that for any $s \in S(\mathbb{Q})$ outside a thin set, the motive M_s has a motivic Galois group of type G_2.*

Proof. That the motives $(N_s)_{s\in S(\mathbb{C})}$ form a family of motives (in the sense of § 3.2) can be seen from the following arguments. Let $\Gamma_\sigma \subseteq X \times_S X$ be the graph of the automorphism σ and let $\Delta(X) \subseteq X \times_S X$ be the diagonal. By Corollary 2.4.2, the projectors Π_s arise from the $\mathbb{Q}$-linear combination of schemes $(\Delta(X) - \Gamma_\sigma)/2$ over S via base change to s. The Künneth projector $\pi^6_{X_s} \in \mathrm{Corr}^0_{\mathrm{mot}}(X_s, X_s)$ is invariant under the action of $\pi_1(S)$. It therefore follows from the theorem of the fixed part, as in [And96, § 5.1], that $\pi^6_{X_s}$ arises from the restriction of the Künneth projector $\pi^6_{\bar{X}}$, where $\bar{X}$ denotes a normal crossings compactification over $\mathbb{Q}$ of the morphism $\pi_X : X \to S$ (which exists by [Hir64]). By [And96, Proposition 2.2], the projector $\pi^6_{\bar{X}}$ is a motivated cycle. Since this cycle gives rise to the Künneth projector $\pi^6_{X_s}$ on one fibre via restriction, we can use the local triviality of the family X/S to show that the restriction of $\pi^6_{\bar{X}} \in \mathrm{Corr}^0(\bar{X} \times \bar{X})$ to $X \times_S X$ gives rise to a family of motives $(X_s, \pi^6_{X_s}, 0)$. A similar argument applies to the projectors p_s. Therefore the motives $M_s = N_s(3)$, $s \in S(\mathbb{C})$, indeed form a family of motives.

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Let $\mathcal{G}^{\mathrm{an}}$ be the local system on $S(\mathbb{C})$ defined by the composition of the natural map $\pi_1^{\mathrm{top}}(S(\mathbb{C}), s) \to \pi_1(S, s)$ with the monodromy representation of $\mathcal{G}$. By the comparison isomorphism between singular and étale cohomology, the local system $\mathcal{G}^{\mathrm{an}}$ coincides with the local system which is defined by the singular $\bar{\mathbb{Q}}_\ell$-realizations $H_{\bar{\mathbb{Q}}_\ell}(N_s)$ of the above family $(N_s)_{s\in S(\mathbb{C})}$. It follows from Theorem 2.4.1 that $\mathcal{G}|_{\mathbb{A}^1_{\mathbb{C}}\setminus\{0,1\}} \simeq \mathcal{H}(\mathbf{1},\mathbf{1})|_{\mathbb{A}^1_{\mathbb{C}}\setminus\{0,1\}}$, where $\mathcal{H}(\mathbf{1},\mathbf{1})$ is as in Theorem 1.3.1. It then follows from Theorem 1.3.1(i) that the image of $\pi_1^{\mathrm{top}}(S(\mathbb{C}), s) \leqslant \pi_1(S, s)$ in $\mathrm{Aut}(H_{\bar{\mathbb{Q}}_\ell}(N_s)) \simeq \mathrm{GL}_7(\bar{\mathbb{Q}}_\ell)$ under the monodromy map is Zariski dense in the group $G_2(\bar{\mathbb{Q}}_\ell)$. Since the algebraic group G_2 is connected, the Zariski closure of the image of every subgroup of finite index of $\pi_1^{\mathrm{top}}(S(\mathbb{C}))$ coincides also with $G_2(\bar{\mathbb{Q}}_\ell)$.

By Theorem 3.2.1(i) and (ii), and the fact that S is open in $\mathbb{P}^1$, there exists a local system $(G_s)_{s\in S(\mathbb{C})}$ of algebraic groups with $G_s \leqslant \mathrm{Aut}(H_{\bar{\mathbb{Q}}_\ell}(M_s))$ such that the motivic Galois group G_{M_s} is contained in G_s and there exists a thin subset $\mathrm{Exc} \subseteq \mathbb{Q}$ such that if $s \in \mathbb{Q}\setminus\mathrm{Exc}$, then $G_s = G_{M_s}$. By Theorem 3.2.1(iii), G_s contains a subgroup of finite index of the image of $\pi_1(S(\mathbb{C}), s)$. Thus, by what was said above, the group G_s contains the group G_2 for all $s \in S(\mathbb{C})$. Let $\mathfrak{g}_2$ denote the Lie algebra of the group G_2. Let $\mathfrak{g}'_s$ denote the Lie algebra of the group G_s. By Theorem 3.2.1(iv), the Lie algebra $\mathfrak{g}_2$ is an ideal of $\mathfrak{g}'_s$. From this and the fact that $N_{\mathrm{GL}_7}(G_2) = \mathbb{G}_m \times G_2$ (where $\mathbb{G}_m$ denotes the subgroup of scalars of GL_7), it follows that $G_{N_s} \leqslant \mathbb{G}_m \times G_2$ for all $s \in S(\mathbb{C})$. The representation ρ_{N_s} of G_{N_s}, which belongs to the motive N_s under the Tannaka correspondence, is therefore a tensor product $\chi \otimes \rho$ where $\chi : G_{N_s} \to \mathbb{G}_m$ is a character and $\rho : G_{N_s} \to \mathrm{GL}_7$ has values in $G_2 \leqslant \mathrm{GL}_7$. Let A_s denote the dual of the motive which belongs to χ under the Tannaka correspondence. Then $G_{N_s\otimes A_s} = G_2$ for all $s \in \mathbb{Q}\setminus\mathrm{Exc}$.

We claim that for $s \in \mathbb{Q}\setminus\mathrm{Exc}$, the motive A_s is the motive $(\mathrm{Spec}(s), \mathrm{Id}, 3)$. The Galois representation which is associated to the motive N_s is equivalent to that of the stalk of $\mathcal{G}$ at s (viewed as a $\bar{\mathbb{Q}}$-point) and is therefore pure of weight 6. By [KW03, Theorem 3.1], any rank-one motive over $\mathbb{Q}$ is a Tate twist of an Artin motive. Therefore, the ℓ-adic realization of any rank-one motive over $\mathbb{Q}$ is a power of the cyclotomic character with a finite character. For $G_{N_s\otimes A_s}$ to be contained in G_2, the ℓ-adic realization of A_s has to be of the form $\epsilon \otimes \chi_\ell^3$, where ϵ is of order at most 2. But if the order of ϵ is equal to 2, we derive a contradiction to Theorem A.1 in the appendix. It follows that A_s is the motive $(\mathrm{Spec}(s), \mathrm{Id}, 3)$ and that the motivic Galois group of $M_s = N_s \otimes A_s = N_s(3)$ is of type G_2. □

Remark 3.3.2. Under the hypothesis of the standard conjectures, Theorem 3.3.1 and Lemma 3.1.1 imply the existence of Grothendieck motives whose motivic Galois group is of type G_2. Moreover, it follows from Remark 3.1.2 that, independent of the standard conjectures, there is a projective smooth variety X over $\mathbb{Q}$ such that the group G_X^{alg} (which is defined in Remark 3.1.2) has a quotient G_2.

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Appendix. On Galois representations with values in G_2

Michael Dettweiler and Nicholas M. Katz

Let ℓ be a prime and let $\mathcal{H}(\mathbf{1},\mathbf{1})$ be the cohomologically rigid $\bar{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1_{\bar{\mathbb{Q}}}$ of rank 7 which is given in Theorem 1.3.1 of the main article (whose notation we adopt in this appendix). The restriction of $\mathcal{H}(\mathbf{1},\mathbf{1})$ to $\mathbb{A}^1_{\bar{\mathbb{Q}}}\backslash\{0,1\}$ is a lisse $\bar{\mathbb{Q}}_\ell$-sheaf whose monodromy is dense in the exceptional algebraic group $G_2(\bar{\mathbb{Q}}_\ell)$ and whose local monodromy at $0, 1$ and ∞ is of type

$$\mathbf{1}^3 \oplus (-\mathbf{1})^4, \quad \mathbf{U}(2)^2 \oplus \mathbf{U}(3) \quad \text{and} \quad \mathbf{U}(7), \quad \text{respectively.} \tag{A.1}$$

By [Kat96, Theorem 5.5.4], there exists a lisse $\mathbb{Q}_\ell$-sheaf $\mathcal{G}_\ell$ on $S_\ell := \mathbb{A}^1_{R_\ell}\backslash\{0,1\}(R_\ell = \mathbb{Z}[1/2\ell])$ which, after the base change $R_\ell \to \bar{\mathbb{Q}}$ and the extension of scalars $\mathbb{Q}_\ell \to \bar{\mathbb{Q}}_\ell$ on the coefficients, becomes the restriction of $\mathcal{H}(\mathbf{1},\mathbf{1})$ to $\mathbb{A}^1_{\bar{\mathbb{Q}}}\backslash\{0,1\}$. (The construction of $\mathcal{G}_\ell$ is given below.) The monodromy representation of the Tate twisted sheaf $\mathcal{G}_\ell(3)$ is denoted by

$$\rho_\ell : \pi_1(S_\ell) \longrightarrow \mathrm{GL}_7(\mathbb{Q}_\ell).$$

Let $s_0 \in S_\ell(\mathbb{Q})$. The morphism $s_0 \to S_\ell$ induces a homomorphism $\alpha : \pi_1(s_0, \bar{s}_0) \to \pi_1(S_\ell, \bar{s}_0)$. Since $\pi_1(s_0, \bar{s}_0)$ is isomorphic to $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, we can view α as a homomorphism $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \pi_1(S_\ell, \bar{s}_0)$. The *specialization of ρ_ℓ to s_0* is then defined as the composition

$$\rho_\ell^{s_0} := \rho_\ell \circ \alpha : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_7(\mathbb{Q}_\ell).$$

Indeed, we may view s_0 as a point of S_ℓ with values in the ring $\mathbb{Z}[1/(2\ell)][s_0, 1/s_0, 1/(s_0-1)]$, so that $\rho_\ell^{s_0}$ is in fact unramified except possibly at $2, \ell$ and those primes p such that either s_0 or $s_0 - 1$ fails to be a p-adic unit. Our main result is the following theorem.

Theorem A.1.

(i) *The representation ρ_ℓ has values in $G_2(\mathbb{Q}_\ell)$.*

(ii) *Let a and b be two coprime integers which each have an odd prime divisor that is different from ℓ, and let $s_0 := 1 + a/b$. Then the image of $\rho_\ell^{s_0}$ is Zariski dense in $G_2(\mathbb{Q}_\ell)$.*

For $s_0 \in S_\ell(\mathbb{Q})$, let M_{s_0} be the motive for motivated cycles which appears in Theorem 3.3.1. By construction, the above Galois representation $\rho_\ell^{s_0}$ is the Galois representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the ℓ-adic realization of the motive M_{s_0} (see the proof of Corollary A.2). As a corollary of Theorem A.1, we find an explicit way of obtaining motives with motivic Galois group of type G_2.

Corollary A.2.

(i) *Let $s_0 = 1 + a/b$ be as in Theorem A.1. Then the motive for motivated cycles M_{s_0} has a motivic Galois group of type G_2.*

(ii) *Let (a, b) and (a', b') be pairs of squarefree odd coprime integers, each greater than or equal to 3, such that $(a, b) \neq (a', b')$. Let $s_0 = 1 + a/b$ and $s_0' = 1 + a'/b'$. For any prime ℓ not dividing the product $aba'b'$, the ℓ-adic representations $\rho_\ell^{s_0}$ and $\rho_\ell^{s_0'}$ are not isomorphic. In particular, the motives M_{s_0} and $M_{s_0'}$ are not isomorphic.*

(iii) *There exist infinitely many non-isomorphic motives M_{s_0} whose motivic Galois group is of type G_2.*

The proofs of Theorem A.1 and Corollary A.2 are given below. First let us recall the construction of $\mathcal{G}_\ell$. The group $\mu_2(R_\ell)$ of the second roots of unity of $R_\ell = \mathbb{Z}[1/(2\ell)]$ acts on the étale covers f_1 and f_2 of $S_\ell = \mathbb{A}^1_{R_\ell} \setminus \{0,1\}$ defined by the equations $y^2 = x$ and $y^2 = x - 1$, respectively. The covers f_1 and f_2 therefore define surjections

$$\eta_i : \pi_1(S_\ell) \longrightarrow \mu_2(R_\ell) \quad \text{for } i = 1, 2.$$

The composition of the embedding $\chi : \mu_2(R_\ell) \to \mathbb{Q}_\ell$ with η_i, for $i = 1, 2$, defines lisse $\mathbb{Q}_\ell$-sheaves $\mathcal{L}_{\chi(x)}$ and $\mathcal{L}_{\chi(x-1)}$ on S_ℓ. Let $j : S_\ell \to \mathbb{A}^1_{R_\ell}$ denote the tautological inclusion, and let

$$\mathcal{F}_3 = \mathcal{F}_5 = \mathcal{F}_7 := j_*(\mathcal{L}_{\chi(x)}) \quad \text{and} \quad \mathcal{F}_2 = \mathcal{F}_4 = \mathcal{F}_6 := j_*(\mathcal{L}_{\chi(x-1)}).$$

Let $\mathcal{H}_0 := j_*(\mathcal{L}_{\chi(x)} \otimes \mathcal{L}_{\chi(x-1)})$ and define inductively

$$\mathcal{H}_i := j_*(\mathcal{F}_{i+1} \otimes j^*(\mathrm{MC}_\chi(\mathcal{H}_{i-1}))) \quad \text{for } i = 1, \ldots, 6, \tag{A.2}$$

where $\mathrm{MC}_\chi(\mathcal{H}_i)$ is as defined in [Kat96, § 4.3] (see also Remark A.5 below). We remark that on each geometric fibre $\bar{S}_\ell$ of S_ℓ, one has

$$\mathrm{MC}_\chi(\mathcal{H}_{i-1})|_{\bar{S}_\ell} = \mathrm{MC}_\chi(\mathcal{H}_{i-1}|_{\bar{S}_\ell}), \tag{A.3}$$

where MC_χ on the right-hand side is the middle convolution functor defined in [Kat96, ch. 5], (or in § 1.1 of the main part of this article). We then define $\mathcal{G}_\ell$ to be the lisse sheaf $\mathcal{H}_6|_{S_\ell}$. It follows from the construction of $\mathcal{H}(\mathbf{1}, \mathbf{1})$ in the proof of Theorem 1.3.1 and from formula (A.3) that

$$(\mathcal{G}_\ell \otimes \bar{\mathbb{Q}}_\ell)|_{\mathbb{A}^1_{\bar{\mathbb{Q}}} \setminus \{0,1\}} = \mathcal{H}(\mathbf{1}, \mathbf{1})|_{\mathbb{A}^1_{\bar{\mathbb{Q}}} \setminus \{0,1\}}. \tag{A.4}$$

Remark A.3. By [Kat96, 5.5.4 part (3)], the weight of $\mathcal{G}_\ell$ is equal to 6, which implies that the Tate twist $\mathcal{G}_\ell(3)$ has weight zero. By [Kat96, 5.5.4 part (2)], the restriction of $\mathcal{H}_6$ to any geometric fibre is irreducible and cohomologically rigid of the same type of local monodromy. Moreover, by the specialization theorem (see [Kat96, 4.2.4]), the geometric monodromy group (of the restriction of $\mathcal{G}_\ell$) on any geometric fibre of S_ℓ is also Zariski dense in G_2.

Proposition A.4. *Let s_0 be a rational number that is not* 0 *or* 1, *and let p be an odd prime number different from ℓ. Then the following hold.*

(i) *If $\mathrm{ord}_p(s_0) < 0$, then the restriction of $\rho_\ell^{s_0}$ to the inertia subgroup $I_p \leqslant \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ at p factors through the tame inertia group at p, $I_p^{\mathrm{tame}} \cong \hat{\mathbb{Z}}^{(\mathrm{not}\ p)}(1)$, and is an indecomposable unipotent block of length* 7.

(ii) *If $\mathrm{ord}_p(s_0 - 1) > 0$, then the restriction of $\rho_\ell^{s_0}$ to I_p factors through the tame inertia group $I_p^{\mathrm{tame}} \cong \hat{\mathbb{Z}}^{(\mathrm{not}\ p)}(1)$ and is the direct sum of an indecomposable unipotent block of length* 3 *and of two indecomposable unipotent blocks of length* 2.

(iii) *If $\mathrm{ord}_p(s_0) > 0$, then I_p acts tamely, by automorphisms of order at most* 2.

(iv) *If both s_0 and $s_0 - 1$ are p-adic units, then I_p acts trivially.*

Proof of Proposition A.4. We first prove (i). Let W_p denote the ring of Witt vectors of an algebraic closure of $\mathbb{F}_p$. Let t be the standard parameter on $\mathbb{A}^1_{R_\ell}$, let $z := 1/t$ denote the parameter at infinity, and consider the formal punctured disc $\Delta_p := \mathrm{Spec}\,(W_p[[z]][1/z])$. Since $\mathrm{Spec}\,(W_p)$ is simply connected, one knows that $\pi_1(\Delta_p)$ is the group $\hat{\mathbb{Z}}^{(\mathrm{not}\ p)}(1)$ (this follows from Abhyankar's

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lemma; see [Kat80, Example on p. 120]). In more concrete terms, all finite connected étale covers of Δ_p are obtained by taking the Nth root of z for some N prime to p. We can read the effect of a topological generator of this group in our representation after extension of scalars from W_p to the complex numbers, so we know that a topological generator gives a single unipotent block of size 7 (since this is the local monodromy of $\mathcal{G}_\ell$ around ∞ on every geometric fibre of S_ℓ over R_ℓ). If we specialize z to a non-zero point z_0 (here $1/s_0$) in the maximal ideal pW_p of W_p, the resulting ring homomorphism $W_p[[z]][1/z] \to K_p := \mathrm{Frac}(W_p)$ induces a homomorphism of fundamental groups $I_p \to \pi_1(\Delta_p)$, which, in view of Abhyankar's lemma, factors through $I_p^{\mathrm{tame}} \cong \hat{\mathbb{Z}}^{(\text{not } p)}(1)$. Identifying both source and target of this map $I_p^{\mathrm{tame}} \to \pi_1(\Delta_p)$ with the group $\hat{\mathbb{Z}}^{(\text{not } p)}(1)$, we see that this map is non-zero (simply because in W_p, z_0 does not have an Nth root for any N not dividing $\mathrm{ord}_p(z_0)$). So after pullback to such a point, the specialized representation of I_p^{tame} remains unipotent and indecomposable (because in characteristic zero, if A is a unipotent automorphism of a finite-dimensional vector space, then A and any non-zero power of A have the same Jordan decomposition). To prove (ii) and (iii), we repeat these arguments, but now with z taken to be $t-1$ and t, respectively, and using the fact that for our sheaf $\mathcal{G}_\ell$, local monodromy around 1 (respectively, around 0) is unipotent of the asserted shape (respectively, involutory). Assertion (iv) was already noted at the beginning of the appendix. □

Proof of Theorem A.1. For fields of cohomological dimension less than or equal to 2 (and hence for ℓ-adic fields), it is known that there exists only one form of the algebraic group G_2, namely the split form (see [Kne65, Ser95]). It therefore follows from Theorem 1.3.1 and formula (A.4) that the geometric monodromy of $\mathcal{G}_\ell(3)$ is Zariski dense in the group $G_2(\mathbb{Q}_\ell)$. Poincaré duality, applied in each step of the convolution construction of $\mathcal{G}_\ell$ given in (A.2), implies that the sheaf $\mathcal{G}_\ell(3)$ is orthogonally self-dual and hence that ρ_ℓ respects a non-degenerate orthogonal form. The normalizer of $G_2(\mathbb{Q}_\ell)$ in the orthogonal group $\mathrm{O}_7(\mathbb{Q}_\ell)$ consists of the scalars $\langle \pm 1\rangle$ only. Since the representation ρ_ℓ has degree 7, it follows that there exists a character $\epsilon_\ell : \pi_1(S_\ell) \to \langle \pm 1\rangle$ such that $\rho_\ell \otimes \epsilon_\ell$ has values in $G_2(\mathbb{Q}_\ell)$.

We have to show that ϵ_ℓ is trivial. To do this, we argue as follows. Because on each fibre the geometric monodromy group is G_2 by Remark A.3, the character ϵ_ℓ is actually a character of $\pi_1(\mathbb{Z}[1/(2\ell)])$. As ℓ varies, the characters ϵ_ℓ form a compatible system (this follows from the compatibility of ρ_ℓ which, in turn, comes from the compatibility of MC_χ proved in [Kat96, 5.5.4(4)]). So, taking ℓ to be 2, one sees that ϵ_ℓ is a quadratic character whose conductor is a power of 2. Given the structure of 2-adic units as the product of $\langle \pm 1\rangle$ with the pro-cyclic group $1+4\mathbb{Z}_2$, one can see that any homomorphism from this group to $\langle \pm 1\rangle$ actually factors through the units modulo 8. Therefore it suffices to show that for p in a set of primes whose reduction modulo 8 meets each non-trivial class of units mod 8, and for one $t \in \mathbb{F}_p \setminus \{0,1\}$), the Frobenius element $\rho_\ell(\mathrm{Frob}_{p,t})$ is contained in $G_2(\mathbb{Q}_\ell)$.

Since the weight of ρ_ℓ is 0 by Remark A.3, the eigenvalues of $\rho_\ell(\mathrm{Frob}_{p,t})$ (with $p \neq 2, \ell$) are Weil numbers of complex absolute value equal to 1. Moreover, any Frobenius element is contained in either $G_2(\mathbb{Q}_2)$ or the coset $-G_2(\mathbb{Q}_\ell)$. Since any semi-simple element in $G_2(\bar{\mathbb{Q}}_\ell) \leqslant \mathrm{GL}_7(\bar{\mathbb{Q}}_\ell)$ is conjugate to a diagonal matrix of the form $\mathrm{diag}(x, y, xy, 1, (xy)^{-1}, y^{-1}, x^{-1})$, it follows (from elementary arguments on trigonometric functions) that the trace of $\rho_\ell(\mathrm{Frob}_{p,t})$ lies in the interval $[-2,7]$ if $\rho_\ell(\mathrm{Frob}_{p,t})$ is contained in $G_2(\mathbb{Q}_\ell)$, or in the interval $[-7,2]$ if $\rho_\ell(\mathrm{Frob}_{p,t})$ is contained in $-G_2(\mathbb{Q}_\ell)$. By compatibility and the discussion above, it therefore suffices to show that for p in a set of primes whose reduction modulo 8 meets each non-trivial class of units mod 8 and for some $t \in \mathbb{F}_p \setminus \{0,1\}$, the trace of $\rho_\ell(\mathrm{Frob}_{p,t})$ lies in the left open interval $]2,7]$ if $p \neq \ell$.

Using the computer system Mathematica, the authors have checked that, in fact, for the primes $p = 137, 139, 149$ and 151, the trace of some $\rho_\ell(\mathrm{Frob}_{p,t})$ is contained in $]2, 7]$ if $\ell \neq p$. (Details of the actual computation are discussed in Remark A.5 below.) This implies that ϵ_ℓ is trivial for all primes ℓ, proving the first assertion of the theorem.

By Scholl [Sch06, Proposition 3], a pure ℓ-adic Galois representation ρ_ℓ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is irreducible if the following conditions are satisfied: there exists a prime $p \neq \ell$ and an open subgroup $I \leqslant I_p$ such that the restriction of ρ_ℓ to I is unipotent and indecomposable, and the restriction of ρ to $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ is Hodge–Tate. By Proposition A.4(i) and the assumption on $s_0 = 1 + a/b$, the restriction of $\rho_\ell^{s_0} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_7(\mathbb{Q}_\ell)$ to I_p is unipotent and indecomposable, where p is any odd prime divisor of b which is different from ℓ. It follows from the motivic interpretation of $\mathcal{G}_\ell$ given in Corollary 2.4.2 that $\rho_\ell^{s_0}$ is a Galois submodule of the sixth étale cohomology group of a smooth projective variety over $\mathbb{Q}$. Since the étale cohomology groups of a smooth projective variety over $\mathbb{Q}_\ell$ are Hodge–Tate by [Fal88], the restriction of $\rho_\ell^{s_0}$ to $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ has the Hodge–Tate property. Since $\rho_\ell^{s_0}$ is pure of weight 0, Scholl's result implies that the representation $\rho_\ell^{s_0}$ is absolutely irreducible. Let q be an odd prime divisor of a which is different from ℓ, and let J_q be the image of a topological generator of I_q^{tame} under $\rho_\ell^{s_0}$. By Proposition A.4, the Jordan canonical form of J_q has two Jordan blocks of length 2 and one of length 3. By [Asc87, Corollary 12], a Zariski closed proper maximal subgroup of $G_2(\bar{\mathbb{Q}}_\ell)$ is reducible, or G is isomorphic to the group $\mathrm{PSL}_2(\bar{\mathbb{Q}}_\ell)$. In the latter case, the non-trivial unipotent elements of the image of $\mathrm{PSL}_2(\bar{\mathbb{Q}}_\ell)$ are conjugate in $\mathrm{GL}_7(\bar{\mathbb{Q}}_\ell)$ to a Jordan block of length 7. Thus the existence of J_q implies that the Zariski closure of $\mathrm{Im}(\rho_\ell^{s_0})$ in $G_2(\bar{\mathbb{Q}}_\ell)$ is equal to $G_2(\bar{\mathbb{Q}}_\ell)$. It follows that $\mathrm{Im}(\rho_\ell^{s_0})$ is Zariski dense in $G_2(\mathbb{Q}_\ell)$, finishing the proof of the second claim of Theorem A.1. □

Proof of Corollary A.2. By construction, the Galois representation $\rho_\ell^{s_0} : G_\mathbb{Q} \to \mathrm{GL}_7(\mathbb{Q}_\ell)$ is isomorphic to the Galois representation on the stalk $(\mathcal{G}_\ell(3))_{\bar{s}_0}$. Moreover, the stalk $(\mathcal{G}_\ell(3))_{\bar{s}_0}$ is the ℓ-adic realization of the motive M_{s_0} which appears in § 3.3 of the article. The motivic Galois group $G_{M_{s_0}}$ of M_{s_0} can be characterized as the stabilizer of the spaces of motivated cycles in the realizations of every subobject of the Tannakian category $\langle M_{s_0}\rangle$ generated by M_{s_0} (this can be seen from using the arguments in [And04, § 6.3]). By Chevalley's theorem, there exists one object $M \in \langle M_{s_0}\rangle$ such that the motivic Galois group $G_{M_{s_0}}$ is characterized as the stabilizer of a line in the realization of M which is spanned by a motivated cycle. This line is fixed by an open subgroup of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Therefore, the group $G_{M_{s_0}}(\mathbb{Q}_\ell)$ contains the image of an open subgroup of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ under $\rho_\ell^{s_0}$. Since the group $G_2(\mathbb{Q}_\ell)$ is connected and the Zariski closure of $\rho_\ell^{s_0}(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}))$ is dense in $G_2(\mathbb{Q}_\ell)$ by Theorem A.1(ii), the group $G_2(\mathbb{Q}_\ell)$ is contained in $G_{M_{s_0}}(\mathbb{Q}_\ell)$. By construction, the motivic Galois group $G_{M_{s_0}}$ of M_{s_0} is contained in the group G_2 (see the proof of Theorem 3.3.1). Together with what was said before, one concludes that $G_{M_{s_0}}(\mathbb{Q}_\ell) = G_2(\mathbb{Q}_\ell)$ and so $G_{M_{s_0}}(\mathbb{Q}_\ell)$ is of type G_2, proving the first claim.

To prove the second claim, we argue as follows. Fix a prime ℓ which does not divide the product $aba'b'$. By Proposition A.4, we recover the odd primes p that divide a as those odd primes p where I_p^{tame} acts unipotently with a block of length 3 and two blocks of length 2, and we recover the odd primes that divide b as those where I_p^{tame} acts unipotently with a single block of length 7. Since $(a, b) \neq (a', b')$, the Galois representations $\rho_\ell^{s_0}$ and $\rho_\ell^{s_0'}$ have a different ramification behaviour at at least one prime divisor p of $a \cdot b$ or of $a' \cdot b'$. Thus the Galois representations $\rho_\ell^{s_0}$ and $\rho_\ell^{s_0'}$ are not isomorphic, so long as ℓ does not divide the product $aba'b'$. For any such ℓ, the

ℓ-adic realizations of M_{s_0} and $M_{s_0'}$ are not isomorphic as Galois representations, which implies that the motives M_{s_0} and $M_{s_0'}$ are not isomorphic. This concludes the proof of (ii). Assertion (iii) is an immediate consequence of (ii). □

Remark A.5. Let $\pi : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ denote the addition map. For $\mathcal{H}_i$, $i = 0, \ldots, 6$, and χ as above, the sheaf $\mathrm{MC}_\chi(\mathcal{H}_i)$ is the image of the !-*convolution*

$$\mathcal{H}_i *_! j_*(\mathcal{L}_\chi) = R\pi_!(\mathcal{H}_i \boxtimes j_*(\mathcal{L}_\chi))$$

in the $*$-convolution $R\pi_*(\mathcal{H}_i \boxtimes j_*(\mathcal{L}_\chi))$ under the 'forget supports map' (cf. [Kat96, § 4.3]). In the case at hand, it happens that each of the sheaves $\mathcal{H}_i$ has unipotent local monodromy at ∞ (in fact, a single Jordan block of length $i+1$). It then follows from [Kat96, 2.9.4 part 3]) that the canonical map

$$\mathcal{H}_i *_! j_*(\mathcal{L}_\chi) \longrightarrow \mathcal{H}_i *_{\mathrm{mid}} j_*(\mathcal{L}_\chi) = \mathrm{MC}_\chi(\mathcal{H}_i)$$

is an isomorphism. At each $\mathbb{F}_p$-rational point $t \in S_\ell(\mathbb{F}_p)$, one may then use the Grothendieck–Lefschetz trace formula to see that the trace of the Frobenius $\mathrm{Frob}_{p,t}$ at t on the stalk $(\mathcal{H}_i *_! j_*(\mathcal{L}_\chi))_{\bar t}$ is given by the convolution

$$f_i * f_2(t) := -\sum_{x \in \mathbb{F}_p} f_i(x) f_2(t-x), \quad i = 1, \ldots, 6, \tag{A.5}$$

where $f_i(x)$ gives the trace of $\mathrm{Frob}_{p,x}$ on $\mathcal{H}_i$ and $f_2(x)$ gives the trace of $\mathrm{Frob}_{p,x}$ on $\mathcal{L}_\chi$. Using a standard computer algebra system, such as Mathematica, it is easy to derive from formula (A.5) the trace of $\mathrm{Frob}_{p,t}$ (for small primes p) for the sequence $\tilde{\mathcal{H}}_0 = \mathcal{H}_0, \tilde{\mathcal{H}}_1, \ldots, \tilde{\mathcal{H}}_6$ of constructible sheaves defined as follows: the 'middle tensor' operation

$$j_*(\mathcal{F}_{i+1} \otimes j^*(\mathrm{MC}_\chi(\mathcal{H}_{i-1}))), \quad i = 1, \ldots, 6, \tag{A.6}$$

on the right-hand side of (A.2) is replaced by the literal tensor product

$$\tilde{\mathcal{H}}_i = \mathcal{F}_{i+1} \otimes (\tilde{\mathcal{H}}_{i-1} *_! j_*(\mathcal{L}_\chi)), \quad i = 1, \ldots, 6, \tag{A.7}$$

of $\mathcal{F}_{i+1}$ with the !-convolution $\tilde{\mathcal{H}}_{i-1} *_! j_*(\mathcal{L}_\chi)$. We derive the traces of the following Frobenius elements on $\tilde{\mathcal{H}}_6(3)$.

$\mathrm{Trace}(\mathrm{Frob}_{137,85})$	$\mathrm{Trace}(\mathrm{Frob}_{139,18})$	$\mathrm{Trace}(\mathrm{Frob}_{149,59})$	$\mathrm{Trace}(\mathrm{Frob}_{151,73})$
$2.88\ldots$	$3.59\ldots$	$3.51\ldots$	$3.03\ldots$

How well do these traces of Frobenii on $\tilde{\mathcal{H}}_6(3)$ approximate the traces of the same Frobenii on $\mathcal{H}_6(3)$? Although the canonical map $\mathcal{H}_i *_! j_*(\mathcal{L}_\chi) \to \mathcal{H}_i *_{\mathrm{mid}} j_*(\mathcal{L}_\chi) = \mathrm{MC}_\chi(\mathcal{H}_i)$ is an isomorphism at each stage, the middle tensor product in (A.6) may differ, by a δ-function at either 0 or 1, from the literal tensor product used in (A.7). Keeping careful track of these δ-functions and their progeny under later stages of the algorithmic construction of $\mathcal{H}_6(3)$ and $\tilde{\mathcal{H}}_6(3)$ leads to the conclusion that the largest error in computing traces at $\mathbb{F}_p$-points when working with !-convolution and literal tensoring instead of middle convolution and middle tensoring is bounded in absolute value by $(8/\sqrt{p}) + (4/p)$. Thus, for $p > 100$, the largest error in trace at an $\mathbb{F}_p$-rational point of $\mathbb{A}^1 \backslash \{0, 1\}$ is 0.84. So from the table above we see that for each p listed, the trace of Frobenius on $\mathcal{H}_6(3)$ at the indicated $\mathbb{F}_p$-rational point does indeed lie in $]2, 7]$.

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