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TRANSLATION PLANES OF DIMENSION TWO WITH ODD CHARACTERISTIC

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1. Introduction. A translation plane of dimension d over its kernel $K = GF(q)$ can be represented by a vector space of dimension $2d$ over K. The lines through the zero vector form a "spread"; i.e., a class of mutually independent vector spaces of dimension d which cover the vector space.

The case where $d = 2$ has aroused the most interest. The more exotic translation planes tend to be of dimension two; a spread in this case can be interpreted as a class of mutually skew lines in projective three-space.

The stabilizer of the zero vector in the group of collineations is a group of semi-linear transformations and is called the translation complement. The subgroup consisting of linear transformations is the linear translation complement.

A central problem in connection with finite translation planes is to identify the linear groups which can act as subgroups of a linear translation complement, to determine how these groups act on the planes, and to identify the planes upon which they act.

In this paper we determine what the groups must be like for $d = 2$ and q odd. We also get some information about the way in which the groups act.

When this is put together with previous results mentioned below, the effect is that for $d = 2$ with the characteristic equal to two or greater than 5 we now know that the groups must come from a relatively small list; we know something about how the groups must act; but we cannot give a complete listing of the planes admitting these groups. In some cases we cannot even say whether there are any finite translation planes admitting a specified group.

Johnson and the author have investigated the case where the characteristic and dimension of the plane are both equal to two [**5**, **6**].

The author has investigated the case of odd characteristic and dimension two when the order of the group is relatively prime to the characteristic [**8**].

The present paper is a study of the problem for odd characteristic and dimension two when the characteristic divides the order of the group.

In all of this, not much attention is paid to the case where the linear translation complement is both solvable and reducible. There are very

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many examples of planes which satisfy these conditions and the number of possibilities seems very great indeed. Yet these conditions do tell us something about the nature of the plane and the group. If one is looking for a plane with certain special properties the solvability and reducibility conditions (the invariant subspaces have dimension one or two) often enable us to nail down the possibilities rather completely.

For instance, the group must at least be homomorphic to a subgroup of $GL(2, q)$ and the subgroups of $GL(2, q)$ are well known. The kernel of the homomorphism is the subgroup fixing the subspace pointwise and the possibilities here are also known.

Our main results for $d = 2$ and characteristic greater than 5 are collected in Theorem (2.9). Actually, the translation plane of order 9 and one of Walker's planes of order 25 both would be counter-examples to (2.9) if we dropped the condition $p > 5$. In part this says that when the linear translation complement is irreducible then $SL(2,p^s)$ (for some s) enters in unless we are in one of the cases discussed in [**8**]. Here p is the characteristic. This part is essentially Lemma (2.6) which is an easy consequence of a result of Suprunenko and Zalesskii. See (1.6). By modifying arguments of [**9**] we determine the action of $SL(2, p^s)$. Also see [**11**]. In the irreducible case there is a Hering subplane of order p^{2s}. Thus, with the possible exception of $p = 3$ or 5 there are no real surprises left for $d = 2$ at least insofar as the abstract groups are concerned and there is not much room for surprises insofar as unusual action is concerned.

We do really seem to be lacking information about translation planes with proper subplanes that are nearfield planes, Hering planes, etc. In the other direction we know very little about cases in which the translation complement is not much larger than the kernel (the scalar transformations).

Realizing the limitations, we suspect that we are about as close to a complete classification of translation planes of dimension two as we are likely to get.

We conclude this section with some preliminary definitions and statements of results we shall use.

(1.1) *Definition*. Let G be a group of linear transformations acting irreducibly on a vector space V. G is said to be *imprimitive* if $V = V_1 \oplus \ldots \oplus V_k$ where the image of each V_i under each element of G is some V_k. The subspaces $V_1, \ldots, V_k$ are called *subspaces of imprimitivity*. Otherwise G is said to be primitive.

(1.2) (See [**1**], Theorem 2.10B.) Let G be an irreducible subgroup of $GL(n, F)$ and let E be a finite extension of F. Then G is completely reducible as a subgroup of $GL(n, E)$ and all of its irreducible components have the same dimension.

(1.3) (See [**1**] Theorem 4.4.) Let V be a vector space of dimension n over an algebraically closed field F. Let G be a primitive subgroup of $GL(V)$ such that $\text{Fit}\, G \neq Z(G)$. (Here $\text{Fit}\, G$ is the Fitting subgroup; $Z(G)$ is the center of G.) Then (a)$|\text{Fit}\, G:Z(G)| = d^2$ for some divisor d of n. (b) the Sylow subgroups of $\text{Fit}\, G/Z(G)$ are elementary abelian.

(1.4) ([**1**], Theorem 4.5.) In (1.3) above $n > 1$ and G is solvable the hypothesis $\text{Fit}\, G \neq Z(G)$ is automatically satisfied. Otherwise $n = 1$ and G is abelian.

(1.5) (See [**8**], Lemmas (2.13) and (2.14).) Let G be a subgroup of the linear translation complement for a translation plane of dimension two over $GF(q)$, q odd. Let $\bar{G}$ be the induced group on the line at infinity. Suppose that $\bar{G}$ is elementary abelian of order 16. Then at least one and at most five of the involutions in $\bar{G}$ have pre-images that are affine homologies.

(1.6) (See [**10**].) Let G be a finite irreducible linear group of degree 4 over an algebraically closed field p of characteristic $p > 5$. Suppose that G coincides with its commutant and cannot be obtained as a result of reduction mod p of a complex linear group of degree 4 isomorphic to G. Then G is one of the following groups:

$$SL(4, F), SU(4, F), Sp(4, F), PSL(2, F)\ |F| \neq p,$$
$$SL(2, F)|F| \neq 7, SL(2, F) \otimes SL(2, 5),$$
$$SL(2, 5) \otimes SL(2, F_1)$$

where F and F_1 are finite subfields of P.

(1.7) (See [**8**], Lemma (2.3)). If π is a translation plane of dimension two over its kernel, if G is a subgroup of the linear translation complement, and if V_1 is a minimal subspace invariant under G, the dimension of V_1 is not equal to 3 and hence is 1, 2, or 4.

2.

(2.1) *Definition.* In the following π always denotes a translation plane of dimension two over $F = GF(q)$ so that the points of π are elements of a four dimensional vector space over F. Unless the contrary is specifically indicated, G is an irreducible subgroup of the linear translation complement. We shall use p to denote the characteristic of F.

The case where $p = 2$ has been investigated in [**5**, **6**] so we shall assume that p is odd.

The case where p does not divide the order of G was investigated in [**8**].

(2.2) *Assumptions.* The characteristic p is greater than 5 and divides $|G|$. G is generated by its p-elements.

(2.3) LEMMA. *G is non-solvable.*

Proof. A Sylow p-group fixes some non-trivial subspace pointwise; its normalizer leaves this subspace invariant. Hence G must have more than one Sylow p-group, since G is irreducible.

If G is not absolutely irreducible, then G is isomorphic to a subgroup of $GL(2, q^a)$ for some a, but a subgroup of $GL(2, q^a)$ generated by p-elements not all in the same Sylow subgroup is non-solvable so the lemma is proved for this case. See (1.2).

A similar situation holds if G is absolutely irreducible but becomes imprimitive over some finite extension of F. That is, when the vector space V on which π is defined is regarded as a vector space over some $GF(q^a)$, we have $V = V_1 \oplus V_2$ or $V = V_1 \oplus V_2 \oplus V_3 \oplus V_4$, and G has a subgroup of index dividing 2 or $|S_4|$ which fixes V_1 and V_2 or V_1, V_2, V_3, V_4 respectively. In either case, the p-groups must fix all of the V_i (we assumed $p > 5$) so G must be reducible if it is generated by its p-groups.

In the remaining case, we may interpret the vector space on which π is defined as a vector space over the algebraic closure of F. If G is solvable, $Z(G)$ consists of scalars and by (1.3) and (1.4)

$$[\text{Fit}\, G{:}Z(G)] = d^2$$

where d divides 4 but $d \neq 1$. That is, $d^2 = 4$ or 16. Also, Fit $G/Z(G)$ is elementary Abelian.

Returning to the interpretation of G as a collineation group of π, $G/Z(G) = \bar{G}$ is the permutation group induced by G on l_∞.

Furthermore the elements of order p in G induce automorphisms on Fit $G(Z/G)$. If $[\text{Fit}\, G{:}Z(G)] = 4$ and $p > 5$ then the p elements in $\bar{G}$ must centralize Fit $G/Z(G)$ so $\bar{G}$ centralizes Fit $G/Z(G)$.

We claim that Fit $G/Z(G)$ is Fit $\bar{G}$. This comes from the fact that if A_0 is nilpotent there is a normal series $A_0 \supset A_1 \supset \ldots \supset A_r = 1$ in which A_{i-1}/A_i is the center of A_0 for each i. If A_0 is a subgroup of G, $\bar{A}_{i-1}/\bar{A}_i = A_{i-1}/A_i$ so the pre-image of Fit $\bar{G}$ is niplotent and in fact must be Fit G. If G is solvable this implies that $\bar{G}$ is abelian and hence has a unique Sylow p-group. See (1.4). By (2.2) G(and $\bar{G}$) are generated by p-elements so in this case G is a p-group. But G is irreducible by Definition (2.1) and p-groups are reducible.

If $[\text{Fit}\, G{:}Z(G)] = 16$ we apply (1.5). The effect is that at least one and at most five of the involutions in $[\text{Fit}\, G{:}Z(G)]$ have pre-images that are involutory homologies. With $p > 5$ we again get a unique Sylow p-group.

Thus we get a contradiction if we assume that G is solvable.

(2.4) LEMMA. *Let G_0 be a minimal normal non-solvable subgroup of G. Then p divides $|G_0|$.*

Proof. G_0 must admit non-trivial automorphisms of order p since each p-element induces an automorphism by conjugation and the p-elements cannot all centralize G_0 and G_0 cannot be in the center of G. This also holds for non-solvable characteristic subgroups and non-solvable factor groups of G_0.

Suppose that G_0 is reducible. By (1.7), the minimal G_0-spaces have dimension 1 or 2. Each 1-space belongs to a unique component so we may assume that G_0 has an invariant 2-space. The only non-solvable subgroup of $PSL(2, q)$ which contains no p-elements is $PSL(2, 5)$. If G_0 is non-solvable and contains no p-elements then G_0 is homomorphic to $PSL(2, 5)$ and the homomorphism will carry G into a group containing p elements which must induce non-trivial automorphisms on $PSL(2, 5)$. With $p > 5$ there are no such automorphisms. Hence p must divide $|G_0|$ if G_0 is reducible.

Suppose that G_0 is irreducible. If G_0 is not absolutely reducible it is reducible over some extension. By (1.2) it must be a subgroup of $GL(2, q^a)$ or $GL(1, q^a)$ for some a. The latter group is solvable. If p does not divide $|G_0|$ we can repeat the argument of the previous paragraph.

Now suppose that G_0 is absolutely irreducible and that p does not divide $|G_0|$. Then G_0 is isomorphic to a 4 dimensional complex group and must be one of the groups listed in [**8**], Theorem (2.17). The condition that every non-solvable characteristic subgroup or factor group admits non-trivial automorphisms of order p eliminates all of these cases.

(2.6) Lemma. *G has a minimal normal non-solvable subgroup isomorphic to $SL(2, p^s)$ for some s.*

Proof. Note that G_0 must be its own commutator subgroup. Suppose that G is primitive. Then Clifford's Theorem implies that the minimal G_0 spaces are all isomorphic as G_0 modules. In this situation an element of G_0 fixing a minimal G_0-space pointwise must fix all of them pointwise and be the identity. Thus if G is primitive G_0 must be faithful on its minimal invariant subspaces.

Suppose that G is not primitive. The number of subspaces of imprimitivity is 2 or 4, so the elements of order p must fix these subspaces. Hence G must be reducible, contrary to assumption. Thus if G_0 is reducible it must be faithful on its minimal invariant subspaces in any case. Since G is irreducible these minimal subspaces have dimension 1 or 2, π is a direct sum of invariant G_0-spaces (which are G-images of each other). Hence $G_0 \underset{\sim}{\subseteq} GL(2, q)$. If G_0 is non-solvable and contains p-elements the minimal condition implies $G_0 = SL(2, p^s)$ for some s. If G_0 is irreducible, it must be one of the groups listed in (1.6): $SL(4, F)$, $SU(4, F)$, $Sp(4, F)$, $PSL(2, F)$, $SL(2, F)$, $SL(2, F) \otimes SL(2,5)$, $SL(2, \mathrm{F}) \otimes SL(2, F_1)$ where F and F_1 are subfields of $GF(q)$.

If G is irreducible, the subgroup of G_0 generated by its p-elements is a normal non-solvable subgroup of G so G_0 is generated by its p-elements. Hence

$$G_0 \neq SL(2, F) \otimes SL(2, 5).$$

The groups $SL(4, F)$, $SU(4, F)$, $Sp(4, F)$ all contain transvections; i.e., elements which fix a 3-space pointwise. But the fixed points of a collineation must be collinear or the points of a subplane so this cannot happen.

$PSL(2, F)$ contains an elementary Abelian subgroup of order 4 in which the involutions are all conjugate and not in the center. An involutory homology in the translation complement which has an affine axis has exactly two fixed components, both of which are invariant under its centralizer. Hence the involutions cannot be affine homologies, since no two of them can have the same axis. An involution fixing a Baer subplane pointwise has another Baer subplane which is invariant and intersects non-trivially precisely those components intersected non-trivially by the first one. (The involution is completely reducible on each fixed component so has another fixed 1-space on each of these components. These invariant 1-spaces are in the "other" invariant Baer subplane.) Again these two subplanes are invariant under the centralizer of the Baer involutions. It turns out that $PSL(2, F)$ cannot be in the translation complement.

The matrix representation of $SL(2, F) \otimes SL(2, F_1)$ consists of matrices

$$\begin{pmatrix} aa_1 & ab_1 & ba_1 & bb_1 \\ ac_1 & ad_1 & bc_1 & bd_1 \\ ca_1 & cb_1 & da_1 & db_1 \\ cc_1 & cd_1 & dc_1 & dd_1 \end{pmatrix}$$

where

$$ab - cd = a_1b_1 - c_1d_1 = 1,$$

$$a, b, c, d, \in F; a_1, b_1, c_1, d_1 \in F_1.$$

Take $a = c = d = a_1 = d_1 = 1$; $b = b_1 = c_1 = 0$ to get the matrix

$$\begin{pmatrix} 10 & 00 \\ 01 & 00 \\ & \\ 10 & 10 \\ 01 & 01 \end{pmatrix}$$

The reader may verify that this matrix has a pointwise fixed subspace of dimension 2. This pointwise fixed subspace is either a component of the spread or is a Baer subplane. Foulser has shown that for $p > 3$ the translation complement cannot contain both affine elations (shears) and Baer p elements [**2**]. By [**4**], the group generated by the affine elations is

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$SL(2, p^s)$. Clearly this group is a minimal normal non-solvable subgroup of G. Foulser has shown, again for $p > 3$, that the Baer p-elements also generate $SL(2, p^s)$.

We conclude that G has a minimal non-solvable normal subgroup isomorphic to $SL(2, F)$, where $F = GF(p^s)$ is a subfield of $GF(q)$. We will refer to this group as G_0.

(2.7) THEOREM. *Let π be a translation plane of dimension two over $F = GF(q)$, where q is a power of the prime $p > 5$. Let G_0 be a reducible subgroup of the translation complement such that G_0 is isomorphic to $SL(2, p^s)$ where $GF(p^s)$ is a subfield of F. Then the p-elements in G_0 consist of affine elations or Baer p-elements.*

Proof. Our present interest is in the case where p is odd. The theorem was proved for $p = 2$ in [**5**]. The arguments are similar to the ones for the case $p = 2$ and, like those, are modifications of arguments used by Schaeffer for the case $F = GF(p^s)$. We assume p is odd.

We shall leave out some of the details for the part of the proof which is independent of the characteristic. We can choose a basis so that G_0 is represented by a set of matrices of the form $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$, where 0 is a two by two zero matrix; A, B, C are 2 by 2 matrices over $F_0 = GF(p^s)$ and the sets of matrices $\{A\}$ and $\{B\}$ form groups isomorphic to G_0. Indeed, we can do this so that if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$C = \begin{pmatrix} a^\alpha & b^\alpha \\ c^\alpha & d^\alpha \end{pmatrix}$$

where α is some automorphism of F_0.

In particular, we can choose a basis so that the element $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ of $SL(2, p^s)$ corresponds to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ b_1 & b_2 & 1 & 0 \\ b_3 & b_4 & 1 & 1 \end{pmatrix}$$

and $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ corresponds to

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & a^\alpha & 0 \\ 0 & 0 & 0 & a^{-\alpha} \end{pmatrix}.$$

Up to here, everything is the same as for the case $p = 2$. Using

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^i = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$$

for $i = 0, 1, \ldots, p - 1$ we can write

$$\begin{pmatrix} 1 & 0 \\ \mathrm{i} & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ i & 1 & 0 & 0 \\ f_1(i) & f_2(i) & 1 & 0 \\ f_3(i) & f_4(i) & i & 1 \end{pmatrix}$$

By induction

$$f_1(i) = ib_1 + i(i-1)b_2/2$$
$$f_2(i) = ib_2$$
$$f_4(i) = ib_4 + i(i-1)b_2/2$$

but

$$\begin{pmatrix} i^{-1} & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & i^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ i^2 & 1 \end{pmatrix}.$$

If we take the corresponding conjugate in our four dimensional representation our isomorphism tells us that

$$f_1(i^2) = b_1 \quad f_2(i^2) = i^{-2}b_2 \quad f_4(i^2) = b_4.$$

Hence $i^2b_2 = i^{-2}b_2$. If $b_2 \neq 0$, $i^4 = 1 \bmod p$ for $i = 1, 2, \ldots, p - 1$. This does not hold for $p > 5$, so $b_2 = 0$. Also $b_1 = b_1 i$, $b_4 = b_4 i$ for $i = 1, 2, \ldots, p - 1$ implies $b_1 = b_4 = 0$.

Hence

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ b_3 & 0 & 1 & 1 \end{pmatrix}.$$

The reader may verify that the pointwise fixed subspace is 2-dimensional and hence is an affine elation or a Baer p-element. But the group generated by elations (or Baer p-elements) is a normal subgroup and its p-elements consist only of Baer p-elements or only of elations. Hence the p-elements are of one of these two kinds.

(2.8) THEOREM. *Let π be a translation plane of dimension two over $GF(q)$ where q is a power of a prime $p > 5$. (π is represented on a vector space of dimension 4 over $GF(q)$). Suppose that the linear translation complement of π has a subgroup G_0 isomorphic to $SL(2, p^s)$. Then one of the following holds.*

(a) *The p elements in G_0 are affine elations. G has an invariant Desarguesian subplane of order p^s.*

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(b) *The p-elements in G_0 are Baer p-elements. π is derived from a plane in which G_0 acts in accordance with case* (a).

(c) *G_0 has an invariant subplane π_0 of order p^{2s}. π_0 is a Hering plane. The action of G_0 on the vector space is irreducible.*

Proof. G_0 is generated by a properly chosen pair of p elements. If they are affine elations, conclusion (a) is part of Theorem 4 in a paper by the author [**7**].

Conclusion (b) is part of Theorem (4.1) in [**2**].

From the previous theorem, G_0 must be irreducible if the p-elements are not affine elations nor Baer p-elements.

Following [**9**], there are just two type of four-dimensional irreducible representations of $SL(2, p^s)$ over $GF(p^s)$. By [**12**] Corollary 3F the irreducible representations of $SL(2, p^s)$ over extensions of $GF(p^s)$ may be realized over $GF(p^s)$. Thus our representation of G_0 over $GF(q)$ can be taken to be a set of matrices whose coefficients are all in $GF(p^s)$ and give an irreducible representation when the vector space is taken to be a 4-tuple over $GF(p^s)$.

The first type of representation is with respect to a module $V_2^{\alpha} \otimes_K V_2^{\beta}$, where $K = GF(p^s)$ while α and β are automorphisms of K and V_2 indicates the two dimension representation. Without loss of generality, we can restrict our attention to the case where $\beta = 1$. The matrix representation corresponding to the element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the two dimensional representation is then the Kronecker product of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a^{\alpha} & b^{\alpha} \\ c^{\alpha} & d^{\alpha} \end{pmatrix}.$$

The second type of representation is on the space of homogeneous polynomials of degree 3. Again there is a field automorphism which may be applied to each element in each matrix; this automorphism may again be taken to be equal to 1.

In the two representations, the four by four matrix corresponding to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ comes out to be

$$\begin{pmatrix} a^{\alpha+1} & a^{\alpha}b & b^{\alpha}a & b^{\alpha+1} \\ a^{\alpha}c & a^{\alpha}d & b^{\alpha}c & b^{\alpha}d \\ c^{\alpha}a & c^{\alpha}b & d^{\alpha}a & d^{\alpha}b \\ c^{\alpha+1} & c^{\alpha}d & d^{\alpha}c & d^{\alpha+1} \end{pmatrix}$$

in the first case and

$$\begin{pmatrix} a^3 & 3a^2b & 3ab^2 & b^3 \\ a^2c & 2abc+a^2d & b^2c+2abd & b^2d \\ ac^2 & 2acd+bc^2 & ad^2+2bcd & bd^2 \\ c3 & 3c^2d & 3cd^2 & d^3 \end{pmatrix}$$

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in the second case. See [**9**], pp. 47–48.

In the first case, the p-elements obtained by letting $a = d = 1$, $b = 0$ have fixed point subspaces which are two-dimensional, so we must use the second representation since we are now assuming that the p-elements are not elations nor Baer p-elements.

In effect, we are taking the points of π to be quartuples (x_1, x_2, x_3, x_4), where the coordinates are chosen from $GF(q)$. If we restrict ourselves to the set of points where the coordinates are in $GF(p^s)$ and if $p^{s+1} \equiv 0 \bmod 3$ we can use the group to define a Hering plane as in Hering's original construction [**3**]. If this is the set of points of a subplane, the subplane admits the group to make it be a Hering plane.

Let π_0 be the set of points (x_1, x_2, x_3, x_4) where all of the coordinates are in $GF(p^s)$. Let us look at the action of G_0 on π and at the intersection of certain components of π with π_0.

We now change notation slightly, writing $(X, Y) = (x_1, x_2, y_1, y_2)$ for a typical point of π.

If we take $a = d = 1$, $c = 0$ we get a Sylow p-group consisting of matrices:

$$\begin{pmatrix} 1 & 3b & 3b^2 & b^3 \\ 0 & 1 & 2b & b^2 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The 1-space $\langle (0, 0, 0, 1) \rangle$ is invariant under this Sylow p-group and must belong to a component of the spread. If $\langle (0, 0, 0, 1), (u_1u_2u_3u_4) \rangle$ is an invariant two-space then there are constants k and c so that

$$(u_1, 3bu_1 + u_2, 3b^2u_1 + 2bu_2 + u_3, b^3u_1 + b^2u_2 + bu_3 + u_4) = k(0, 0, 0, 1) + c(u_1, u_2, u_3, u_4)$$

so that

$$\begin{aligned} u_1 &= cu_1 \\ 3bu_1 + u_2 &= cu_2 \\ 3b^2u_1 + 2bu_2 + u_3 &= cu_3 \\ b^3u_1 + b^2u_2 + bu_3 + u_4 &= k + cu_4. \end{aligned}$$

If $u_1 \neq 0$, $c = 1$ and $3bu_1 = 0$ therefore $u_1 = 0$.

If $u_2 \neq 0$, and $u_1 = 0$ then $c = 1$ and $2bu_2 = 0$ therefore $u_2 = 0$. Thus $X = 0$ is the only invariant 2-space which contains the 1-space $\langle (0, 0, 0, 1) \rangle$. Hence $X = 0$ is a component of the spread. Similarly $Y = 0$ is a component and, corresponding to the $p^s - 1$ remaining Sylow p-groups, there are $p^s - 1$ components of the form $Y = XM$ where the matrices M are two by two matrices over $GF(p^s)$.

The reader may verify that the element of order 3 obtained by taking $a = c = -1$, $b = 1$, $d = 0$ has a two-dimensional pointwise fixed sub-

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space of the form $Y = XM$ for $\begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} = M$. Let the above matrix of order 3 be denoted by W. If one first calculates $W^2 + W + I$, it is quite direct to compute that the null space of $W^2 + W + I$ is $Y = X\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.

If $x^2 + x + 1$ is an irreducible polynomial over $GF(q)$ there can be no W-space other than $Y = X\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ which is disjoint from $Y = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix}$. That is, W acts as a homology. Furthermore we must have (in this case) that $q + 1 \equiv 0 \bmod 3$. It follows that $\langle W \rangle$ has $\frac{1}{2}p^s(p^s - 1)$ conjugates in G_0.

Thus if $x^2 + x + 1$ is irreducible, π has a total of $(p^s + 1) + 2 \times \frac{1}{2}p^s(p^s - 1)$ components of the form $Y = XM$ with the coefficients in $GF(p^s)$. This does not count the components $X = 0$ or $Y = 0$. Thus we have a total of $p^{2s} + 1$ components each intersecting π_0 in a 2-space over $GF(p^s)$. Hence π_0 is a subplane invariant under G_0. (Note that G_0 is still irreducible as a group of linear transformations over $GF(q)$.)

The case remains where $x^2 + x + 1$ is reducible. Then $GF(q)$ must contain an element W of order 3 and some vector (x_1, x_2, y_1, y_2) in π has the property that

$$(x_1, x_2, y_1, y_2)W = (x_1\omega, x_2\omega, y_1\omega, y_2\omega)$$

and there also is an eigenvector for ω^2. The eigenspaces for ω and ω^2 respectively are 1-dimensional; the components to which they belong must be invariant. But W fixes a component pointwise and is not an elation; so it must be a homology. Hence there can be just one invariant component besides the axis of W. Thus this component must be the one generated by the eigenspaces corresponding to the eigenvalues ω and ω^2. Clearly the component in question must again be the null space of $W^2 + W + I$. The proof of the lemma is completed by repeating the argument used in the case where $W^2 + W + I$ was irreducible.

If we drop Assumptions (2.2), then Lemma (2.6), Theorem (2.7) and Theorem (2.8) imply the following:

(2.9) THEOREM. *Let π be a translation plane of dimension two over $GF(q)$, where q is a power of a prime $p > 5$.* (a) *Suppose that the linear translation complement of π is irreducible. Then either $|G|$ and p are relatively prime and we have one of the cases of Theorem* (2.17) *in* [**8**] *or G has a normal subgroup isomorphic to $SL(2, p^s)$ for some s. In this case, π has a Hering subplane of order p^{2s}.*

(b) *If G is reducible and non-solvable then either $|G|$ and p are relatively prime or G has a normal subgroup G_0 isomorphic to $SL(2, p^s)$ for some s. The p-elements in G_0 are elations or Baer p-elements.*

Remark. A necessary condition for the existence of a Hering plane of order q^2 is that $q \equiv -1 \bmod 3$ so we must have $p^s \equiv 1 \bmod 3$ for the latter part of (a) to happen.

Remark. In a sense which we will not bother to make precise, "most" of the known planes of dimension two have a normal subgroup of index 1 or 2 which is (a) solvable, (b) reducible, (c) not faithful, on at least one of its invariant subspaces. Excluding the Hall planes this includes the generalized Andre planes, the generalized Hall planes and probably all of the semi-field planes and some less well known cases where there are affine elations or homologies.

When G is reducible, we may assume by (1.7) that there is an invariant two-space since an invariant 1-space is included in a unique component. The induced group (i.e., the factor group mod the subgroup fixing the 2-space pointwise) is then a subgroup of $GL(2, q)$. The subgroups of $GL(2, q)$ are well known. (The author likes to think of them as collineation groups of a Desarguesian affine plane.) The non-solvable ones which contain no p-elements have $SL(2, 5)$ as a minimal non-solvable normal subgroup. Some of the irregular nearfield planes and other planes related to them admit groups of homologies containing $SL(2, 5)$.

Added in proof. It should be pointed out that Walker's thesis contains a proof of (2.8) for the case $p^s = q$.

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