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COMPACTNESS PROPERTIES OF CARLEMAN AND HILLE-TAMARKIN OPERATORS

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Introduction. In this paper we study integral operators with domain a Banach function space L_{ρ_1} and range another Banach function space L_{ρ_2} or the space L_0 of all measurable functions. Recall that a linear operator T from L_{ρ_1} into L_0 is called an integral operator if there exists a $\mu \times \nu$-measurable function $T(x, y)$ on $X \times Y$ such that

$$\int |T(x, y)f(y)|d\nu(y) < \infty \text{ a.e. for all } f \in L_{\rho_1} \quad \text{and}$$

$$Tf(x) = \int T(x, y)f(y)d\nu(y) \text{ a.e. for every } f \in L_{\rho_1}.$$

Such an integral operator is called a Carleman integral operator if for almost every $x \in X$ the function

$$y \mapsto T(x, y) = T_x(y)$$

is an element of the associate space L'_{ρ_1}, i.e.,

$$\rho'_1(T_x(y)) < \infty \quad \text{for almost every } x \in X.$$

If in addition the function

$$x \mapsto \rho'_1(T_x(y))$$

is an element of L_{ρ_2}, i.e., $\rho_2(\rho'_1(T_x(y))) < \infty$, then T is a Hille-Tamarkin operator from L_{ρ_1} into L_{ρ_2}. Hille-Tamarkin operators are also known as integral operators of finite double norm. Characterizations of the above classes of operators are discussed in **[6]** and [7]. It follows from the results in [7] that an order continuous linear operator from L_{ρ_1} into L_0 is an integral operator if and only if the image under T of an order interval is equimeasurable (in the sense of Grothendieck), i.e., if we denote

$$H = T[0, f] = \{Tg:0 \leqq g \leqq f, g \in L_{\rho_1}\},$$

then for every $X_0 \subset X$ of finite measure and every $\epsilon > 0$ there exists $X_\epsilon \subset X_0$ such that $\mu(X_0 \setminus X_\epsilon) < \epsilon$ and $\chi_{X_\epsilon} \circ H$ is a relatively norm compact subset of L_∞. It can be derived from the results in **[1]** that if ρ'_1 is order continuous, then an order continuous linear operator T from L_{ρ_1} into L_0 is

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a Carleman integral operator if and only if the image under T of the unit ball is equimeasurable. We reformulate this result as a compact factorization theorem: if ρ_1' is order continuous, then an order continuous linear operator T from L_{ρ_1} into L_0 is a Carleman integral operator if and only if there exists a factorization $T = R \cdot S$ with S a compact operator from L_{ρ_1} into L_∞ and R a multiplication operator from L_∞ into L_0. We also prove a similar characterization of Hille-Tamarkin operators. Let ρ_1' and ρ_2 be order continuous norms. Then an order continuous linear operator T from L_{ρ_1} into L_{ρ_2} is a Hille-Tamarkin operator if and only if $T = R \cdot S$ with S a compact operator from L_{ρ_1} into L_∞ and R a multiplication operator from L_∞ into L_{ρ_2}. In the third section of this paper we discuss the relation between Hille-Tamarkin operators and majorizing operators. The results of this paper extend and complement those of [**5**].

1. Carleman integral operators. Throughout this paper we shall denote by (X, μ) and (Y, ν) σ-finite measure spaces and by

$$L_{\rho_1} = L_{\rho_1}(Y, \nu) \quad \text{and} \quad L_{\rho_2} = L_{\rho_2}(X, \mu)$$

we shall denote Banach function spaces. For information on Banach function spaces we refer to [**8**], Chapter 15. We begin with a definition.

Definition 1.1. Let $H \subset L_0(X, \mu)$. Then H is called *relatively uniformly compact in* L_0 if there exists $g \in L_0(X, \mu)$ with $g(x) > 0$ a.e. such that $\frac{1}{g} \cdot H$ is a relatively norm compact subset of $L_\infty(X, \mu)$.

Similarly $H \subset L_0(X, \mu)$ is called *relatively uniformly compact in* L_ρ if there exists $g \in L_\rho$ with $g(x) > 0$ a.e. such that $\frac{1}{g} \cdot H$ is a relatively norm compact subset of $L_\infty(X, \mu)$.

We note that the above notions are related to relative compactness in the topology of relative uniform convergence (see [**2**] for relative uniform convergence of sequences). We now present a description of equimeasurable sets.

THEOREM 1.2. *Let* $H \subset L_0(X, \mu)$. *Then* H *is equimeasurable if and only if* H *is relatively uniformly compact in* $L_0(X, \mu)$.

Proof. Assume first that H is relatively uniformly compact in L_0. Let g be as in definition 1.1, $\epsilon > 0$ and $X_0 \subset X$ of finite measure. Then there exists $\delta > 0$ such that

$$\mu\{x \in X_0 : g(x) < \delta\} < \epsilon.$$

Let

$$X_\epsilon = \{x \in X_0 : g(x) \geqq \delta\}.$$

Then $\mu(X_0 \setminus X_\epsilon) < \epsilon$ and it is easy to see that $\chi_{X_\epsilon} \circ H$ is relatively norm compact in L_∞, so H is equimeasurable. Assume now that H is equimeasurable. Then we can find disjoint X_n such that

$$X = \bigcup_{n=1}^{\infty} X_n$$

and $\chi_{X_n} \cdot H$ is relatively norm compact in $L_\infty(X, \mu)$ for all n. Let

$$c_n = \sup(\|h\chi_{X_n}\|_\infty : h \in H)$$

and define

$$g(x) = 1 + \sum_{n=1}^{\infty} nc_n\chi_{X_n}.$$

Then $0 < g(x)$ a.e. and

$$|h(x)| \leqq \frac{1}{n} g(x) \text{ a.e. on } X_n \text{ for all } h \in H.$$

A diagonal argument now shows that $\frac{1}{g} \cdot H$ is relatively norm compact in $L_\infty(X, \mu)$, i.e., H is relatively uniform compact in $L_0(X, \mu)$.

We recall that a linear operator $T:L_\rho \to L_0$ is called *order continuous* if $f_n \in L_\rho$ and $f_n(x) \downarrow 0$ a.e. implies that $Tf_n(x) \to 0$ a.e.

The following theorem complements Theorem 2.2 of [**6**].

THEOREM 1.3. *Let $L_\rho = L_\rho(Y, \nu)$ be a Banach function space and let $T:L_\rho \to L_0$ be a linear operator. Then the following are equivalent.*

(i) *T is a Carleman integral operator.*

(ii) *If $f_n \in L_\rho$ and $f_n \to 0$ $\sigma(L_\rho, L'_\rho)$, then $Tf_n(x) \to 0$* a.e. *If in addition ρ' is order continuous, then each of the above is equivalent to*

(iii) *T is order continuous and $T(U)$ is equimeasurable, where*

$$U = \{f \in L_\rho, \rho(f) \leqq 1\}.$$

(iv) *T is order continuous and T has a factorization $T = RS$ where $S:L_\rho \to L_\infty$ is compact and $R:L_\infty \to L_0$ is multiplication by $g \in L_0$ with* $g(x) > 0$ a.e.

Proof. The implication (i) $\Rightarrow$ (ii) is immediate from the inequality

$$\rho'(T_x(y)) < \infty \text{ a.e.}$$

Assume now that (ii) holds. Then $\rho(f_n) \to 0$ implies

$$f_n \to 0 \ \sigma(L_\rho, L'_\rho),$$

so $\rho(f_n) \to 0$ implies $Tf_n(x) \to 0$ a.e. It follows that there exists $0 \leqq$

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$g \in L_0$ such that

$$|Tf(x)| \leqq g(x)\rho(f) \text{ a.e.}$$

(see the proof of Theorem 2.2 of [**6**]). Let now $0 \leqq f_n \leqq f$ in L_ρ such that $f_n \to 0$ in measure on every set of finite measure. Then $f_n \to 0$ $\sigma(L_\rho, L'_\rho)$, so

$$Tf_n(x) \to 0 \text{ a.e.}$$

We conclude from Theorem 3.3 of [**5**] that T is an integral operator. It follows now from

$$|Tf(x)| \leqq g(x)\rho(f) \text{ a.e. for all } f \in L_\rho,$$

as in the proof of Theorem 2.2 of [**6**], that T is a Carleman integral operator. Assume now that ρ' is order continuous and assume that (i) holds. We shall show that (iii) holds. The order continuity of T is obvious, since T is an integral operator. To show that $T(U)$ is equimeasurable we may assume that $\mu(X) < \infty$. Let $\epsilon > 0$. Then, since

$$\rho'(T_x(y)) < \infty \text{ a.e.},$$

we can remove a set of measure less than ϵ, so

$$\rho'(T_x(y)) \leqq M < \infty \text{ a.e. on } X.$$

It follows then that

$$\|\rho'(T_x(y))\|_1 \leqq M \cdot \mu(X) < \infty.$$

Since the norm $\| \|_1 \times \rho'$ is order continuous, it follows that there exist simple functions $t_n(x, y)$ of the form

$$\sum_1^n \alpha_h \chi_{A_h}(x)\chi_{B_h}(y)$$

such that

$$\|\rho'(T_x(y) - t_n(x, y))\|_1 \to 0$$

(see also the remark before Theorem 3.2). By passing to a subsequence we can assume that

$$\rho'(T_x(y) - t_n(x, y)) \to 0 \text{ a.e. on } X.$$

An application of Egoroff's theorem now yields a set $X_0 \subset X$ with $\mu(X \setminus X_0) \leqq \epsilon$ such that

$$\|\chi_{X_0}(\rho'(T_x(y) - t_n(x, y)))\|_\infty \to 0 \quad \text{as } n \to \infty.$$

This implies directly that $\chi_{X_0} \circ T$ is a norm limit of finite rank operators with respect to the operator norm on $\mathscr{L}(L_\rho, L_\infty)$. Hence (iii) holds. The equivalence of (iii) and (iv) follows from Theorem 1.2. Assume therefore that (iii) and (iv) hold. Then (iii) implies that T is an integral operator (Theorem 3.3 of [7]) and (iv) implies via Theorem 1.2 that there exists $0 < g \in L_0$ such that

$$|Tf(x)| \leqq g(x)\rho(f) \text{ a.e. for all } f \in L_\rho.$$

As above this implies that T is a Carleman integral operator.

Remarks. 1. The implications (i) $\Leftrightarrow$ (iii), provided ρ' is order continuous, are a variant of a result of Gretsky and Uhl ([**1**]). We point out however that the definition of a Carleman operator as given in [**1**] is not exactly the same as the definition of a Carleman integral operator as given above. The reason is that if $\rho'(T_x(y)) < \infty$ for a.e. $x \in X$, then it need not be true that the function

$$x \mapsto T(x, y) \in L'_\rho(Y, \nu)$$

is strongly measurable. Under the hypothesis that ρ' is order continuous we could have proved that $x \mapsto T(x, y)$ is strongly measurable, but the proof of this is longer than the above direct proof.

2. If ρ is order continuous, then we showed in [**6**] that (i) above is equivalent with:

(ii)′ If $f_n \in L_\rho$ and $\rho(f_n) \to 0$, then $Tf_n(x) \to 0$ a.e.

In case ρ is not order continuous, then (ii)′ need not imply (i) as can be seen from the identity operator on L_∞. Therefore condition (ii) above is the right condition.

2. Hille-Tamarkin operators. In the following theorem we derive a characterization of order continuity of the norm of a Banach lattice, closely related to Meyer-Nieberg's characterization. For information on Banach lattice we refer to [**3**] or [**9**]. We would like to thank the referee for pointing out the present simple proof, which replaces our original more complicated proof.

THEOREM 2.1. *Let E be a Banach lattice. Then the following are equivalent.*

(i) *If $0 \leqq f_n \leqq f$ in E and $f_n \wedge f_m = 0$ if $n \neq m$, then there exist $\lambda_n \in \mathbf{R}$ with $\lambda_n \uparrow \infty$ and $g \in E$ such that*

$$g = \sum_1^\infty \lambda_n f_n.$$

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(ii) *If* $0 \leqq f_n \leqq f$ *in* E *and* $f_n \wedge f_m = 0$ *if* $n \neq m$, *then there exist* $\lambda_n \in \mathbf{R}$ *with* $\lambda_n \uparrow \infty$ *such that*

$$\sup_n \left\| \sum_1^n \lambda_k f_k \right\| < \infty.$$

(iii) *The norm of* E *is order continuous.*

Proof. If (i) holds, then obviously (ii) holds. Assume (ii) holds. To prove (iii) let $0 \leqq f_n \leqq f$ with $f_n \wedge f_m = 0$ if $n \neq m$. Then by (ii) there are $\lambda_n \uparrow \infty$ and $M \in \mathbf{R}$ such that

$$\left\| \sum_1^n \lambda_k f_k \right\| \leqq M \quad \text{for all } n.$$

Hence

$$\|f_n\| \leqq M/\lambda_n \to 0 \text{ as } n \to \infty.$$

It follows now from Meyer-Nieberg's theorem (see **[9]**) that the norm on E is order continuous. Assume now that (iii) holds. Let $0 \leqq f_n \leqq f$ as in (i). Then there exist natural numbers k_n with $k_n < k_{n+1}$ for all n such that

$$\left\| \sum_{i=k_n}^{k_n+p} f_i \right\| < \frac{1}{n2^n} \quad \text{for all } p \in \mathbf{N}.$$

It follows that

$$\left\| \sum_{i=k_n}^{k_{n+1}} nf_i \right\| < 1/2^n \quad \text{for all } n,$$

which implies (i).

Recall that $H \subset L_\rho$ is called relatively uniform compact in L_ρ if there exists $0 < g \in L_\rho$ such that $\frac{1}{g}H$ is relatively norm compact in L_∞. The following theorem extends the result of Theorem 1.2.

THEOREM 2.2. *Let* L_ρ *be a Banach function space with order continuous and let* $H \subset L_\rho$. *Then* H *is relatively uniformly compact in* L_ρ *if and only if* H *is equimeasurable and order bounded in* L_ρ.

Proof. Assume first that H is relatively uniformly compact in L_ρ. From Theorem 1.2 we then conclude that H is equimeasurable. Let $0 < g \in L_\rho$ such that $\frac{1}{g}H$ is relatively norm compact in L_∞. Then there exists $M \in \mathbf{R}$ such that

$$\left\| \frac{h}{g} \right\|_\infty \leqq M \quad \text{for all } h \in H.$$

Hence $|h| \leqq Mg$ for all $h \in H$ and thus H is order bounded in L_ρ. Assume now that $H \in L_\rho$ is equimeasurable and order bounded in L_ρ. We assume first that $\chi_X \in L_\rho$ and $|h| \leqq \chi_X$ for all $h \in H$. Then we can find disjoint $X_n \subset X$ with

$$\bigcup_{n=1}^{\infty} X_n = X$$

such that $\chi_{X_n} \cdot H$ is relatively norm compact in L_∞. From the above theorem it follows that there exist $\lambda_n \in \mathbf{R}$ with $1 \leqq \lambda_n \uparrow \infty$ such that

$$g = \sum_1^\infty \lambda_n \chi_{X_n} \in L_\rho.$$

Now

$$1 \leqq g(x) \text{ a.e. and}$$

$$|h(x)| \leqq \frac{1}{\lambda_n} g(x) \text{ a.e. on } X_n \text{ for all } h \in H.$$

Again a diagonal argument shows that $\frac{1}{g} \cdot H$ is relatively norm compact in L_∞. Now we shall remove the assumption that $\chi_X \in L_\rho$ and $|h| \leqq \chi_X$ for all $h \in H$. Assume $|h| \leqq g_0 \in L_\rho$ for all $h \in H$. Then define $\rho_1(f) = \rho(fg_0)$. Then ρ_1 is an order continuous function norm (we may assume that $g_0 > 0$ a.e. on X). Let

$$H_1 = \left\{\frac{h}{g_0} : h \in H\right\}.$$

Then $H_1 \subset L_{\rho_1}$ and $|h| \leqq \chi_X$ for all $h \in H_1$. From Theorem 1.2 it is immediate that H_1 is also equimeasurable. Hence by the above argument there exists $g \in L_{\rho_1}$ such that $\frac{1}{g} \cdot H_1$ is relatively norm compact in L_∞. Put $g_1 = g \cdot g_0$. Then $\rho(g_1) = \rho_1(g_0) < \infty$ and obviously $\frac{1}{g_1} H$ is relatively norm compact in L_∞.

To prove the analogue of Theorem 1.3 for Hille-Tamarkin operators we first need some additional definitions. Let $f_n \in L_\rho$. Then f_n is relatively uniformly convergent to $f \in L_\rho$ if there exists $g \in L_\rho$ such that $|f - f_n| \leqq \epsilon_n g$ for some sequence of scalars $\epsilon_n \downarrow 0$. We also recall that a function norm has the Fatou property if it follows from $0 \leqq f_1 \leqq f_2 \leqq \ldots \uparrow f$ with all $f_n \in L_\rho$ that $\rho(f_n) \uparrow \rho(f)$ (where $\rho(f) = \infty$ if $f \notin L_\rho$).

THEOREM 2.3. *Let $L_{\rho_1} = L_{\rho_1}(Y, \nu)$ and $L_{\rho_2} = L_{\rho_2}(X, \mu)$ be Banach function spaces and assume that ρ_2 is order continuous and has the Fatou* *property. Let* $T:L_{\rho_1} \to L_{\rho_2}$ *be a linear operator. Then the following are equivalent.*

(i) *T is a Hille-Tamarkin operator.*

(ii) *If* $f_n \in L_{\rho_1}$, $\rho_1(f_n) \leqq 1$ *and* $f_n \to 0$ $\sigma(L_{\rho_1}, L'_{\rho_1})$, *then* $Tf_n \to 0$ *relatively uniformly in* L_{ρ_2}.

If in addition ρ'_1 *is order continuous, then each of the above is equivalent to*:

(iii) *T is order continuous and $T(U)$ is relatively uniformly compact in* L_{ρ_2}, *where*

$$U = \{f \in L_{\rho_1}:\rho_1(f) \leqq 1\}.$$

(iv) *T is order continuous and T has a factorization $T = RS$ where* $S:L_{\rho_1} \to L_\infty$ *is compact and* $R:L_\infty \to L_{\rho_2}$ *is a multiplication operator.*

Proof. Assume (i) holds. To prove (ii) let $f_n \in L_{\rho_1}$, $\rho_1(f_n) \leqq 1$ such that

$$f_n \to 0 \ \sigma(L_{\rho_1}, L'_{\rho_1}).$$

Then by Theorem 1.3

$$Tf_n(x) \to 0 \text{ a.e.} \quad \text{and} \quad |Tf_n(x)| \leqq \rho'_1(T_x(y)) \text{ a.e.}$$

Hence $Tf_n \to 0$ in order in L_{ρ_2}. It follows ([**2**], Theorems 11.8 and 16.3) that $Tf_n \to 0$ relatively uniformly. Assume now that (ii) holds. It follows from the proof of Theorem 1.3 that T is a Carleman integral operator. It remains to show that if $T(x, y)$ denotes the kernel of T, then

$$\rho'_1(T_x(y)) \in L_{\rho_2}.$$

From Corollary 2.3 of [**6**] it follows that

$$\rho'_1(T_x(y)) = \sup(|Tf| : \rho_1(f) \leqq 1),$$

where the supremum is taken in the space $L_0(X, \mu)$. We now first show that there exists a constant M such that

$$\rho_2(\sup_{i \leqq n} |Tf_i|) \leqq M$$

for all finite sequences $(f_i)_{i \leqq n}$ in L_{ρ_1} with $\rho_1(f_i) \leqq 1$ for $i \leqq n$. If the assertion would be false, then there would exist finite sequences $(f_{i,k})_{i \leqq n_k}$ for each k such that

$$\rho_1(f_{i,k}) \leqq 1 \quad \text{and} \quad \rho_2(\sup_{i \leqq n_k} |Tf_{i,k}|) \geqq k \cdot 2^k.$$

Replacing $f_{i,k}$ by $f_{i,k}/2^k$ and concatenating all the finite sequences into one single sequence $(f_n)_{n=1}^\infty$ we have

$$\rho_1(f_n) \to 0 \quad \text{and} \quad \rho_2(\sup_{k \leqq n} |Tf_k|) \to \infty \text{ as } n \to \infty.$$

This contradicts (ii), so there exists a constant M with the above described property. From the σ-finiteness of μ it follows that there exists $f_n \in L_{\rho_1}$ with $\rho_1(f_n) \leqq 1$ such that

$$\rho_1'(T_x(y)) = \sup_n(\sup_{k \leqq n} |Tf_k(x)|) \text{ a.e. on } X.$$

From the Fatou property of ρ_2 it now follows that

$$\rho_2(\rho_1'(T_x(y))) = \sup_n \rho_2(\sup_{k \leqq n} |Tf_k|) \leqq M < \infty.$$

Assume now that ρ_1' is order continuous. The equivalence of (iii) and (iv) is obvious and the implication (i) $\Rightarrow$ (iii) follows from the previous theorem and Theorem 1.3. Assume therefore that (iii) holds. Then by Theorem 1.3 T is a Carleman integral operator and as before

$$\rho_1'(T_x(y)) = \sup(|Tf| : f \in L_{\rho_1}, \rho_1(f) \leqq 1).$$

Now $\{|Tf| : f \in L_{\rho_1}, \rho_1(f) \leqq 1\}$ is relatively uniformly compact in L_{ρ_2}, so that there exist $g \in L_{\rho_2}$ and $C \in \mathbf{R}$ such that

$$|Tf| \leqq Cg \text{ for all } f \in L_{\rho_1} \text{ with } \rho_1(f) \leqq 1.$$

Hence

$$\rho_1'(T_x(y)) \leqq Cg \in L_{\rho_2},$$

so

$$\rho_1'(T_x(y)) \in L_{\rho_2}$$

and thus T is a Hille-Tamarkin operator.

Remarks. 1. We note that the assumption that ρ_2 has the Fatou property is only used in the proof that (ii) implies (i). We give now an example to show that this hypothesis can not be dropped. Let $I:l_1 \to c_0$ be the identity operator. Then obviously I is not a Hille-Tamarkin operator, since

$$\sup_n |Ie_n| \notin c_0.$$

We shall indicate that I satisfies condition (ii) of the above theorem. Let $f_n \in l_1$ such that

$$f_n \to 0 \ \sigma(l_1, l_\infty).$$

Then $\|f_n\|_1 \to 0$, so also $\|f_n\|_\infty \to 0$. Let

$$\epsilon_n = \|f_n\|_\infty^{1/2}$$

and assume all $\epsilon_n \neq 0$. Then

$$\|\frac{1}{\epsilon_n} f_n\|_\infty \to 0,$$

which implies that

$$g = \sup_n \frac{1}{\epsilon_n}|f_n| \in c_0.$$

Now $|f_n| \leqq \epsilon_n g$, so that $f_n \to 0$ relatively uniformly in c_0.

2. If ρ_1' and ρ_2 are order continuous, then the above theorem implies that every Hille-Tamarkin operator from L_{ρ_1} into L_{ρ_2} is compact. It is well known from examples on L_1 and L_∞ that to obtain (iii) from (i) we can not drop either one of the order continuity assumptions in the above theorem. This raises the question whether we can characterize the integral operators which satisfy (iii) in case ρ_1' and ρ_2 are not both order continuous. In the next section we shall derive an answer to this question.

3. Majorizing integral operators. Let $T:L_{\rho_1} \to L_{\rho_2}$ be a linear operator. Then we shall call T *majorizing* if the set

$$\{Tf:f \in L_{\rho_1}, \rho_1(f) \leqq 1\}$$

is order bounded in L_{ρ_2}. We note that our usage of the term majorizing is in general slightly more restrictive than in [**3**]. By $\mathscr{L}^m(L_{\rho_1}, L_{\rho_2})$ we denote the set of all majorizing operators. Define

$$\|T\|_m = \inf(\rho_2(g):g \in L_{\rho_2} \text{ such that } |Tf| \leqq g \cdot \rho_1(f) \text{ for all } f \in L_{\rho_1})$$

for $T \in \mathscr{L}^m(L_{\rho_1}, L_{\rho_2})$. It is easily seen that $\|T\|_m$ defines a latticenorm on $\mathscr{L}^m(L_{\rho_1}, L_{\rho_2})$ such that $\mathscr{L}^m(L_{\rho_1}, L_{\rho_2})$ becomes a Banach lattice. Let $\mathscr{H}_{\rho_1,\rho_2}$ denote the set of all Hille-Tamarkin operators from L_{ρ_1} into L_{ρ_2} and let $\rho_2 \otimes \rho_1'$ denote the double norm on $\mathscr{H}_{\rho_1,\rho_2}$, i.e., if $T(x, y)$ is the kernel of $T \in \mathscr{H}_{\rho_1,\rho_2}$ then

$$\rho_2 \otimes \rho_1'(T) = \rho_2(\rho_1'(T_x(y))).$$

The formula

$$\rho_1'(T_x(y)) = \inf(g: |Tf| \leqq g\rho_1(f))$$

implies immediately that for all $T \in \mathscr{H}_{\rho_1,\rho_2}$ we have

$$\|T\|_m = \rho_2 \otimes \rho_1'(T).$$

In general $\mathscr{H}_{\rho_1,\rho_2}$ will be a proper subset of $\mathscr{L}^m(L_{\rho_1}, L_{\rho_2})$, but it follows from Theorem 3.2 of [**6**] that

$$\mathscr{H}_{\rho_1,\rho_2} = \mathscr{L}^m(L_{\rho_1}, L_{\rho_2})$$

whenever ρ_1 is order continuous. It is easy to verify that $\mathscr{H}_{\rho_1,\rho_2}$ is a Banach lattice in general, so that $\mathscr{H}_{\rho_1,\rho_2}$ is always a closed subspace of $\mathscr{L}^m(L_{\rho_1}, L_{\rho_2})$. We can consider now $L_{\rho_1}' \otimes L_{\rho_2}$ as a subspace of $\mathscr{H}_{\rho_1,\rho_2}$. If $L_{\rho_1}' \bar{\otimes}_m L_{\rho_2}$ denotes the completion of $L_{\rho_1}' \otimes L_{\rho_2}$ with respect to $\|T\|_m$, then $L_{\rho_1}' \bar{\otimes}_m L_{\rho_2}$ is a closed subspace of $\mathscr{H}_{\rho_1,\rho_2}$.

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THEOREM 3.1. *Let L_{ρ_1} and L_{ρ_2} be Banach function spaces and let $T:L_{\rho_1} \to L_{\rho_2}$ be a linear operator. Then the following are equivalent.*

(i) $T \in L'_{\rho_1} \bar{\otimes}_m L_{\rho_2}$.

(ii) *T is order continuous and*

$$\{Tf:f \in L_{\rho_1}, \rho_1(f) \leqq 1\}$$

is relatively uniformly compact in L_{ρ_2}.

Proof. Assume (i) holds. Then there exist $S_n \in L'_{\rho_1} \otimes L_{\rho_2}$ with $\|S_n\|_m < 1/4^n$ such that $T = \sum_1^\infty S_n$, with the series convergent in the $\|\cdot\|_m$-norm. Let $0 < g_n \in L_{\rho_2}$ such that

$$\rho_2(g_n) < 1/4^n \quad \text{and} \quad |S_n f| \leqq g_n \rho_1(f) \text{ for all } f \in L_{\rho_1}.$$

Put

$$g = \sum_1^\infty 2^n g_n.$$

Then $|Tf| \leqq g\rho_1(f)$ for all $f \in L_{\rho_1}$, so $\frac{1}{g} \cdot T$ maps L_{ρ_1} into L_∞. Now $\frac{1}{g} \cdot S_n$ is a finite rank operator from L_{ρ_1} into L_∞ and

$$\left|\frac{1}{g}Tf - \frac{1}{g}\sum_1^N S_n f\right| \leqq \frac{1}{g} \sum_{n=N+1}^\infty |S_n f|$$
$$\leqq \frac{1}{g}\left(\sum_{n=N+1}^\infty g_n\right)\rho_1(f) \leqq \left(\sum_{n=N+1}^\infty \frac{1}{2^n}\right)\rho_1(f).$$

Hence $\frac{1}{g}T$ is a compact mapping from L_{ρ_1} into L_∞, which proves (ii). Assume now that (ii) holds. Then there exists $0 < g \in L_{\rho_2}$ such that

$$\frac{1}{g} \cdot T:L_{\rho_1} \to L_\infty$$

is compact. Since L_∞ has the approximation property it follows that there exist $S_n:L_{\rho_1} \to L_\infty$ of finite rank such that $S_n \to \frac{1}{g} \cdot T$ in the operator norm of $\mathscr{L}(L_{\rho_1}, L_\infty)$. Put

$$T_n f = gS_n f.$$

Then $T_n:L_{\rho_1} \to L_{\rho_2}$ is of finite rank and $\|T - T_n\|_m \to 0$. Hence

$$T \in L'_{\rho_1} \bar{\otimes}_m L_{\rho_2}.$$

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We now remark that the above result is an order continuous version of Proposition 8.2 of [**3**]. We now introduce another class of Hille-Tamarkin operators. Let $\mathscr{E}_{\rho_1,\rho_2}$ be the set of all order continuous T from L_{ρ_1} into L_{ρ_2} such that $\{Tf:\rho_1(f) \leqq 1\}$ is equimeasurable and order bounded in L_{ρ_2}. From Theorem 1.3 it follows easily that

$$\mathscr{E}_{\rho_1,\rho_2} \subset \mathscr{H}_{\rho_1,\rho_2}.$$

From Theorem 2.2 and Theorem 3.1 it follows that

$$\mathscr{E}_{\rho_1,\rho_2} = L'_{\rho_1} \bar{\otimes}_m L_{\rho_2}$$

whenever ρ_2 is order continuous. From Theorem 1.3 it follows that

$$\mathscr{E}_{\rho_1,\rho_2} = \mathscr{H}_{\rho_1,\rho_2}$$

whenever ρ_1' is order continuous, so in particular

$$L'_{\rho_1} \bar{\otimes}_m L_{\rho_2} = \mathscr{H}_{\rho_1,\rho_2}$$

whenever ρ_1' and ρ_2 are order continuous. This result generalizes a result obtained in [**4**] for L_p spaces.

THEOREM 3.2. *$\mathscr{E}_{\rho_1,\rho_2}$ is a closed sublattice of $\mathscr{H}_{\rho_1,\rho_2}$.*

Proof. The fact that $\mathscr{E}_{\rho_1,\rho_2}$ is a sublattice immediately follows from the known properties of compact operators with range in L_∞ (see [**3**]). It remains to be shown that $\mathscr{E}_{\rho_1,\rho_2}$ is closed in $\mathscr{H}_{\rho_1,\rho_2}$. Let $T_n \in \mathscr{E}_{\rho_1,\rho_2}$ and $T \in \mathscr{H}_{\rho_1,\rho_2}$ such that

$$\|T - T_n\|_m < 1/2^n \quad \text{for all } n.$$

Thus there exist $0 \leqq g_n \in L_{\rho_2}$ with $\rho_2(g_n) \leqq 1/2^n$ such that

$$|(T - T_n)f| \leqq g_n \quad \text{for all } f \in U := \{f:\rho_1(f) \leqq 1\}.$$

Then $g = \sum_1^\infty g_n$ exists in L_{ρ_2}, so that $g_n(x) \to 0$ a.e. on X. Let $X_0 \subset X$ be a set of finite measure and $\epsilon > 0$. Then Egoroff's theorem implies that there exists $X_{0,\epsilon} \subset X_0$ such that

$$\mu(X_0 \setminus X_{0,\epsilon}) < \epsilon/2$$

and $g_n \to 0$ uniformly on $X_{0,\epsilon}$. Now $T_n \in \mathscr{E}_{\rho_1,\rho_2}$ implies that there exist $X_n \subset X_{0,\epsilon}$ with

$$\mu(X_{0,\epsilon} \setminus X_n) < \epsilon/2^{n+1}$$

such that $\chi_{X_n} \cdot T(U)$ is relatively norm compact in L_∞. Let

$$X_\epsilon = \bigcap_{n=1}^\infty X_n.$$

Then $\mu(X_0 \setminus X_\epsilon) < \epsilon$ and $\chi_{X_\epsilon} \cdot T_n(U)$ is relatively norm compact in L_∞ for all n. Let now $\delta > 0$. Then there exists $n \in N$ such that

$$\|\chi_{X_\epsilon} \cdot g_n\|_\infty < \delta/4.$$

For this n there exist $f_i \in U \quad (1 \leqq i \leqq m)$ such that for all $f \in U$ there exists $f_k \in \{f_i\}$ such that

$$\|\chi_{X_\epsilon}(T_n f - T_n f_k)\|_\infty < \delta/2.$$

Hence

$$\|\chi_{X_\epsilon}(Tf - Tf_k)\|_\infty \leqq \|\chi_{X_\epsilon}(T - T_n)(f - f_k)\|_\infty + \|\chi_{X_\epsilon} T_n(f - f_k)\|_\infty \leqq 2\|\chi_{X_\epsilon} g_n\|_\infty + \frac{\delta}{2} < \delta.$$

It follows that $\chi_{X_\epsilon} \cdot T(U)$ is relatively norm compact in L_∞ and thus $T \in \mathscr{E}_{\rho_1,\rho_2}$.

In conclusion we remark that all the inclusions in

$$L'_{\rho_1} \bar{\otimes}_m L_{\rho_2} \subset \mathscr{E}_{\rho_1,\rho_2} \subset \mathscr{H}_{\rho_1,\rho_2}$$

can be proper, for if $T:L_1 \to L_\infty$ is continuous but not compact in measure, then $T \in \mathscr{H}_{1,\infty}$ but $T \notin \mathscr{E}_{1,\infty}$ and if $T:L_1 \to L_\infty$ weakly compact but not compact, then $T \in \mathscr{E}_{1,\infty}$ but $T \notin L_\infty \bar{\otimes}_m L_\infty$.

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