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THE HILBERT TRANSFORM OF GENERALIZED FUNCTIONS AND APPLICATIONS

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Introduction. The theory of Fourier transforms of tempered distributions as developed by Laurent Schwartz [**17**] is quite simple and elegant and has wide variety of applications, but there does not exist a corresponding neat and simple theory for the Hilbert transform of generalized functions (distributions) having wide applications. One of the objectives of this paper is to develop such a theory for the Hilbert transform of generalized functions and indicate its applicability to a variety of problems.

In problems of physics sometimes we need to find harmonic functions $u(x, y)$ in the region $y > 0$ whose limit as $y \to 0+$ does not exist in pointwise sense but does exist in the distributional sense. The theory of Hilbert transform of generalized functions that we are going to develop will provide an answer to the existence and uniqueness of this problem.

Hilbert transform for distributions in various subspaces of $\mathcal{D}'$ were investigated by a number of authors in [**2**], [**3**], [**5**]-[**13**], [**15**]-[**16**]. We note that Mitrovic in [**7**]-[**10**] has extended Hilbert transform to O'_α [**3**] and Orton in [**12**]-[**13**], to an arbitrary element of $\mathcal{D}'$. Methods followed by Orton [**12**]-[**13**] are dependent upon the analytic representation of distribution [**3**] which is not quite constructive for distributions which do not have compact support and as such the methods used by her [**12**] cannot be applied with sufficient ease to applied problems which involve computations of Hilbert transforms of distributions having non-compact support.

The work done by Mitrovic [**7**]-[**10**], concerning the Hilbert transform of distributions of O'_α suffers from similar drawbacks. Though the analytic representation $F(z)$ of an arbitrary $f \in O'_\alpha$ is straight forward; i.e., under suitable restrictions on F we have

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$$(1) \qquad F(z) = \frac{1}{2\pi i} < f(t), \frac{1}{t-z} > \qquad \operatorname{Im} z \neq 0,$$

the corresponding computation of the Hilbert transform of an arbitrary $f \in O'_\alpha$ is not straightforward. We will show that the classical Hilbert transform H is a homeomorphism from $\mathscr{D}_{L^p}$, $p > 1$ onto itself and then define the Hilbert transform Hf of $f \in (\mathscr{D}_{L^p})'$ as a generalized function in $(\mathscr{D}_{L^p})'$ by the relation

$$(2) \qquad < Hf, \varphi > = < f, -H\varphi > \quad \forall\, \varphi \in \mathscr{D}_{L^p}, p > 1$$

which is an analogue of a classical result proved in [**18**, p. 132]. The formula (2) enables us to compute Hilbert transform of any $f \in (\mathscr{D}_{L^p})'$ and thus in view of the inversion formula

$$(3) \qquad -\frac{1}{\pi^2} H^2 f = f \quad \forall\, f \in (\mathscr{D}_{L^p})', \quad p > 1$$

which we will prove in the sequel, it becomes quite simple to apply our technique to solve some singular integral equations in $(\mathscr{D}_{L^p})'$, $p > 1$. To our knowledge none of the theories of Hilbert transform of distributions including those of Orton and Mitrovic will be so well suited to solve singular integral equations. Besides, contrary to Orton and Mitrovic and others we will be interpreting limits in distributional boundary value problems in $(\mathscr{D}_{L^p})'$ (and not in $\mathscr{D}'$) and the weak convergence in $\mathscr{D}'$ of our solutions to boundary value problems follows as a simple corollary.

There have been considerable amounts of work on the solution to Hilbert problems in generalized function spaces by numerous authors in [**7**]-[**10**], [**12**]-[**13**], [**15**]-[**16**]. As a by-product of our theory, its uses in solving certain Hilbert problems in the space $(\mathscr{D}_{L^p})'$ will also be shown. We will prove that for any $f \in (\mathscr{D}_{L^p})'$, $p > 1$ the analytic function $F(z)$, $\operatorname{Im} z \neq 0$ defined by

$$(4) \qquad F(z) = \frac{1}{2\pi i} < f(t), \frac{1}{t-z} >$$

is an analytic representation of $f \in (\mathscr{D}_{L^p})'$ in the sense that

$$\lim_{y \to 0+} < F(x+iy) - F(x-iy), \varphi(x) > = < f, \varphi > \qquad \forall\, \varphi \in \mathscr{D}_{L^p}.$$

Again, it may be noted that though our space $(\mathscr{D}_{L^p})'$ is not quite as large as that of Orton and Mitrovic our definition of analytic representation turns out to be more general.

We will also solve the problem of distributional representation of analytic functions [10], for the space $(\mathscr{D}_{L^p})'$; that is given an analytic function $F(z)$ in the region $\operatorname{Im} z > 0$ and satisfying the uniform bound

$$(5) \qquad \sup_{\substack{-\infty < x < \infty \\ y \geqq \delta}} |F(x + iy)| \leqq A_\delta < \infty$$

and the uniform asymptotic order (with respect to x)

$$(6) \qquad |F(x + iy)| = o(1), \quad y \to \infty$$

such that the distributional limit (in $(\mathscr{D}_{L^p})'$) of $F(x + iy)$ as $y \to 0+$ exists and equals $f^+(t)$ then we wish to prove that for $\operatorname{Im} z > 0$,

$$(7) \qquad F(z) = \frac{1}{2\pi i} < f^+(t), \frac{1}{t - z} > .$$

The distributional representation (7) of the analytic function $F(z)$ defined by (5) and (6) is arrived at in a way similar to that arrived at by Mitrovic for the space O'_α [10].

The representation (7) will be exploited in proving the uniqueness of the solution to a Dirichlet boundary-value problem in the space $(\mathscr{D}_{L^p})'$. The existence of the solution will be proved in a very constructive way.

It is known that if $f \in L^p$, $1 < p < \infty$, then its Hilbert transform

$$(8) \qquad F(x) = P \int_{-\infty}^{\infty} \frac{f(t)}{t - x}\, dt \equiv Hf$$

(where the integral is taken in Cauchy principal-value sense), belongs to the space L^p [18, p. 132]. Therefore, if g is another function belonging to L^q where $1/p + 1/q = 1$ and $G(t)$ is its Hilbert transform then the integrals

$$\int_{-\infty}^{\infty} F(x)\, g(x)\, dx \quad \text{and} \quad \int_{-\infty}^{\infty} G(x) f(x)\, dx$$

both exist and

$$(9) \qquad \int_{-\infty}^{\infty} F(x)\, g(x)\, dx = -\int_{-\infty}^{\infty} G(t) f(t)\, dt. \quad [19, \text{p. } 170].$$

In adjoint notation this result can be written as

$$(10) \qquad < Hf, g > = < f, -Hg >$$

where Hf stands for the Hilbert transform of f in the classical and Cauchy principal-value sense. Our Hilbert transform of generalized functions will be defined in analogy to the result in (10). To this end we have to construct a testing function space of infinitely differentiable functions which is mapped by the operation H of the classical Hilbert transformation (as described before) into another (or the same) testing function space of infinitely differentiable functions in a continuous way.

The notation and terminology follow that of [**20**]. The set of real numbers is denoted by $\mathbf{R}$, and t, x and y are real variables unless otherwise stated. If f is a generalized function, then the notation $f(t)$ is used to indicate that the testing functions on which f is defined have t as their variable. The pairing between a testing function space and its dual is denoted by $< f, \varphi >$. The space of C^∞ functions on $\mathbf{R}$ having compact support is denoted by $\mathcal{D}$. Its dual is the space of Schwartz distributions on $\mathbf{R}$ [**17**].

Schwartz testing function space $\mathcal{D}_{L^p}$ $(1 < p < \infty)$. An infinitely differentiable function φ defined over $\mathbf{R}$ is said to belong to the space $\mathcal{D}_{L^p}$ if $\varphi^{(k)}(t)$ belongs to L^p for each $k = 0, 1, 2, 3, \ldots$. We assign a topology to the space $\mathcal{D}_{L^p}$ in such a way that the sequence $\varphi_j \in \mathcal{D}_{L^p}$ converges to 0 in $\mathcal{D}_{L^p}$ if $\varphi_j^{(k)}$ converges to 0 in L_p for each $k = 0, 1, 2, 3, \ldots$ as $j \to \infty$ [**17**, p. 199].

$\mathcal{D}$ is dense in $\mathcal{D}_{L^p}$ $(1 < p < \infty)$ and convergence in $\mathcal{D}$ implies convergence in $\mathcal{D}_{L^p}$ and consequently the restriction of $f \in (\mathcal{D}_{L^p})'$ to $\mathcal{D}$ is in $\mathcal{D}'$; i.e., $\mathcal{D}' \supset (\mathcal{D}_{L^p})'$ [**17**, p. 199]. An element φ of $\mathcal{D}_{L^p}$ is bounded on $\mathbf{R}$ and belongs to L^q, $q \geqq p$ and converges to zero as $|x| \to \infty$ [**17**, p. 200].

If $\{\varphi_j\}$ is a sequence in $\mathcal{D}_{L^p}$ converging to zero in $\mathcal{D}_{L^p}$, as $j \to \infty$ then $\varphi_j^{(k)} \to 0$ uniformly on every compact subset of $\mathbf{R}$ and each $k = 0, 1, 2, \ldots$. Consequently every distribution of compact support belongs to $(\mathcal{D}_{L^p})'$.

THEOREM 1. *Let H be the mapping defined by* (8). *Then for* $1 < p < \infty$, *H is a linear homeomorphism from $\mathcal{D}_{L^p}$ onto itself and*

$$H^{-1} = -\frac{1}{\pi^2} H.$$

Proof. If $\varphi \in \mathcal{D}_{L^p}$, then, denoting $H\varphi$ by F, we have

$$F(x) = P \int_{-\infty}^{\infty} \frac{\varphi(t)}{t - x}\, dt$$

$$= P \int_{-\infty}^{\infty} \frac{\varphi(t + x)}{t}\, dt$$

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$$= P \int_{-N}^{N} \frac{\varphi(t + x) - \varphi(x)}{t} \, dt + \left(\int_{-\infty}^{-N} + \int_{N}^{\infty} \right) \frac{\varphi(t + x)}{t} \, dt$$

$$= \int_{-N}^{N} \Psi(x, t) \, dt + \left(\int_{-\infty}^{-N} + \int_{N}^{\infty} \right) \frac{\varphi(t + x)}{t} \, dt,$$

where

$$\Psi(x, t) = \frac{\varphi(t + x) - \varphi(x)}{t}, \quad t \neq 0,$$

$$= \varphi'(x) \quad \text{when } t = 0.$$

We can show that $\Psi(x, t)$, along with all of its partial derivatives, is a continuous function of x, t for all real x and t.

Using Holder's inequality for integrals and Weierstrass' M-test for uniform convergence of integrals, we can show that for $N > 0$ each of the integrals

$$\int_{N}^{\infty} \frac{\varphi'(t + x)}{t} \, dt \quad \text{and} \quad \int_{-\infty}^{-N} \frac{\varphi'(t + x)}{t} \, dt$$

converges uniformly for all real x. Therefore, using standard results on interchange of order of integration and differentiation, we have

$$F'(x) = \int_{-N}^{N} \frac{\partial \Psi}{\partial x}(x, t) \, dt + \left(\int_{-\infty}^{-N} + \int_{N}^{\infty} \right) \frac{\varphi'(t + x)}{t} \, dt$$

$$= P \int_{-N}^{N} \frac{\varphi'(t + x) - \varphi'(x)}{t} \, dt$$

$$+ \left(\int_{-\infty}^{-N} + \int_{N}^{\infty} \right) \frac{\varphi'(t + x)}{t} \, dt$$

$$= P \int_{-N}^{N} \frac{\varphi'(t + x)}{t} \, dt + \left(\int_{-\infty}^{-N} + \int_{N}^{\infty} \right) \frac{\varphi'(t + x)}{t} \, dt$$

$$= P \int_{-\infty}^{\infty} \frac{\varphi'(t + x)}{t} \, dt$$

$$= P \int_{-\infty}^{\infty} \frac{\varphi'(t)}{t - x} \, dt = H \, \varphi'(t).$$

Since $\varphi' \in L^p$, $F' \in L^p$ [**18**, pp. 132-33], using a similar technique and the method of induction we can prove that

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(11) $(H\varphi)^{(k)} = (H\varphi)^{(k)}, \quad k = 1, 2, 3, \ldots .$

Therefore $H\varphi \in \mathscr{D}_{L^p}$. The linearity of H is trivial. Continuity of H follows by virtue of the fact that

(12) $\|F^{(k)}(x)\|_p \leqq M_p\|\varphi^{(k)}\|_p$

where M_p is a constant independent of φ [**18**, p. 133]. The fact that H is one to one and onto follows by virtue of the Riesz-Titchmarsh classical inversion formula i.e.,

(13) $$-\frac{1}{\pi^2} H^2 \varphi = \varphi.$$

Therefore H^{-1} exists and $H^{-1} = -H / \pi^2$ and so H^{-1} is continuous.

Hilbert transform of distributions. In analogy to the relation (10) we define the Hilbert transform Hf of $f \in (\mathscr{D}_{L^p})'$, $1 < p < \infty$ by the relation

(14) $$< Hf, \varphi > = < f, -H\varphi > \quad \forall\, \varphi \in \mathscr{D}_{L^p},$$

where $H\varphi$ in (14) stands for the Hilbert transform of φ in the principal-value sense as defined in (8).

In words, we can say that the Hilbert transform (Hf) of a generalized function $f \in (\mathscr{D}_{L^p})'$ is a generalized function in $(\mathscr{D}_{L^p})'$ which assigns the same number to an element $\varphi \in \mathscr{D}_{L^p}$ as f assigns to $-H\varphi$ for all $\varphi \in \mathscr{D}_{L^p}$, $1 < p < \infty$. Note that the functional Hf so defined belongs to $(\mathscr{D}_{L^p})'$ in view of Theorem 1 [**20**, pp. 25-31].

The regular generalized function generated by $f(x) \in L^q$, when $q > 1$ and $1/p + 1/q = 1$ is denoted by f and is defined by the relation

(15) $$< f, \varphi > = \int_{-\infty}^{\infty} f(x)\, \varphi(x)\, dx \quad \forall\, \varphi \in \mathscr{D}_{L^p}.$$

The fact that $f \in (\mathscr{D}_{L^p})'$ is obvious [**20**, pp. 53-54]. Denoting the Hilbert transform of the regular generalized function f by Hf and using the definition (14) and the result (9) we can show that the generalized Hilbert transform Hf of $f \in L^q$, $q > 1$ is the same as the regular generalized function generated by

$$P \int_{-\infty}^{\infty} \frac{f(t)}{t - x}\, dt .$$

Using (13) and (14) as well as [**20**, pp. 27-28] we prove

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COROLLARY 1.

$$-\frac{1}{\pi^2} H^2 f = f, \quad 1 < p < \infty \quad \forall f \in (\mathcal{D}_{L^p})'. \tag{16}$$

COROLLARY 2. *The operator* $H:(\mathcal{D}_{L^p})' \to (\mathcal{D}_{L^p})'$, $1 < p < \infty$, *is an isomorphism.*

Example 1.

$$\begin{aligned} < H\,\delta, \varphi > &= < \delta, -P \int_{-\infty}^{\infty} \frac{\varphi(t)}{t - x}\, dt > \\ &= -P \int_{-\infty}^{\infty} \frac{\varphi(t)}{t}\, dt \\ &= < -PV\frac{1}{t}, \varphi > \quad \forall\, \varphi \in \mathcal{D}_{L^p}. \end{aligned}$$

Therefore

$$H\,\delta = -PV\frac{1}{t}$$

and using (16) we have

$$HPV\frac{1}{t} = \pi^2\delta$$

which can be arrived at by direct computation too.

Example 2. Now let us use these facts to solve the integral equation

$$Hy = P \int_{-\infty}^{\infty} \frac{f(t)}{t - x}\, dt + \delta(x) \tag{17}$$

in the space $(\mathcal{D}_{L^p})'$ for $f \in L^q$, where $p, q > 1$ satisfy the relation

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The relation (17) can be rewritten as

$$Hy = Hf + \delta(x) \tag{18}$$

where the operator H is as defined in (14).

Operating on (17) by H we have

$$H^2 y = H^2 f + H\,\delta(x) = -\pi^2 f - PV\frac{1}{x}$$

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or

$$y = f + \frac{1}{\pi^2} PV \frac{1}{x}.$$

Example 3. Let us now consider the solution to the integral equation (operator equation)

(19) $\quad y(x) = Hy + f(x)$

where $f \in (\mathscr{D}_{L^p})'$, $p > 1$ and H is defined by (14).

Operating on both sides of (19) by H we have

$$Hy = H^2 y + Hf$$
$$Hy = -\pi^2 y + Hf$$

or

$$y - f(x) = -\pi^2 y + Hf.$$

Therefore,

$$y = \frac{1}{1 + \pi^2} [f + Hf].$$

Let us define the derivative f' of a generalized function $f \in (\mathscr{D}_{L^p})'$, $1 < p < \infty$, by the relation

$$< f', \varphi > = < f, -\varphi' > \quad \forall\, \varphi \in \mathscr{D}_{L^p}, \quad [\mathbf{20}, \text{p. } 30].$$

i.e., f' is an element of $(\mathscr{D}_{L^p})'$ which assigns the same number to $\varphi \in \mathscr{D}_{L^p}$ as f assigns to $-\varphi'$. It now readily follows that

(20) $\quad Hf^{(k)} = (Hf)^{(k)}, \quad k = 1, 2, 3, \ldots.$

For, if $\varphi \in \mathscr{D}_{L^p}$ then

$$\begin{aligned} < Hf', \varphi > &= < f', -H\varphi > \\ &= < f, (H\varphi)' > \\ &= < f, H\varphi' > \\ &= < -Hf, \varphi' > \\ &= < (Hf)', \varphi >. \end{aligned}$$

The result (20) is now proved for $k = 1$. Using this result and the method of induction the result (20) follows for any positive integer k.

Example 4. Consider the following differential equation

$$\frac{dy}{dx} = H\delta'(x)$$

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in the space $(\mathscr{D}_{L^p})'$, $1 < p < \infty$.

The given differential equation can be written as

$$\frac{dy}{dx} = \frac{d}{dx} H\delta$$

or

$$\frac{d}{dx}(y - H\delta) = 0.$$

Hence, $y - H\delta = 0$, for the only constant generalized function in $(\mathscr{D}_{L^p})'$, $p > 1$ is the zero distribution [**17**, p. 201].

Therefore,

$$y = H\delta = -PV\frac{1}{x}.$$

Approximate Hilbert transform. Let $f \in (\mathscr{D}_{L^p})'$, $1 < p < \infty$, and x be real. Define a numerical valued function $F_\eta(x)$ of x for each fixed $\eta > 0$ by the relation

$$F_\eta(x) = \left\langle f(t), \frac{t - x}{(t - x)^2 + \eta^2} \right\rangle \equiv (H_\eta f)(x). \tag{21}$$

Note that for each fixed real x, and $\eta > 0$,

$$\frac{t - x}{(t - x)^2 + \eta^2} \in \mathscr{D}_{L^p}.$$

Therefore the function $F_\eta(x)$, as defined by (21), exists. It can easily be seen that

$$\frac{\partial^k}{\partial x^k}\left(\frac{(t - x)}{(t - x)^2 + \eta^2}\right) \in \mathscr{D}_{L^p}$$

as a function of t for each $k = 1, 2, 3, \ldots$. Again, by using the same technique as used in proving [**14**, Theorem 1] or the structure formula for f, [**17**, p. 201] we can show that

$$\frac{\partial^k}{\partial x^k} F_\eta(x) = \left\langle f(t), \frac{\partial^k}{\partial x^k} \frac{t - x}{(t - x)^2 + \eta^2} \right\rangle. \tag{22}$$

LEMMA 1. *Let $f \in (\mathscr{D}_{L^p})'$, $p > 1$ and $F_\eta(x)$ be the approximate Hilbert transform of f as defined by* (21). *Then for each $k = 0, 1, 2, 3, \ldots,$*

$$F_\eta^{(k)}(x) \in L^q, \quad \textit{where } q > 1, \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. We will sketch the proof of the fact that $F_\eta(x) \in L^q$, $q > 1$, (i.e., for $k = 0$) as the proof for the cases $k = 1, 2, 3, \ldots$ can be given in similar fashion. In view of the structure formula [**17**, p. 201] there exists a non-negative and finite integer r and functions $f_i \in L^q$ such that

$$(23)\quad F_\eta(x) = \sum_{i=1}^{r} \int_{-\infty}^{\infty} f_i(t)\left(-\frac{\partial}{\partial t}\right)^{i-1} \left[\frac{t-x}{(t-x)^2+\eta^2}\right] dt.$$

Now, using the results in [**18**, p. 134] and the fact that

$$\left|\frac{t-x}{(t-x)^2+\eta^2}\right| \leqq \frac{1}{2|\eta|}, \quad (\eta \text{ real } \neq 0),$$

it follows that each of the terms in the summation in (23) for fixed real $\eta \neq 0$ is an element of L^q. This completes the proof of Lemma 1.

Hence using Lemma 1, we have

$$< F_\eta(x), \varphi(x) > \equiv \int_{-\infty}^{\infty} F_\eta(x)\,\varphi(x)\,dx \quad \text{exists } \forall\, \varphi \in \mathscr{D}_{L^p}$$

and we can show that the regular generalized function generated by $F_\eta(x)$ is in $(\mathscr{D}_{L^p})'$.

By using the structure formula for f as in (23) and Lemma 1, and Fubini's theorem it follows quite readily that

$$(24)\quad \int_{-\infty}^{\infty} F_\eta(x)\,\varphi(x)\,dx = < f(t), \int_{-\infty}^{\infty} \frac{\varphi(x)(t-x)}{(t-x)^2+\eta^2}\,dx > \qquad \forall\, \varphi \in \mathscr{D}_{L^p}$$

or

$$(25)\quad < H_\eta f, \varphi > = < f, -H_\eta\,\varphi > \forall\, \varphi \in \mathscr{D}_{L^p}.$$

Since $\varphi \in L^p$, in view of results proved by Riesz and Titchmarsh, $H_\eta\,\varphi \in L^p$ [**18**, pp. 132-37]. Again, since $\varphi \to 0$ as $x \to \pm\infty$, using the technique of integration by parts, we can see that

$$(26)\quad \frac{d^k}{dx^k}\,(H_\eta\,\varphi)(x) = (H_\eta\,\varphi^{(k)})\,(x) \in L^p.$$

Therefore, $(H_\eta\varphi)(x) \in \mathscr{D}_{L^p}$ (and hence the expression on the right hand side of (24) or (25) is meaningful). Using the relation (26) and the results [**18**, p. 136] we can see that $(H_\eta\varphi)(x) \to (H\varphi)(x)$ as $\eta \to 0+$ in the topology of $\mathscr{D}_{L^p}$. Therefore, letting $\eta \to 0+$ in (24) or (25) we get

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$$\lim_{\eta \to 0+} < H_\eta f, \varphi > = < f, -H\varphi >$$
$$= < Hf, \varphi > \quad \text{(by definition).}$$

We have now proved:

THEOREM 2. *For $1 < p < \infty$ let $f \in (\mathscr{D}_{L^p})'$. For each $\eta > 0$, let $H_\eta f$ be its approximate Hilbert transform as defined by (21). Then the regular distribution $H_\eta f$ as generated by $(H_\eta f)(x)$ tends to Hf in the weak topology of $(\mathscr{D}_{L^p})'$ as $\eta \to 0+$.*

By using a similar technique we can show that

$$\ll f(t), \frac{y}{\pi((t-x)^2 + y^2)} >, \varphi(x) > \to < f, \varphi >,$$

as

$$y \to 0+ \quad \forall \varphi \in \mathscr{D}_{L^p}, \quad 1 < p < \infty,$$

i.e.,

$$< f(t), \frac{y}{\pi((t-x)^2 + y^2)} > \to f \text{ in } (\mathscr{D}_{L^p})'$$

as $y \to 0+$ [**18**, pp. 132-133], [**21**, pp. 44-45].

Analytic representation. Let $f \in (\mathscr{D}_{L^p})'$, $p > 1$ and $F(z)$ be complex-valued functions of z defined in the region $\operatorname{Im} z \neq 0$ by

$$(27) \quad F(z) = \frac{1}{2\pi i} < f(t), \quad \frac{1}{t-z} >$$

then using the structure formula for f and the same technique as used in proving (22) we can show that

$$(28) \quad F'(z) = \frac{1}{2\pi i} < f(t), \quad \frac{1}{(t-z)^2} >.$$

The analytic function $F(z)$, $\operatorname{Im} z \neq 0$ in view of Theorem 2 satisfies the relation

$$(29) \quad \lim_{\epsilon \to 0} < F(x + i\epsilon) - F(x - i\epsilon), \varphi(x) > = < f, \varphi >$$
$$\forall \varphi \in \mathscr{D}_{L^p}.$$

Therefore, $F(z)$ defined by (27) is an analytic representation of $f \in (\mathscr{D}_{L^p})'$.

Distributional representation of analytic functions. Using the structure formula for $f \in (\mathcal{D}_{L^p})'$ we can easily show that the function $F(z)$ defined by (27) satisfies the condition (5) and the uniform asymptotic order (uniformity with respect to x is assumed here)

$$(29) \quad |F(x + iy)| = O\left[\frac{1}{y^\delta}\right], \quad y \to \infty$$

where $\delta = (p - 1) / p$.

Let us now reverse the problem. Let $F(z)$ be analytic in the upper half plane and satisfy the relation (5) and the uniform asymptotic order (6), such that $F(t + i\epsilon) \in L^q$ for fixed $\epsilon > 0$ and

$$(30) \quad \lim_{\epsilon \to 0} < F(t + i\epsilon) = f^+(t) \quad \text{in } (\mathcal{D}_{L^p})'.$$

Can we find $f \in (\mathcal{D}_{L^p})'$ satisfying

$$(31) \quad F(z) = < f(t), \frac{1}{t - z} >, \quad \text{Im } z > 0 \text{ ?}$$

The answer is affirmative. In view of the asymptotic order (6), the bound (5) and the Cauchy's theorem it can be proved that

$$(32) \quad \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(t + i\epsilon)}{t - z} dt \begin{array}{ll} = F(t + i\epsilon), & \text{Im } z > 0 \\ = 0, & \text{Im } z < 0. \end{array}$$

Now, $1/(t - z) \in \mathcal{D}_{L^p}$ for fixed z satisfying $\text{Im } z \neq 0$. Therefore, from (32) we have

$$(33) \quad \frac{1}{2\pi i} < F(t + i\epsilon), \frac{1}{t - z} > = F(z + i\epsilon), \quad \text{Im } z > 0.$$

Since $F(z)$ is holomorphic in the region $\text{Im } z > 0$ letting $\epsilon \to 0+$ in (33) we have

$$< f^+(t), \frac{1}{2\pi i} \frac{1}{t - z} > = F(z), \quad \text{Im } z > 0.$$

Therefore, (31) holds with

$$(34) \quad f(t) = \frac{1}{2\pi i} f^+(t)$$

and we have

$$(35) \quad F(z) = \frac{1}{2\pi i} < f^+(t), \frac{1}{t - z} >, \quad \text{Im } z > 0 .$$

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We now conclude the following.

THEOREM 3. *Let $F(z)$ be an analytic complex-valued function of the complex variable $z = x + iy$ in the set Ω where Ω is the region of the upper-half plane* (im $z > 0$) *in the complex z-plane, satisfying*

(i) a) *for fixed* $y > 0$, $q = \dfrac{p}{p-1}$, $p > 1$ *we have* $F(x + iy) \in L^q$

(36) b) $\lim\limits_{y \to 0+} F(z) = f^+(x)$ *in* $(\mathcal{D}_{L^p})'$

(37) (ii) $|F(z)| = o(1)$ *as* $y \to \infty$

uniformly for all $x \in \mathbf{R}$ *and*

(38) (iii) $\sup\limits_{\substack{-\infty < x < \infty \\ |y| \geqq \delta}} |F(z)| = A_\delta < \infty$

then

(39) $F(z) = < f^+(t), \dfrac{1}{2\pi i} \dfrac{1}{t - z} > \quad \forall\, z \in \Omega .$

As a consequence of Theorem 3 we have

THEOREM 4. *Let Ω be the region as defined in Theorem 3. Then there exists one and only one function $F(z)$ analytic in the region Ω satisfying the conditions* (36), (37), (38) *and that $F(z)$ has the representation formula* (39).

Existence and uniqueness of the solution to a Diritchlet boundary value problem.

Definition. A harmonic function $w(x, y)$ defined in the upper half plane (z : Im $z > 0$) is said to *belong to the space $\mathscr{H}$* if and only if

(40) (i) (a) $w(x, y)$ for fixed $y > 0$ belongs to L^q, $q = p/(p - 1)$, $p > 1$ when treated as a function of x

(b) $w(x, y)$ converges to some distributional limit in $(\mathcal{D}_{L^p})'$ as $y \to 0+$

(41) (ii) $\sup\limits_{\substack{-\infty < x < \infty \\ y \geqq \delta > 0}} |w(x, y)| = A_\delta < \infty$

(42) (iii) $w(x, y) = o(1)$ as $y \to \infty$

the asymptotic order being uniform with respect to $\forall\, x \in \mathbf{R}$.

THEOREM 5. *Let $\mathscr{H}$ be the space of harmonic functions defined above. Assume that $u(x, y)$ and $v(x, y)$ are conjugate harmonic functions belonging to the space $\mathscr{H}$ and converging to the weak distributional limits f and g respectively in $(\mathscr{D}_{L^p})'$ as $y \to 0+$, then we have*

$$(43) \qquad \begin{aligned} \frac{1}{\pi} Hf &= -g \\ \frac{1}{\pi} Hg &= f \end{aligned}$$

and

$$(44) \quad u(x, y) = \frac{1}{\pi} < f(t), \frac{y}{(t - x)^2 + y^2} >$$

$$(45) \quad v(x, y) = \frac{1}{\pi} < f(t), \frac{t - x}{(t - x)^2 + y^2} > .$$

More clearly, u and v as defined by (44) *and* (45) *are the only harmonic functions belonging to $\mathscr{H}$ satisfying*

$$(46) \quad \lim_{y \to 0+} u(x, y) = f$$

$$(47) \quad \lim_{y \to 0+} v(x, y) = g = -\frac{1}{\pi} Hf.$$

Proof. Let u and v both belong to $\mathscr{H}$ and let f and g be their distributional limits respectively in $(\mathscr{D}_{L^p})'$ as $y \to 0+$.

Therefore, an appeal to Theorem 4 shows that there is only one conjugate pair u and v in $\mathscr{H}$ satisfying such a requirement and that

$$(48) \quad u(x, y) + iv(x, y) = < f(t) + ig(t), \frac{1}{2\pi i} \frac{1}{t - z} > .$$

Letting $y \to 0+$ in (48) and using Theorem 2 and the result stated immediately after it we have

$$(49) \quad f + ig = \frac{-iH}{\pi} (f + ig).$$

Since f and g are real functionals (i.e., functionals which assign real numbers to real valued functions of $\mathscr{D}_{L^p}$) equating the real and imaginary parts in both sides of (49) we get the reciprocity relation (43). Again, equating the real and imaginary parts of both sides of (48) (on the assumption that f and g are real functionals) and then using the definition of the Hilbert transform of generalized functions and the reciprocity relation (43) we get,

$$(50)\quad u(x, y) = \frac{1}{\pi} < f(t), \frac{y}{(t-x)^2 + y^2} >$$

$$(51)\quad v(x, y) = \frac{1}{\pi} < f(t), \frac{t-x}{(t-x)^2 + y^2} >.$$

The relation (51) can also be written in the form

$$(52)\quad v(x, y) = \frac{1}{\pi} < g(t), \frac{y}{(t-x)^2 + y^2} >.$$

It can easily be verified that u and v obtained in (50) and (51) are conjugate Harmonic functions. An appeal to the structure formula for $f \in (\mathcal{D}_{L^p})'$ shows that

$$(53)\quad \begin{aligned} u(x, y) &= O\left[\frac{1}{y^{(p-1)/p}}\right], y \to \infty \\ v(x, y) &= O\left[\frac{1}{y^{(p-1)/p}}\right], y \to \infty \end{aligned}$$

uniformly $\forall\, x \in R$, and

$$(54)\quad \begin{aligned} \sup_{\substack{-\infty<x<\infty \\ y \geqq \delta > 0}} |u(x, y)| &= A_\delta < \infty \\ \sup_{\substack{-\infty<x<\infty \\ y \geqq \delta > 0}} |v(x, y)| &= B_\delta < \infty. \end{aligned}$$

Therefore, u and v as defined by (50) and (51) belong to the space $\mathcal{H}$. The uniqueness has already been proved.

Example 5. Consider the Dirichlet boundary-value problem

$$\Delta^2 u = 0 \text{ in } \Omega,\ u \in \mathcal{H}$$

where,

$$\Omega = \{z : \operatorname{Im} z > 0\}$$

$$\lim_{y \to 0+} u(x, y) = -PV\left(\frac{1}{x}\right) \text{ in } (\mathcal{D}_{L^p})'$$

This problem can be rewritten as

$$\Delta^2 u = 0 \text{ in } \Omega,\ u \in \mathcal{H}$$

$$\lim_{y \to 0+} u(x, y) = H\delta.$$

The obvious solution is

$$\nu(x, y) = \left< \delta(t), \frac{t - x}{(t - x)^2 + y^2} \right> = \frac{-x}{x^2 + y^2}.$$

The Hilbert problem for generalized functions.

Description. Let $g(x)$ be a given function of the real variable x defined on the real line. Let $g^+(z)$ and $g^-(z)$ be analytic functions holomorphic in the upper half plane ($\operatorname{Im} z > 0$) and the lower half plane ($\operatorname{Im} z < 0$) respectively. Assume that $g^+(x)$ and $g^-(x)$ are the limits of $g^+(z)$ and $g^-(z)$ respectively; i.e.,

$$\left.\begin{aligned} g^+(x) &= \lim_{y \to 0+} g^+(x + iy) \\ g^-(x) &= \lim_{y \to 0-} g^-(x + iy) \end{aligned}\right\} \tag{55}$$

Our object is now to find analytic functions $g^+(z)$ and $g^-(z)$ as described before, satisfying the relation

$$g^+(x) + g^-(x) = g(x)\,. \tag{56}$$

In many problems of mathematical physics a slightly more difficult problem appears, viz.

$$g^+(x) + k(x)\, g^-(x) = g(x) \tag{57}$$

where $k(x)$ is a known function defined on the real axis. By factorizing $k(x)$ as

$$k(x) = k^+(x)\, k^-(x) \tag{58}$$

the problem (57) which is known as Hilbert problem can be written as

$$\frac{g^+(x)}{k^+(x)} + k^-(x)\, g^-(x) = \frac{g(x)}{k^+(x)} \tag{59}$$

which is of the form (56).

In [**6**] Lauwerier has considered the simpler problem (56) and has called it Hilbert Problem. In [**6**, p. 158], Lauwerier gives the solution to the Hilbert problem in the form

$$\left.\begin{aligned} g^+(z) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{g(t)}{t - z}\, dt + P(z), \qquad \operatorname{Im} z > 0 \\ g^-(z) &= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{g(t)}{t - z}\, dt - P(z), \quad \operatorname{Im} z < 0 \end{aligned}\right\} \tag{60}$$

where $P(z)$ is an arbitrary polynomial and $g(x)$ is an appropriately chosen function.

Solutions to the Hilbert problem (56) when $g \in (\mathscr{D}_{L^p})'$, $p > 1$ and limits are being interpreted in the weak distributional sense (or in $(\mathscr{D}_{L^p})'$) can be written as

$$(61) \qquad \begin{aligned} g^+(z) &= \frac{1}{2\pi i} < g(t), \frac{1}{t - z} > + P(z) & \operatorname{Im} z > 0 \\ g^-(z) &= -\frac{1}{2\pi i} < g(t), \frac{1}{t - z} > - P(z), & \operatorname{Im} z < 0 \end{aligned}$$

where $P(z)$ is a polynomial in z and $< g(t), 1/(t - z) >$ is the Hilbert transform of $g \in (\mathscr{D}_{L^p})'$, $p > 1$ with complex parameter z such that $\operatorname{Im} z \neq 0$. The fact that $< g(t), 1/(t - z) >$ is analytic in the upper and lower half planes follows quite readily in view of the structure formula for g [**17**, p. 201] or [**20**, pp. 58-59]. The fact that our solution obtained actually verifies (56) follows in view of the result stated immediately following Theorem 2.

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