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COGROWTH SERIES OF FREE PRODUCTS OF FINITE AND FREE GROUPS

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1. Introduction. Let $A = \{a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}\}$ and *iteration* of A denoted by $A^\star$ to be the set of words in A (including the empty word). Let $S \subseteq A^\star$; then the *growth function* of the set S is the function $\Gamma(l) =$ number of words in S of length l. For $m \leq n$ let $\vec{a} = (a_{i_1}, \ldots, a_{i_m})$, where $i_k \in \{1, \ldots, n\}$ are different; then the *relative* growth function with respect to $\vec{a}$ is the function $\Gamma_{\vec{a}}(l, l_1, \ldots, l_m) =$ number of words in S of length $l + l_1 + \cdots + l_m$ having (for each k) l_k total occurrences of a_{i_k} and $a_{i_k}{}^{-1}$. Then, the *growth series* of S and the *relative growth series* of S with respect to $\vec{a}$ are the functions

$$\gamma(S; t) = \sum_{l=0}^{\infty} \Gamma(l) t^l$$

and

$$\gamma_{\vec{a}}(S; t, t_{i_1}, \ldots, t_{i_m}) = \sum_{l, l_1, \ldots, l_m = 0}^{\infty} \Gamma_{\vec{a}}(l, l_1, \ldots, l_m) t^l t_{i1}^{l1} \cdots t_{im}^{lm},$$

respectively. The set S or variables $t, t_{i_1}, \ldots, t_{i_m}$ will be omitted when understood. Thus, we will sometimes write $\gamma_{\vec{a}}(S)$ or $\gamma_{\vec{a}}(t, t_{i_1}, \ldots, t_{i_m})$ instead of $\gamma_{\vec{a}}(S; t, t_{i_1}, \ldots, t_{i_m})$. Clearly, $\gamma(t) = \gamma_{\vec{a}}(t, \ldots, t)$. Let

$$1 \to N \to \mathbf{F}_n \xrightarrow{\phi} G \to 1$$

be a presentation for the group G, where $\mathbf{F}_n = \langle a_1, \ldots, a_n \rangle$ is the free group. Also let $W(k)$ be the set of freely reduced words w in $\mathbf{F}_n$ having length k and representing the identity of G. The function $\Upsilon(k) = |W(k)|$ is called the *cogrowth function* for this presentation (it is also the growth function of N), and the function

$$\upsilon(t) = \sum_{j=0}^{\infty} \Upsilon(j) t^j$$

is called the *cogrowth series* of the presentation. Let

$$\alpha(k) = \sum_{j=0}^{k} \Upsilon(j) \text{ and } \alpha = \lim_{k \to \infty} \alpha(k)^{1/k}.$$

Then α is called the *cogrowth* (or *cogrowth exponent*) of the presentation. By a result of Cohen [**1**] and Grigorchuk [**2**] a group G is *amenable* if and only if $\alpha = 2n-1$. Since

$1/\alpha$ is equal to the radius of convergence of the cogrowth series, which is determined by the critical point of the cogrowth series having smallest absolute value, the cogrowth series gives a way to determine whether a group is amenable, although in most cases this is not the most convenient way. The cogrowth and the cogrowth series were studied in [**1**, **2**, **3**, **4**, **6**, **9**, **10**].

In this paper, we present a way of calculating the cogrowth series of free products of finite and free groups. We explicitly calculate the cogrowth series of the presentations $D_\infty = \langle x, y | x^2 = y^2 = id \rangle$, $PSL_2 = \langle x, y | x^2 = y^3 = id \rangle$, and $(x_1, \ldots, x_n | x_1^k, \ldots, x_n^k)$ for $k = 2,3$ (see Section 3). In some other cases, we use the discriminant to calculate the cogrowth.

The return generating function (see [**8**]) for a group can be defined in the same way as the cogrowth series if we take the presentation

$$1 \to N \to \mathbf{N}_n \to G \to 1,$$

where $\mathbf{N}_n$ is the free monoid on A. A similar result for those functions was obtained by Gregory Quenell [**8**].

2. The method. Since the cogrowth series of the presentation of G is the same as the growth series of the set $W = \bigcup_{k \in \mathbf{Z}} W(k)$, we need to find the growth series of W. Let B and C be alphabets, and let $x \in C$, $X \subseteq B^\star$, $Y \subseteq C^\star$. Define the operation of *substitution* of the set X for the letter x in every word of the set Y by

$$Y|_{x=X} = \{w_1 \upsilon_1^{\varepsilon 1} w_2 \upsilon_2^{\varepsilon 2} \ldots w_n | n \in \boldsymbol{Z}, \upsilon_1, \ldots, \upsilon_{n-1} \in X, w = w_1 x^{\varepsilon 1} w_2 x^{\varepsilon 2} \ldots w_n \in Y\},$$

where $\varepsilon_i = \pm 1$ and $\upsilon^{-1} = b_l^{-1} \ldots b_1^{-1}$ if $\upsilon = b_1 \ldots b_l$, where the b_i's are in the alphabet of X (if x^{-1} is present in a word of Y then we assume that for every letter b in the alphabet of X, the letter b^{-1} is in the alphabet of X as well). Multiple substitutions are defined by

$$Y|_{(x_1 = X_1, \ldots, x_m = X_m)} = (\ldots (Y|_{x_1 = X_1})| \ldots)|_{x_m = X_m},$$

where $X_i \subseteq B$, $i = 1, \ldots, m$. Let $\mathcal{P}(X)$ denote the power set of X.If $w \in (A \cup \{x_j^{\pm 1}\}_{j \in M})^\star$, where M is a set of indices, then we will let $I(w) \subseteq M$ denote the set of all indices i such that x_i or x_i^{-1} is a letter of w.

The method we use is as follows.

1. We find a collection of sets S_i, $i = 0,1,2,\ldots$, where $S_i \subseteq S_{i+1} \subseteq A^\star$ (for all i) and partition each S_i into substets S_i^j, $j \in M = \{1, \ldots, m\}$ (where $S_i^j \subseteq S_{i+1}^j$ for all i and j), so that the following 2 conditions are satisfied.
2. We suppose that we have a function

$$f : M \to \mathcal{P}((A \bigsqcup \{x_j^{\pm 1} | j \in M\})^\star),$$

where $x_1, \ldots, x_m$ are letters not in A, such that

(a) f generates S_{i+1} from S_i in the following sense:

$$f(j)|_{(x_1 = S_i^1, \ldots, x_m = S_i^m)} = S_{i+1}^j, \text{ for all } i \geq 0 \text{ and } j \in M.$$

(b) f is 1-to-1 as explained further. Denote $\bigcup_{i=0}^{\infty} S_i^j$ by S^j. Then this property can be described as follows: a word in $f(j)|_{(x_1=S^1,\ldots,x_m=S^m)}$ can be obtained from only one word $w \in f(j)$ by substituting a word from S^j for every x_j; also, there is only one combination of these words from S_j which gives the original word. Formally, the property can be written as follows. Suppose we have words

$$w = w_1 x_{j_1} w_2 x_{j_2} \ldots w_{l-1} x_{j_{l-1}} w_l \in f(j),$$

$$w' = w'_1 x_{j'_1} w'_2 x_{j'_2} \ldots w'_{l'-1} x_{j_{l'-1}} w'_l \in f(j),$$

where w_i, $w_i' \in A^\star$, and $\upsilon_\alpha \in \bigcup_{i=0}^{\infty} S_i^{j_\alpha}$ for all $\alpha \in I(w)$, $\upsilon'_\alpha \in \bigcup_{i=0}^{\infty} S_i^{j'_\alpha}$ for all $\alpha \in I(w')$. Suppose we also have

$$w_1 \upsilon_{j1} w_2 \ldots \upsilon_{j_l} w_l = w'_1 \upsilon'_{j'_1} w'_2 \ldots \upsilon'_{j'_{l'}} w'_{l'}.$$

Then

$$(w, w_1, \upsilon_{j1}, w_2, \ldots, \upsilon_{j_l}, w_l) = (w', w'_1, \upsilon'_{j'_l}, w'_2, \ldots, \upsilon'_{j'_{l'}}, w'_l)$$

as ordered sequences. For simplicity, the letter x (with indices) will be reserved for substitution variables, as in the above.

3. We suppose that we can calculate the growth series of W from the growth series of the $S^j = \bigcup_{i=0}^{\infty} S_i^j$.

Then we can calculate the cogrowth series of the presentation using the following result:

THEOREM 1. *In the situation described above f induces the following system of functional equations on the series $\gamma_j = \gamma(S^j)$, $j \in M$:*

$$\begin{cases} \gamma_1 &= f_1^\star(\gamma_1, \ldots, \gamma_m) \\ \vdots & \vdots \qquad \vdots \\ \gamma_m &= f_m^\star(\gamma_1, \ldots, \gamma_m) \end{cases} \tag{1}$$

where $f_j^\star$ is the operator induced by $f(j)$: if $g_1(t), \ldots, g_m(t)$ are functions, then

$$f_j^\star(g_1(t), \ldots, g_m(t)) = \gamma_{(x_1,\ldots,x_m)}(f(j); t, g_1(t), \ldots, g_m(t)).$$

Proof. By the generating property 2(a) of f we have:

$$f(j)|_{(x_1=S_i^1,\ldots,x_m=S_i^m)} = S_{i+1}^j \text{ for all } j \in M.$$

Hence,

$$f(j)|_{(x_1=S^1,\ldots,x_m=S^m)} = S^j \text{ for all } j \in M,$$

and so,

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$$\gamma(f(j)|_{(x_1=S^1,\ldots,x_m=S^m)}) = \gamma_j.$$

The result follows from the equality

$$\gamma(f(j)|_{(x_1=S^1,\ldots,x_m=S^m)}) = \gamma_{(x_1,\ldots,x_m)}(f(j); t, \gamma_1, \ldots, \gamma_m).$$

The last equality is obtained by repeatedly applying the following result.

LEMMA 1. *Let X be a set of words in some alphabet containing x, let Y be another set of words, and let $Z = X|_{x=Y}$. Suppose that if $w = w_1xw_2x\ldots w_{n1+1}$, $w' = w'_1 xw'_2x\ldots w'_{n2+1} \in X$ and $\upsilon_1, \ldots, \upsilon_{n1}, \upsilon'_1, \ldots, \upsilon'_{n2} \in Y$, with*

$$w_1\upsilon_1w_2\upsilon_2\ldots w_{n_1+1} = w'_1\upsilon'_1w'_2\upsilon'_2\ldots w'_{n_1+1}$$

then

$$(w, \upsilon_1, \ldots, \upsilon_{n_1}) = (w', \upsilon'_1, \ldots, \upsilon'_{n_2})$$

(*this is a reformulation of the 1-to-1 property*). *Then*

$$\gamma(Z; t) = \gamma_x(X; t, \gamma(Y; t)).$$

Proof. We have:

$$Z = X|_{x=Y} = \{w|_{x=Y} | w \in X\} = \bigsqcup_{w\in X} \{w\}|_{x=Y}.$$

The union is disjoint because of the given property of words in Z. Hence,

$$\gamma(Z) = \sum_{w\in X} \gamma(\{w\}|_{x=Y}). \tag{2}$$

Let $w = b_1xb_2x\ldots b_{l-1}xb_l \in X$, where $b_i \neq x$ are in the alphabet of X or empty words. Then

$$\{w\}|_{x=X} = \{b_1\upsilon_1b_2\ldots\upsilon_lb_l | \forall i \upsilon_i \in Y\} = \bigsqcup_{(\upsilon_1,\ldots,\upsilon_l)\in Y\times\ldots\times Y} \{b_1\upsilon_1b_2\ldots\upsilon_lb_l\}.$$

Again, the union is disjoint because of the given property of words in Z. Hence,

$$\begin{aligned}
\gamma(\{w\})|_{x=Y}) &= \sum_{(\upsilon_1,\ldots,\upsilon_l)\in Y\times\ldots\times Y} \gamma(\{b_1\upsilon_1b_2\ldots\upsilon_lb_{l+1}\}) \\
&= \sum_{\upsilon_1\in Y}\sum_{\upsilon_2\in Y}\cdots\sum_{\upsilon_l\in Y} \gamma(\{b_1\})\gamma(\{\upsilon_1\})\ldots\gamma(\{\upsilon_l\})\gamma(\{b_{l+1}\}) \\
&= \gamma(\{b_1b_2\ldots b_{l+1}\})\left(\sum_{\upsilon\in Y}\gamma(\{\upsilon\})\right)^l \\
&= \gamma(\{b_1b_2\ldots b_{l+1}\})\gamma^l(Y) = \gamma_x(\{b_1xb_2\ldots xb_{l+1}\}; t, \gamma(Y)).
\end{aligned}$$

The last inequality is true since

$$\gamma_x(\{b_1xb_2\dots xb_{l+1}\};\, t, x) = \gamma(\{b_1b_2\dots b_{l+1}\};\, t)x^l.$$

Substituting the above expression for $\gamma(\{w\}|_{x=Y})$ into (2), we obtain

$$\begin{aligned}\gamma(Z;\, t) &= \sum_{b_1xb_2\dots xb_l\in X} \gamma(\{b_1xb_2\dots xb_l\}|_{x=Y};\, t)\\ &= \sum_{b_1xb_2\dots xb_l\in X} \gamma_x(\{b_1xb_2\dots xb_l\};\, t, \gamma(Y))\\ &= \gamma_x\left(\bigsqcup_{b_1xb_2\dots xb_l\in X} \{b_1xb_2\dots xb_l\};\, t, \gamma(Y)\right)\\ &= \gamma_x(X;\, t, \gamma(Y)).\end{aligned}$$

This proves the lemma and concludes the proof of Theorem 1.

Remark. The following three statements can be useful.

1. The solution of the system of equations (1) can be considered as a fixed point of the operator F:

$$\begin{pmatrix} g_1 \\ \vdots \\ g_m \end{pmatrix} \to \begin{pmatrix} f_1^\star(g_1, \dots, g_m) \\ \vdots \\ f_m^\star(g_1, \dots, g_m) \end{pmatrix}.$$

2. The system of equations (1) can be considered as iterative: as can be easily seen from the above proof, if we take as a beginning approximation $\gamma_j = 0$, and substitute it in the right hand side of (1), then we get $\gamma(S_0)$. Substituting in $\gamma(S_0)$ gives $\gamma(S_1)$, and so on.
3. In general a system of functional equations can have more than one solution, but as the previous consideration shows, there exists a unique solution with Maclaurin series having the free coefficient 0.

In the presentation described in Section 1, $a_i^{\pm 1}$, $i=1,\dots,n$ are called *generators*. A word in the generators is called a *relator* if it represents the identity element of G under the map ψ: $A^\star \to G$ induced by the epimorphism ϕ: $F_n \to G$. An element of W is called a *freely reduced relator*. A freely reduced relator is called *simple* if it is not a concatenation of 2 or more non-trivial ($\neq id$) relators.

The following statement is obvious.

Lemma 2. *Every relator υ is a product of simple relators $\upsilon_1,\dots,\upsilon_k$. Furthermore, the correspondence $\upsilon \leftrightarrow (\upsilon_1,\dots,\upsilon_k)$, $k \geq 0$ between relators and ordered finite sets of simple relators is a bijection.*

3. Cogrowth series of finite groups.

Theorem 2. *Let $G=\{g_1,\dots,g_N\}$ be a finite group, $g_1 = id$, and $P = \langle a_1,\dots,a_n | R\rangle$ be a presentation of G. Then the relative cogrowth of P with respect to $\vec{a}=(a_1,\dots,a_n)$ is $\upsilon_{\vec{a}}(P) = \sum_{k,\varepsilon} \gamma^{\varepsilon}_{g1k} + 1$, where the $\gamma^{\varepsilon}_{g1k}$ are determined by the following system of linear equations in the variables $\gamma^{\varepsilon}_{\beta k}$, where $\beta \in G$; $k=1,\dots,n$; $\varepsilon = \pm$ (here $\tilde{\varepsilon} = \varepsilon 1$)*

$$\left\{ \gamma^{\varepsilon}_{\beta k} = (\sum_{\substack{k' \neq k \\ \varepsilon'}} \gamma^{\varepsilon'}_{\beta\phi(a_k)^{-\tilde{\varepsilon}},k'} + \gamma^{\varepsilon}_{\beta\phi(a_k)^{-\tilde{\varepsilon}},k} + \delta^{\phi(a_k)^{\tilde{\varepsilon}}}_{\beta}) a_k \right\}_{\beta,k,\varepsilon}.$$

In particular the relative cogrowth of P is a rational function.

Proof. In this proof all growth series are assumed to be relative growth series with respect to $\vec{a}$. The result will follow from Theorem 1. Thus we follow the steps of the method of Section 2.

1. Define S_i to be the set of non-empty freely reduced words in $A^\star$ (where $A = \{a_1^{\pm 1}, \ldots, a_n^{\pm 1}\}$ as in Section 1) of length $\leq i+1$. Partition each S_i into $S_i^{\beta k \varepsilon}$ where $\beta \in G$; $1 \leq k \leq n$; $\varepsilon = \pm 1$. Here $S_i^{\beta k \varepsilon}$ is defined as the set of words $w \in S_i$ such that $\psi(w) = \beta$, and $a_k^{\tilde{\varepsilon}}$ (where $\tilde{\varepsilon} = \varepsilon 1$) is the last letter of w. Obviously, $S_i^{\beta k \varepsilon} \subseteq S_{i+1}^{\beta k \varepsilon}$.
2. We define

$$f(\beta, k, \varepsilon) = \left(\bigcup_{\substack{k' \neq k \\ \varepsilon'}} \{x_{\beta b_k^{-\tilde{\varepsilon}},k',\varepsilon'} a_k^{\tilde{\varepsilon}}\} \right) \bigcup \{x_{\beta b_k^{-\tilde{\varepsilon}},k,\varepsilon} a_k^{\tilde{\varepsilon}}\} \bigcup_{\text{if } \beta = b_k^{\tilde{\varepsilon}}} \{a_k^{\tilde{\varepsilon}}\},$$

   where $b_k = \phi(a_k)$. Since the right hand side represents the different ways that a word of length $i+1$ which represents $\beta \in G$ can be obtained from a word of length i, properties 2(a) and 2(b), required by the method, are satisfied.
3. Since $W = \cup_{k,\varepsilon} \cup_{i=0}^{\infty} S_i^{g_1 k \varepsilon} \cup \{id\}$, the formula for υ follows.

Theorem 1 gives the stated system, where $\gamma^{\varepsilon}_{\beta k} = \gamma_{\vec{a}}(S^{\beta k \varepsilon})$ (as in Section 2, $S^{\beta k \varepsilon} = \cup_{i=0}^{\infty} S_i^{\beta k \varepsilon}$).

4. The infinite cyclic group in the presentation $P_{01} = \langle a,b|b \rangle$. The cogrowth of this presentation has been calculated using a completely different technique in [**3**]. However, we will need the relative cogrowth with respect to (a, b). Define the *height* $h(w)$ of a relator w in P_{01} inductively as follows: if $w = w_1 \ldots w_k$, where w_i's are simple relators then $h(w) = \max\{h(w_i) | i = 1, \ldots, k\}$; if w is a simple relator then $w = b^{\pm 1}$ or $w = a^{\varepsilon} w' a^{-\varepsilon}$, where $\varepsilon = \pm 1$. In the first case we define $h(w) = 1$, and in the second case we define $h(w) = h(w') + 1$. Now we follow the method of Section 2.

1. Define S_i to be the set of simple relators beginning with a of height not more than $i+2$. Further subdivision of S_i will not be necessary.
2. In this case, f is a function from a one point set. Identify f with the image of this point. Let W_1 be the set of non-empty words in $b^{\pm 1}$ and x, in which x doesn't follow itself and b^{ε} doesn't follow $b^{-\varepsilon}$ ($\varepsilon = \pm 1$). Define f to be the set of words in $a^{\pm 1}$, $b^{\pm 1}$, and x such that every word w has the form $a w_1 a^{-1}$, where $w_1 \in W_1$. Then property 2(a) is obvious and 2(b) follows from Lemma 2.
3. By Lemma 2, $W = X|_{x = \cup_i S_i}$, where X is the set of words in $b^{\pm 1}$, $x^{\pm 1}$ in which x^{ε} doesn't follow itself, and b^{ε} doesn't follow $b^{-\varepsilon}$ ($\varepsilon = \pm 1$). It is easy to see that

$\gamma_{(x,b)}(X) = \upsilon_{(x,b)}(P_{11})$, where P_{11} is the presentation of the identity group $\langle x,b|x,b\rangle$. Hence, using Lemma 1 and the results of Section 3, we have

$$\gamma_{(x,b)}(X;\, t,\, x,\, b) = \frac{1 + bx - b - x}{1 - (3bx + b + x)}$$

and

$$\gamma_{(a,b)}(W;\, t,\, a,\, b) = \gamma_{(x,b)}(X;\, t,\, \gamma_{(a,b)}(\cup_i S_i;\, t,\, a,\, b),\, b).$$

In the above equations, we used x, a, and b as letters of the alphabet in the subscripts of γ and as variable names otherwise. Whenever it does not cause confusion, such usage allows faster recognition of the variable that corresponds to a given letter of the alphabet.

Finally

$$f^\star = \gamma_{(a,b,x)}(f) = a^2\gamma_{(x,b)}(W_1),$$

and calculating $\gamma := \gamma_{(x,b)}(W_1)$ similarly to $\gamma_{(x,b)}(P_{11})$, we get the system:

$$\begin{cases} \gamma &= \gamma_b^+ + \gamma_b^- + \gamma_x^+, \\ \gamma_b^+ &= (\gamma_x^+ + \gamma_b^+ + 1)b, \\ \gamma_b^- &= (\gamma_x^+ + \gamma_b^- + 1)b, \\ \gamma_x^+ &= (\gamma_b^+ + \gamma_b^- + 1)x. \end{cases}$$

Solving this system and applying Theorem 1, we get the following equation on $\gamma := \gamma_{(a,b)}(\cup_i S_i)$:

$$\gamma = a^2\frac{2b + 3b\gamma + \gamma}{1 - (2\gamma + 1)b}.$$

Solving this equation (using also the fact that γ should have Maclaurin series with first coefficient 0) and substituting the result in the function obtained at step 3, after simplifications, we get

$$\gamma_{(a,b)}(P_{01}) = \frac{(b+1)\sqrt{1 - a^2}}{\sqrt{(1 + a - b + 3ab)(1 - a - b - 3ab)}}.$$

5. Free product of finite and free groups. Suppose we have a free product of finite and free groups G in the presentation $P = P_1 * \ldots * P_n$, where $P_j = (a_{j1},\ldots,a_{jn_j}|R_j)$, $j = 1,\ldots,n'$; are presentations of finite groups, and $P_j = \langle a_{j1}\rangle$, $j = n'+1,\ldots,n$ are presentations of free groups on one generator. Also assume that the generators of different P_j's are distinct. Then we can identify any element of P_j with its image in P (Prop. 4.1 in [**5**]). Let w be a simple relator in this presentation. The following procedure defines the 0'th *level* of w:

1. Include the first letter of w in the 0'th level and let $w = b_1w_1$; where b_1 is a generator, w_1 is a word. Let j be such that $b_1 \in P_j$.
2. If we have $w = b_1\upsilon_1 b_2\ldots\upsilon_{l-1}b_l$ ($l \geq 1$), where $b_1,\ldots,b_l$ are generators included in the 0'th level, then we are done.

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3. If we have $w=b_1\upsilon_1 b_2\ldots\upsilon_{l-1}b_l w_l$ $(l\geq 1)$, where $b_1,\ldots,b_l$ are generators included in the 0'th level, and $w_l\neq id$, then let υ_l be the shortest initial subword of w_l (possibly empty) which is a relator, with the property that the generator b_{l+1} which follows it is in P_j (where j has been fixed in the first step). Note, that υ_l exists since w is a simple relator (see Exercise 1 of 4.1 in [**5**]). Thus, we have $w=b_1\upsilon_1\ldots\upsilon_l b_{l+1}w_{l+1}$ for some word w_{l+1} (possibly empty). Include b_{l+1} in the 0'th level and repeat beginning with step 2.

This process will stop because the length of each w_l is at least 1 less than the length of w_{l-1}. Define the 0'th *level* of a relator w to be the union of the 0'th levels of the simple relators of which w is the concatenation.

Now we can state the inductive definition of the *level* of a relator as follows. The empty word is a word of *level-1*. The non-empty relator $w=b_1\upsilon_1\ldots b_{l-1}\upsilon_{l-1}b_l$, where $b_1,\ldots,b_l$ are letters that form the 0'th level is said to be of *level i* if the maximum level of the υ_j's is $i-1$.

To calculate the cogrowth series of the presentation P, we follow the steps of the method of Section 2:

Step 1. Define S_i^j, $i\geq 0$, $j=1,\ldots,n$ to be the set of non-empty freely reduced relators of level not more than i, the 0'th level of which forms a relator in P_j. Define $S_i=\sqcup_{j=1}^n S_i^j$, $i\geq 0$. Obviously, $S_i\subseteq S_{i+1}$ and $S_i^j\subseteq S_{i+1}^j$ for all i and j. Later in the section we will need the following result.

Lemma 3. *Any relator r in P can be uniquely written as a concatenation $r_1\ldots r_l$ of relators satisfying*

(*i*) *the 0'th level of each r_i forms a relator in P_j for some $j=j(i)$ and*

(*ii*) $j(i)\neq j(i+1)$.

Proof. We have, $r=s_1\ldots s_p$ for some simple relators s_k. Let $r_1=s_1\ldots s_l$, where $s_1,\ldots,s_l$ form a relator in one of the P_j's, and l is such that $s_{l+1}\notin P_j$ for that j. The result follows by induction on p.

Step 2. Define $f(j)$ to be the set of non-empty freely reduced words in $a_{j1}^{\pm 1},\ldots,a_{jn_j}^{\pm 1}$, $x_1,\ldots,\hat{x_j},\ldots,x_n$ such that the following three conditions $(*)$ are satisfied:

(i) neither x_i nor x_i^{-1} follows itself (for $i=1,\ldots,n$),
(ii) if we remove all the x_i's, we get a relator in P_j; and
(iii) let w' be any initial subword having last letter x_i, and let w'' be the result of removing all the x_i's from w'. Then w'' is not a relator in P_j.

We claim that the function f defined above is the f referred to in Section 2. We need to cheek the two properties of f.

(a) Suppose we have $w=b_1w_1b_2w_2\ldots b_l\in S_{i+1}^j$, where $b_1,\ldots,b_l$ form the 0'th level of w. Let j be such that $b_1,\ldots,b_l\in P_j$. Then it follows from the procedure of finding the 0'th level that w_k, $k=1,\ldots,l-1$ are products of simple relators the 0'th level of which is not in P_j and of level not more then i. Hence,

$$w\in f(j)|_{(x_1=S_i^1,\ldots,x_n=S_i^n)}.$$

On the other hand suppose w was obtained from some word

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$$y = b_1 y_1 b_2 y_2 \dots b_l \in f(j)$$

(where y_k's are some words in x_k's) by the substitutions $(x_k = S_i^k)_k$. Then it is of level not more than $i+1$ and $b_1,\dots,b_l$ still form the 0'th level of w. Hence $w \in S^j_{i+1}$ and the property

$$f(j)|_{x_1=S_i^1,\dots,x_n=S_i^n} = S^j_{i+1}$$

is proven.

(b) If we have $w_1 v_{11}\dots = w'_1 v'_{11}\dots$ as in 2(b) of the method, then the 0'th level of $w_1 v_1 \dots$ is equal to the 0'th level of $w'_1 v'_1 \dots$. This means that

$$(w_1, v_{11}\dots v_{1n_1}, w_2, \dots) = (w'_1, v'_{11}\dots v'_{1n'_1}, w'_2, \dots).$$

Applying Lemma 3 for $v_{k1}\dots v_{kn_k} = v'_{k1}\dots v'_{kn'_k}$, we get

$$(v_{k1}, \dots, v_{kn_k}) = (v'_{k1}, \dots, v'_{kn'_k}).$$

Thus the one to one property is satisfied.

Step 3. By Lemma 3, $W = X|_{(x_1=S^1,\dots,x_n=S^n)}$, where X is the set of words in $x_1,\dots,x_n$, such that x_i doesn't follow itself (X includes the empty word). Hence, by Lemma 1, we get

$$\gamma(W) = \gamma_{\vec{x}}(X)(\gamma(\cup_i S_i^1), \dots, \gamma(\cup_i S_i^n)).$$

To calculate $\gamma_{\vec{x}}(X)$, we again use the method of Section 3. By a proof analogous to the one in Section 3, we get that $\gamma_{\vec{x}}(X) = \tilde{\gamma}_n$, where $\tilde{\gamma}_n$ is determined by the following linear system on $\tilde{\gamma}_n, \gamma_1,\dots,\gamma_n$:

$$\begin{cases} \tilde{\gamma}_n & = & \gamma_{x_1} + \dots + \gamma_{x_n}, \\ \gamma_{x_1} & = & (\tilde{\gamma}_n - \gamma_{x_1} + 1)x_1, \\ \vdots & \vdots & \vdots \\ \gamma_{x_n} & = & (\tilde{\gamma}_n - \gamma_{x_n} + 1)x_n. \end{cases}$$

Now, we need to calculate the induced functions. Recall that $f(j)$ consists of freely reduced words in $a_{j1}^{\pm 1}, \dots, a_{jn_j}^{\pm 1}, x_1,\dots, \widehat{x_j},\dots,x_n$ such that conditions (*) are satisfied. Consider words in $f(j)$ in the following way: we have the set of relators of P_j; then, in each relator of P_j, we insert words in $x_1,\dots, \widehat{x_j},\dots,x_n$ such that no x_i follows itself, so that the relator is freely reduced and conditions (*) are still satisfied. The relative growth of non-empty words which we insert can be calculated as in step 3, and is $\tilde{\gamma}_{n-1} = \tilde{\gamma}_{n-1}(x_1, \dots, \widehat{x_j}, \dots, x_n)$. After that, in the case $1 \leq j \leq n'$ we follow the reasoning of Section 3, inducting on the total number of $a_{ji}^{\pm 1}$'s instead of length. It is easy to derive the following system of linear equations in the variables $\gamma^{\varepsilon}_{\beta k}$, where $\beta \in G_j; k = 1,\dots,n_j; k' = 1,\dots,n_j; \varepsilon, \varepsilon' = \pm$ (use $\tilde{\varepsilon} = \varepsilon 1$):

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$$\begin{cases} \left\{ \gamma^{\varepsilon}_{\beta k} = \left(\sum_{\frac{k' \neq k}{\varepsilon'}} \gamma^{\varepsilon'}_{\beta a^{-\tilde{\varepsilon}}_{jk}, k'} + \gamma^{\varepsilon}_{\beta a^{-\tilde{\varepsilon}}_{jk}, k} \right) (\tilde{\gamma}_{n-1} + 1) a_{jk} + \gamma^{-\varepsilon}_{\beta a^{-\tilde{\varepsilon}}_{jk}, k'} \tilde{\gamma}_{n-1} a_{jk} \right\}_{\beta \neq a^{\tilde{\varepsilon}}_{jk}, k, \varepsilon} \\ \gamma^{\varepsilon}_{a^{\tilde{\varepsilon}}_{jk}} = \left(\sum_{\frac{k' \neq k}{\varepsilon'}} \gamma^{\varepsilon'}_{id,k'} + \gamma^{\varepsilon}_{id,k} + 1 \right) a_{jk} \end{cases}$$

with

$$f^{\star}_j(x_1, \ldots, x_n) = \gamma_{\bar{x}}(f(j)) = \sum_{\varepsilon,k} \gamma^{\varepsilon}_{id,k}|_{(a_{j1}=t,\ldots,a_{jn}=t)}.$$

In the case $n'+1 \leq j \leq n$, we use the ideas of Section 4: we look on the relative growth series of the freely reduced words in $a^{\pm 1}_{j1}$, y satisfying the conditions (*), where y stands for words in x_i's. Since there is no y^{-1}, and y shall not follow itself the system from Section 4 will be changed to (changing all a's to a_{j1}'s and b's to y's)

$$\begin{cases} \gamma &= \gamma^+_y + \gamma^+_x, \\ \gamma^+_y &= (\gamma^+_x + 1)y, \\ \gamma^+_x &= (\gamma^+_y + 1)x. \end{cases}$$

Also, since the level 0 shall be a relator in P_j,

$$\gamma_{(a_{j1},y)}(W) = \gamma_z(X)|_{z=\gamma_{(a_{j1},y)}}(\cup_i S_i),$$

where $X = \{z^k | k \in \mathbf{Z} \backslash \{0\}\}$ with $\gamma_z(X) = 2z/(1-z)$. Thus, we finally get

$$f^{\star}_j = \left(\frac{y(1-t^2) - \sqrt{(1-t^2)(1-(t+2ty)^2)}}{-1+t^2+y+3t^2y} - 1 \right)|_{(y=\tilde{\gamma}_{n-1}(x_1,\ldots,\widehat{x_j},\ldots,x_n))}.$$

Remark. The method gives n rational equations; (the equations induced by $f(j)$, $j > n'$ can be easily rewritten in rational form as well). However, if we have some $P_j = P_{j'}$, then the corresponding equations will be the same, so we can eliminate all but one of them. Also, the equation for $P_j = \langle a_{j1} | a_{j1} \rangle$ is $\gamma_j = t$. So, we can eliminate it as well. Furthermore, if one of the presentations of the finite groups is not repeated, we will have an equation $\gamma_j = F(\gamma_1, \ldots, \widehat{\gamma_j}, \ldots, \gamma_n)$, and, we can use it for substitution. Finally, by clearing denominators, we can make the equations polynomial.

6. Examples.

Example 1. Let $P_j = \langle a_j | a^2_j \rangle$; i.e. we have the presentation $P = \langle a_1, \ldots, a_n | a_1{}^2, \ldots, a^2{}_n \rangle$. In this case, all the induced equations are the same, and so we need to solve only one equation with one variable $\gamma = \gamma_1 = \ldots = \gamma_n$.

The induced function will be

$$f^{\star}(x, \ldots, x) = 2\frac{(1+2\tilde{\gamma}_{n-1})t^2}{1-(1+2\tilde{\gamma}_{n-1})t^2},$$

where

$$\tilde{\gamma}_{n-1} = \tilde{\gamma}_{n-1}(x, \ldots, x) = \frac{(n-1)x}{1-(n-2)x}.$$

The equation given by Theorem 1 will be (after simplification)

$$(nt^2 + n - 2)\gamma^2 + (2nt^2 + t^2 - 1)\gamma + 2t^2 = 0.$$

Solving this equation and choosing the solution with the correct Maclaurin series, we get

$$\gamma = \frac{1-(2n+1)t^2 - \sqrt{1-2(6n-7)t^2+(2n-1)^2t^4}}{2(n-2+nt^2)}.$$

Finally, using the fact that the cogrowth series is $\tilde{\gamma}_n(\gamma_1,\ldots,\gamma_n)$, we get

$$\upsilon = \frac{(2-n)((2n-1)t+1) + n\sqrt{1-2(6n-7)t^2+(2n-1)^2t}}{2(1-(2n-1)^2t^2)}.$$

For the case $n=2$, we have the infinite dihedral group with

$$\upsilon(D_\infty) = \sqrt{\frac{1-t^2}{1-9t^2}},$$

which is amenable. Analysing the singular points of the cogrowth series when $n>2$, we find that the cogrowth of this presentation is

$$\sqrt{6n-7-4\sqrt{(2n-3)(n-1)}},$$

and thus, the group is not amenable.

Example 2. Let $P_j = \langle a_j | a_j^3 \rangle$, and $n>1$; i.e. we have the presentation $P = \langle a_1,\ldots,a_n | a_1^3,\ldots,a_n^3 \rangle$. In this case we see that the induced function is

$$f^\star(x,\ldots,x) = \frac{2(\tilde{\gamma}_{n-1} + (1+2\tilde{\gamma}_{n-1})t)t^2}{1-(\tilde{\gamma}_{n-1}t + \tilde{\gamma}_{n-1}t^2 + (1+2\tilde{\gamma}_{n-1})t^3)}$$

and all the rest is quite similar to the previous example. We get that the cogrowth series of P is

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$$\upsilon = \frac{-(1+t)((n-2)+nt+(n-2)(2n-1)t^2 - n\sqrt{R(t)})}{2(1-(2n-1)t)(1+nt+(2n-1)t^2)},$$

where $R(t) = 1 - 2t + (7-4n)t^2 - 2(2n-1)t^3 + (2n-1)^2 t^4$;

the cogrowth is

$$\frac{1 + 2\sqrt{2n-2} - \sqrt{4\sqrt{2n-2} - 3}}{2(2n-1)}$$

and thus, the group is not amenable.

EXAMPLE 3. Let $n=2$, $P_1 = \langle a | a^2 \rangle$, $P_2 = \langle b | b^3 \rangle$; i.e. we have the presentation $P = \langle a,b | a^2, b^3 \rangle$, which is a presentation of the projective special linear group $PSL_2(\mathbf{Z})$. In this case the cogrowth series is

$$\upsilon = \frac{(t+1)(9t^5 - 3t^4 + 8t^3 - t^2 + t - (6t^2 - t + 2)\sqrt{R(t)})}{2(3t-1)(3t^2+1)(3t^2+3t+1)(3t^2-t+1)},$$

where $R(t) = 81t^8 - 54t^7 + 9t^6 - 18t^5 - 8t^4 - 6t^3 + t^2 - 2t + 1$.

The cogrowth is $1/r \approx 2.924984549$, where r is the root of $R(t)$ which is closest to .3418821478. The group is not amenable.

EXAMPLE 4. Let again $n=2$, $P_1 = \langle a | \rangle$ and $P_2 = \langle b | b^2 \rangle$; i.e. we have the presentation $P = \langle a,b | b^2 \rangle$. In this case after simplification, we will have the following equation for the growth series γ of freely reduced relators the 0'th level of which is a word in a, a^{-1}:

$$8t^4 + (-2 + 4t^2 + 30t^4)\gamma + (-1 + 33t^4 + 8t^2)\gamma^2 + 2t^2(5t^2+3)\gamma^3 = 0.$$

It is possible to solve this equation in square and cubic roots, but the solution is too long to be written here. Nevertheless, it is relatively simple to find the cogrowth as follows. We know that the cogrowth is $1/r$, where r is the smallest by absolute value singular point of γ, since the cogrowth of P is equal to the growth of the freely reduced relators having the 0'th level consisting of letters a and a^{-1}. We know that the singular points of a solution of a polynomial equation are among the roots of the discriminant. Also, we have that the cogrowth of P shall be in the interval $[\sqrt{3}, 3]$, since the cogrowth of a presentation having $2n$ generators is in the interval $[\sqrt{2n-1}, 2n-1]$ (see [**1**]), and the corresponding singular point will be positive (since we have series with positive coefficients). The above identifies the cogrowth as $1/r \approx 2.668565567$, where r is the root of

$$729t^{12} + 2430t^{10} - 945t^8 - 1052t^6 - 105t^4 + 30t^2 + 1$$

closest to .3747331572. This constant also was calculated in [**1**].

EXAMPLE 5. Let us consider how the cogrowth of the presentations $P_k = \langle x,y | x^2, y^k \rangle$ behaves when k increases. If we calculate the cogrowth of P_k as in the Example 4, we will get

α_2	=	3.00000000000000,	α_{12}	=	2.67569761213378,
α_3	=	2.92498454967618,	α_{13}	=	2.67334760233820,
α_4	=	2.84166308599088,	...	...	....
α_5	=	2.78382043885166,	α_{20}	=	2.66883920237601,
α_6	=	2.74550209084132,	...	...	....
α_7	=	2.72015120914404,	α_{30}	=	2.66856981326703,
α_8	=	2.70326872061827,	...	...	....
α_9	=	2.69195078417217,	...	...	....
α_{10}	=	2.68432683301238,	...	...	....
α_{11}	=	2.67917768540835,	...	...	....
α_{12}	=	2.67569761213378,	α_∞	=	2.66856556650517,

where α_k is the cogrowth of P_k. As one can see, α_k is approaching α_∞, when $k \to \infty$.

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