GROUPS WITH A CERTAIN CONDITION ON CONJUGATES

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1. Introduction. In this paper, we shall show that if $\mathfrak{G}$ is a nilpotent [**5**] group and if M, a positive integer, is a uniform bound on the number of conjugates that any element of $\mathfrak{G}$ may have, then there exist "large" integers n for which $x \rightarrow x^n$ is a central endomorphism of $\mathfrak{G}$. If $\mathfrak{G}$ is not necessarily nilpotent, if the above condition on the conjugates is retained, and if we can find a member of the lower central series [**1**], every element of which lies in some member of the ascending central series, then we shall show that every non-unity element of the "high" derivatives has finite order.

2. Commutator relations. In a group $\mathfrak{G}$, let $(x, y) = xyx^{-1}y^{-1}$. In general, commutator notation is to be that of [**5**]. Let $\{x, y\}$ be that subgroup of $\mathfrak{G}$ which has generators x and y. By $\mathfrak{T} = \mathfrak{T}(x, y)$ we mean[1] the smallest normal subgroup of $\{x, y\}$ which contains both $((x, y), x)$ and $((x, y), y)$. If (x, y) commutes with both x and y, then

$$(x, y)^n = (x^n, y) = (x, y^n),$$

for every positive integer n, as an induction will show. Similarly

$$(xy)^n = x^n y^n (y, x)^\theta \qquad (\theta = \tfrac{1}{2}n(n-1)).$$

In $\{x, y\}/\mathfrak{T}$, $(x, y)\mathfrak{T}$ commutes with $x\mathfrak{T}$ and $y\mathfrak{T}$. Hence the above commutator formulae can be modified to $(x, y)^n \equiv (x^n, y) \equiv (x, y^n) \bmod \mathfrak{T}(x, y)$ and to $(xy)^n \equiv x^n y^n (y, x)^\theta \bmod \mathfrak{T}(x, y)$ for every $x, y \in \mathfrak{G}$.

3. The uniform bound. In this section, we assume that $\mathfrak{G}$ is a non-trivial group and that M is a positive integer such that the number of conjugates for any element $x \in \mathfrak{G}$ cannot exceed M. We shall call such a group a u.b. group, or say that the group is u.b.; M will be called the u.b. of $\mathfrak{G}$. Let $\mathfrak{Z}^{(1)}$ be the centre of $\mathfrak{G}$. Suppose that $\mathfrak{Z}^{(i)}$ is defined. Then $\mathfrak{Z}^{(i+1)}$ is to be that subgroup of $\mathfrak{G}$ for which $\mathfrak{Z}^{(i+1)}/\mathfrak{Z}^{(i)}$ is the centre of $\mathfrak{G}/\mathfrak{Z}^{(i)}$, and we have described the ascending central series [**1**] of $\mathfrak{G}$. We say that a group is a *torsion group* if every non-unity element thereof has finite order. If every element of a group $\mathfrak{G}$ has infinite order, we say that the group is *torsion-free.*

The group $\mathfrak{G}$ is said to have *uniform torsion* and is called u.t. if there exists a positive integer a such that $x^a = 1$ for all $x \in \mathfrak{G}$; a might be called the *exponent*

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[1]The proof of the principal result has been simplified as suggested by the referee, whereby properties of the $\mathfrak{G}/\mathfrak{T}$ are used.

of $\mathfrak{G}$. If $\mathfrak{G}$ is u.b. with bound M then $\mathfrak{G}/\mathfrak{Z}^{(1)}$ is u.t. with exponent α dividing $M!$ For, if $g, h \in \mathfrak{G}$, the set

$$\{h^{-i}gh^i\} \qquad (i = 0, 1, 2, \ldots, M)$$

cannot have $M + 1$ distinct elements. Equating a suitable pair of these, we find an integer m, $1 \leqslant m \leqslant M$, such that $h^m g = gh^m$. Now $m \mid M! = \mu$ so that $h^\mu g = gh^\mu$. The result is well known. For later use, we recall the fact that, for any group $\mathfrak{G}$ and positive integer i,

$$(\mathfrak{G}, \mathfrak{Z}^{(i+1)}) \subset \mathfrak{Z}^{(i)}.$$

Suppose that $\mathfrak{G}/\mathfrak{Z}^{(1)}$ is u.t. with exponent α and that $\mathfrak{N}$ is any normal subgroup of $\mathfrak{G}$. For $x \in \mathfrak{G}$, $y \in N$, $\mathfrak{T}(x, y) \subset (\mathfrak{G}, (\mathfrak{G}, \mathfrak{N}))$ so that

$$(x, y)^\alpha \equiv (x^\alpha, y) \equiv 1 \qquad \text{mod } (\mathfrak{G}, (\mathfrak{G}, \mathfrak{N})),$$

by the first of the commutator relations above. Let $\mathfrak{S}$ be the set of all $s \in (\mathfrak{G}, \mathfrak{N})$ for which $s^\alpha \in (\mathfrak{G}, (\mathfrak{G}, \mathfrak{N}))$. Then the members of $\mathfrak{S}$ form a set of generators for $(\mathfrak{G}, \mathfrak{N})$, and $\mathfrak{S}$ contains the inverse of each of its elements. Now let s and t be elements of $\mathfrak{S}$. Then

$$(s, t) \in ((\mathfrak{G}, \mathfrak{N}), (\mathfrak{G}, \mathfrak{N})) \subset (\mathfrak{G}, (\mathfrak{G}, \mathfrak{N})).$$

By the second of the commutator relations, $(st)^\alpha \equiv 1 \text{ mod } (\mathfrak{G}, (\mathfrak{G}, \mathfrak{N}))$, and $\mathfrak{S} = (\mathfrak{G}, \mathfrak{N})$. We have the proof of the first part of the following

LEMMA. $(\mathfrak{G}, \mathfrak{N})/(\mathfrak{G}, (\mathfrak{G}, \mathfrak{N}))$ *is* u.t. *with exponent dividing* α *whenever* $\mathfrak{G}/\mathfrak{Z}^{(1)}$ *is* u.t. *with exponent* α *and* $\mathfrak{N}$ *is a normal subgroup of* $\mathfrak{G}$; $(\mathfrak{G}, \mathfrak{Z}^{(i+1)})$ *is* u.t. *with exponent* $\alpha(i)$, *where* $\alpha(i) \mid \alpha^i$ *and where* $\alpha(i) \mid \alpha(i+1)$.

That $\alpha(i) \mid \alpha(i+1)$ is obvious. To show that $\alpha(i) \mid \alpha^i$, we note that the result holds if $i = 0$; and if it holds for $i = k - 1$, take $\mathfrak{N}$ above to be $\mathfrak{Z}^{(k+1)}$. Then $(\mathfrak{G}, \mathfrak{N}) \subset \mathfrak{Z}^{(k)}$, and

$$(\mathfrak{G}, \mathfrak{Z}^{(k+1)})/[(\mathfrak{G}, \mathfrak{Z}^{(k)}) \cap (\mathfrak{G}, \mathfrak{Z}^{(k+1)})]$$

is u.t. with exponent dividing α. Hence $(\mathfrak{G}, \mathfrak{Z}^{(k+1)})$ is u.t. with exponent $\alpha(k)$ where $\alpha(k) \mid \alpha \,.\, \alpha(k-1)$. The induction assumption includes $\alpha(k-1) \mid \alpha^{(k-1)}$, so that $\alpha(k) \mid \alpha^k$.

THEOREM. *If* $\mathfrak{G}/\mathfrak{Z}^{(1)}$ *is* u.t. *and if* $\gamma(i) = \alpha \,.\, \alpha(i-1)$ (*where* $\alpha(i-1)$ *is the exponent of* $(\mathfrak{G}, \mathfrak{Z}^{(i)})$), *then the mapping* $x \to x^{\gamma(i)}$ *on* $\mathfrak{G}$ *induces an endomorphism of* $\mathfrak{Z}^{(i)}$ *into* $\mathfrak{Z}^{(1)}$.

Proof. If $x, y \in \mathfrak{Z}^{(i)}$, $(xy)^\alpha = x^\alpha y^\alpha z$, where

$$z \in (\mathfrak{Z}^{(i)}, \mathfrak{Z}^{(i)}) \cap \mathfrak{Z}^{(1)} \subset (\mathfrak{G}, \mathfrak{Z}^{(i)}) \cap \mathfrak{Z}^{(1)}.$$

Hence $(xy)^{\gamma(i)} = x^{\gamma(i)} y^{\gamma(i)}$. For, $z \in (\mathfrak{Z}^{(i)}, \mathfrak{Z}^{(i)})$ by the second of the commutator relations, using the fact that $\mathfrak{T}(x, y) \subset (\mathfrak{Z}^{(i)}, \mathfrak{Z}^{(i)})$; and $z \in \mathfrak{Z}^{(1)}$, since $w^\alpha \in \mathfrak{Z}^{(1)}$ for every $w \in \mathfrak{G}$. Since $(\mathfrak{G}, \mathfrak{Z}^{(i)})$ is u.t. with exponent $\alpha(i-1)$, $\gamma(i)$ has the indicated property.

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4. The consequences of the theorem.

COROLLARY 1. *Let* $\mathfrak{G}/\mathfrak{Z}^{(1)}$ *be* u.t., *and let* $\mathfrak{G}$ *be nilpotent of class c. Then the mapping* $x \rightarrow x^{\gamma(c)}$ *is a central endomorphism of* $\mathfrak{G}$.

Proof. Take $i = c$ in the theorem.

COROLLARY 2. *If* $\mathfrak{G}/\mathfrak{Z}^{(1)}$ *is* u.t. *and if any member of the ascending central series is torsion-free, then the ascending central series collapses and contains only the centre.*

Proof. If $\mathfrak{Z}^{(n)}$ is torsion-free and if $g \in \mathfrak{Z}^{(n+1)}$, $n \geqslant 1$, then

$$gxg^{-1}x^{-1} \in (\mathfrak{G}, \mathfrak{Z}^{(n+1)}) \subset \mathfrak{Z}^{(n)}$$

for every $x \in \mathfrak{G}$, and the u.t. property of $(\mathfrak{G}, \mathfrak{Z}^{(n+1)})$ shows that $gxg^{-1}x^{-1} = 1$, the unity of $\mathfrak{G}$. Then $gx = xg$ for every $x \in \mathfrak{G}$, and $\mathfrak{Z}^{(n+1)} \subset \mathfrak{Z}^{(1)}$.

COROLLARY 3. *A non-Abelian nilpotent group* $\mathfrak{G}$ *with torsion-free centre cannot be* u.b.

For a given group $\mathfrak{G}$ let $\mathfrak{Z} = \mathfrak{Z}(\mathfrak{G})$ be the set sum of the $\mathfrak{Z}^{(i)}$ $(i = 1, 2, 3, \ldots)$. $\mathfrak{Z}$ is a normal subgroup of $\mathfrak{G}$; and $\mathfrak{G} = \mathfrak{Z}$ if $\mathfrak{G}$ is nilpotent. The converse of the latter statement need not hold. If $\mathfrak{G} = \mathfrak{Z}$ we call $\mathfrak{G}$ *weakly nilpotent.* From the principal theorem, if $\mathfrak{G}/\mathfrak{Z}^{(1)}$ is u.t., then $(\mathfrak{G}, \mathfrak{Z})$ is a torsion subgroup of $\mathfrak{G}$. Similarly, we have the following results:

LEMMA. *If* $\mathfrak{G}/\mathfrak{Z}^{(1)}$ *is* u.t.[2] *and if* $\mathfrak{G}$ *is weakly nilpotent, then* $(\mathfrak{G}, \mathfrak{G})$ *is a torsion subgroup of* $\mathfrak{G}$.

LEMMA. *If* $\mathfrak{G}/\mathfrak{Z}^{(1)}$ *is* u.t. *and if* $\mathfrak{Z} \supset {}^{i}\mathfrak{G}$, *a member of the lower central series of* $\mathfrak{G}$, *then* (a) *the* ${}^{i+k}\mathfrak{G}$, $k > 0$, *are torsion subgroups; and* (b) *for "large" j, the* $\mathfrak{G}^{(j)}$, *members of the derived series are torsion subgroups.*

Proof. (See [**5**] and [**1**] for definitions.) (a) $\mathfrak{Z} \supset {}^{i}\mathfrak{G}$ implies

$$(\mathfrak{G}, \mathfrak{Z}) \supset (\mathfrak{G}, {}^{i}\mathfrak{G}) = {}^{i+1}\mathfrak{G} \supset {}^{i+k}\mathfrak{G} \qquad (k \geqslant 2).$$

(b) It is known [**1**] that $\mathfrak{G}^{(j)} \subset {}^{k}\mathfrak{G}$ $(k = 2^j - 1)$. Choose $j \geqslant \log_2(i + 2)$ for the desired result.

It is well known [**3**] that the integers n for which $x \rightarrow x^n$ is a central endomorphism form an ideal. It would be of interest to extend the work of Levi and van der Waerden and of Bruck [**2**], concerning central endomorphisms of the form $x \rightarrow x^3$, to the general central power endomorphism. But the methods, as in [**2**], seem to depend on the fact that 3 is "small."

[2]For a related result when $\mathfrak{G}$ is u.b. see [**4**].

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