Bull. Austral. Math. Soc. 49Q05
Vol. 53 (1996) [369–372]

FOR THE MINIMAL SURFACE EQUATION, THE SET OF SOLVABLE BOUNDARY VALUES NEED NOT BE CONVEX

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One might think that if the minimal surface equation had a solution on a smooth domain $D \subset \mathbf{R}^n$ with boundary values φ, it would have a solution with boundary values $t\varphi$ for all $0 \leqslant t \leqslant 1$. We give a counterexample in $\mathbf{R}^2$.

We show by example that the minimal surface equation can have a solution on a smooth nonconvex domain $D \subset \mathbf{R}^2$ with smooth boundary values φ, but not with boundary values $\varphi/2$. Thus the set of solvable smooth boundary values need not be convex or star-shaped about 0.

Although a number of friends recall seeing and hearing this problem, I have been unable to locate the source.

For a convex domain, the minimal surface equation has a unique solution for any continuous boundary values. For any smooth domain, the minimal surface equation has a solution for small boundary values of Lipschitz constant at most 1 [**6**].

Theorem. *There are a smooth planar domain D and a smooth function φ on ∂D such that the minimal surface equation has a solution with boundary values $t\varphi$ for $t = 1$ but not for $t = 1/2$.*

Proof: Take two minimal surfaces $\{z = f_1(x, y)\}$, $\{z = f_2(x, y)\}$ over a smooth domain D about the origin such that

$$f_2|\partial D = 2f_1|\partial D \text{ but } f_2(0) \neq 2f_1(0).$$

An easy explicit example is provided by two pieces of catenoids over an annulus, but almost any prescribed boundary values on a convex domain probably will work. We may assume D contains the unit disc $\mathbf{B}(\mathbf{0}, 1)$.

Consider a sequence $1 \gg \varepsilon_1 > \varepsilon_2 > \ldots \to 0$ and domains $D_k = D - \mathbf{B}(\mathbf{0}, \varepsilon_k)$. The function $f_2|D_k$ has a nice minimal graph. We claim that for k large, there is no minimal surface $M_k = \{z = u_k(x, y)\}$ with $u_k|\partial D_k = (1/2)f_2|\partial D_k$.

Received 22 June 1995
I would like to thank Klaus Ecker and Graham Williams for inspiration, and the Australian National University and Melbourne University for hospitality. This work was partially supported by a National Science Foundation grant.

Otherwise (replacing u_k with a subsequence if necessary) we may assume the u_k converge to a function u_∞ on $D - \{0\}$, uniformly on each D_k, which satisfies the minimal surface equation and $u_\infty|\partial D = f_1|\partial D$ (see [3, Chapters 12, 13]). Since a solution to the minimal surface equation cannot have an isolated singularity ([1], or see [5, p.98]), u_∞ extends to all of D. By uniqueness, $u_\infty = f_1$.

Let $A = \max\{\|f_1\|_{C^1}, \|f_2\|_{C^1}, 1\}$, $a = \min\{|(1/2)f_2(0) - f_1(0)|/A, 1\}$. By replacing u_k with a subsequence if necessary, we may assume

$$\varepsilon_2 < \varepsilon_1 \ll a \tag{1}$$

and

$$|u_2(x) - f_1(x)| \leqslant .1a \text{ for } |x| \geqslant \varepsilon_1. \tag{2}$$

Choose $p = (x_0, u_2(x_0))$ with $\varepsilon_2 < |x_0| < \varepsilon_1$ and

$$u_2(x_0) = \left(f_1(0) + \frac{1}{2}f_2(0)\right)\Big/2.$$

We claim that $\mathbf{B}^3(p, .1a)$ does not intersect ∂M_2. The height of the inside boundary differs from $(1/2)f_2(0)$ by at most

$$A\varepsilon_2/2 < Aa/2 \leqslant \left|\frac{1}{2}f_2(0) - f_1(0)\right|\Big/2,$$

while the height of p differs from $f_1(0)$ by at most $|(1/2)f_2(0) - f_1(0)|/2$. The horizontal coordinate gets nowhere near the outside boundary. Therefore $\mathbf{B}^3(p, .1a)$ does not intersect ∂M_2. Hence by monotonicity [4, 9.3],

$$\text{area}\left(M_2 \cap \mathbf{B}^3(p, .1a)\right) \geqslant \pi(.1a)^2. \tag{3}$$

Next we claim that

$$M_2 \cap \mathbf{B}^3(p, .1a) \subset \mathbf{B}^2(0, \varepsilon_1) \times \mathbf{R}. \tag{4}$$

Otherwise there is some

$$(x, u_2(x)) \in \mathbf{B}^3(p, .1a) - \mathbf{B}^2(0, \varepsilon_1) \times \mathbf{R}.$$

Since $(x, u_2(x)) \in \mathbf{B}^3(p, .1a)$, therefore $|x| < \varepsilon_1 + .1a < .2a$ and

$$|u_2(x) - f_1(0)| \geqslant |u_2(x_0) - f_1(0)| - .1a \geqslant .5Aa - .1a \geqslant .4Aa.$$

https://doi.org/10.1017/S0004972700017123 Published online by Cambridge University Press

Since $(x, u_2(x)) \notin \mathbf{B}^2(\mathbf{0}, \varepsilon_1) \times \mathbf{R}$, therefore $\varepsilon_1 < |x| < .2a$ and

$$|u_2(x) - f_1(\mathbf{0})| \leqslant |u_2(x) - f_1(x)| + |f_1(x) - f_1(\mathbf{0})| \leqslant .1a + (.2a)A \leqslant .3aA,$$

by (2). This contradiction establishes (4).

By (3) and (4),

$$\text{area}\left(M_2 \cap \mathbf{B}^2(\mathbf{0}, \varepsilon_1) \times \mathbf{R}\right) \geqslant \pi(.1a)^2. \tag{5}$$

On the other hand, since M_2 minimises area among graphs [4, 6.1], area $\left(M_2 \cap \mathbf{B}^2(\mathbf{0}, \varepsilon_1) \times \mathbf{R}\right)$ is less than the area of $\mathbf{B}^2(\mathbf{0}, \varepsilon_1) \times \{(1/2)f_2(\mathbf{0})\}$, plus the area of a cylinder of radius ε_2 and height at most $(1/2)\|f_2\|_{C^1}\, \varepsilon_2 \leqslant .5Aa$, plus the area of a cylinder of radius ε_1 and height at most

$$\left|\frac{1}{2}f_2(\mathbf{0}) - f_1(\mathbf{0})\right| + \|f_1\|_{C^1}\, \varepsilon_1 \leqslant Aa + A\varepsilon_1 \leqslant 2Aa.$$

Thus

$$\text{area}\left(M_2 \cap \mathbf{B}^2(\mathbf{0}, \varepsilon_1) \times \mathbf{R}\right) \leqslant \pi\varepsilon_1^2 + 2\pi\varepsilon_2(.5Aa) + 2\pi\varepsilon_1(2Aa). \tag{6}$$

Now (5) and (6) contradict hypothesis (1), proving the theorem.

REMARK 1. We conjucture there are wildly oscillating smooth boundary values φ on the unit disc D such that the solutions u_t to the minimal surface equation with boundary values $t\varphi$ have the property that $u_t(\mathbf{0})$ intersects a linear function λt for arbitrarily many values $t_1, \ldots, t_k$ and that for small $\delta > 0$ the set of t for which the minimal surface equation has a solution on $D - \mathbf{B}(\mathbf{0}, \delta)$ with boundary values $t\varphi$ on ∂D and λt on $\partial\mathbf{B}(\mathbf{0}, \delta)$ has k components. To obtain these solutions it may be helpful to choose φ even, so that $Du_t(\mathbf{0}) = 0$.

REMARK 2. Since our argument depends on deleting a small disc to obtain our domain, the conjecture remains open for simply connected planar domains. Our argument does extend to smooth nonconvex balls in $\mathbf{R}^n$ $(n \geqslant 3)$, where we can delete a thin finger instead of a small disc. Then the limiting argument produces a solution of the minimal surface equation with a curve of possible singularities, which are removable for $n \geqslant 3$ ([2], or see [3, Theorem 16.9]). □

ADDED IN PROOF. Fred Almgren has given a simplified counterexample for simply connected planar domains.

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