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SMALL COMPACT ACTIONS ON CHAINABLE CONTINUA

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1. Introduction. In 1931, Newman [**9**] showed that a connected manifold cannot accept arbitrarily small period-n homeomorphisms, for any $n > 1$. In this paper we are concerned with the existence of chainable continua with arbitrarily small homeomorphisms.

For a long time the only known periodic homeomorphisms of chainable continua had periods 1, 2 or 4 [**4**]. Recently, Wayne Lewis [**8**] showed that the pseudo-arc admits periodic homeomorphisms of every order, as well as p-adic cantor group actions. We will see that such homeomorphisms can be made arbitrarily small.

In Section 4, a different chainable indecomposable continuum accepting arbitrarily small period-2 homeomorphisms is constructed.

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2. Preliminaries. A *continuum* is a compact connected metric space. A *chain* D covering a continuum X is a collection of open sets $\{d_i\}_{i=1}^n$, called links, such that $d_i \cap d_j \neq \emptyset$ if and only if $|i - j| \leqq 1$ and such that

$$X \subset \bigcup_{i=1}^{n} d_i.$$

A chain D *c-covers* X if and only if every link of D contains a point of X not in the closure of any other link of D. An *ϵ-chain* D is a chain such that

$$\operatorname{diam}(d_i) < \epsilon \quad \text{for all } i = 1, \dots, n.$$

A continuum is *chainable* if and only if for every $\epsilon > 0$, it can be covered by an ϵ-chain. The pseudo-arc (for a definition see [**2**]) is a chainable continuum. We will use $\bar{n}$ to denote the set $\{1, 2, 3, \dots, n\}$. A function $f:\bar{n} \to \bar{m}$ is a *chain function* if and only if $|i - j| \leqq 1$ implies

$$|f(i) - f(j)| \leqq 1.$$

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The set $H(X)$ of all homeomorphisms of a continuum X is a topological group when provided with the compact-open topology. Since X is compact, this topology coincides with the "sup" topology defined for any given metric for X (see [**5**], p. 168). A topological subgroup G of $H(X)$ is called an *action* on X. The set

$$O_G(x) = \{g(x) | g \in G\}$$

is called the *orbit* of $x \in X$ under G and the set of orbits

$$X/G = \{O_G(x) | x \in X\}$$

endowed with the quotient topology is called the *orbit space* of X under G. We say that G is *small* if and only if $\operatorname{diam}(O_G(x))$ is small for every $x \in X$. Let $N_\epsilon(x)$ denote the set

$$\{y \in X | \rho(x, y) < \epsilon\},$$

where ρ is the metric for X.

3. Compact actions on the pseudo-arc. In this section we will prove that any compact action on the pseudo-arc can be made arbitrarily small.

Throughout this section G will denote an action on a continuum X. Let $\pi: X \to X/G$ be the natural projection. If $U \subset X$ is an open set then

$$\pi^{-1}\pi(U) = \bigcup_{g \in G} g(U)$$

is open. Therefore, π is an open map.

THEOREM 3.1. *If G is compact then X/G is metric.*

Proof. If G is compact then G is equicontinuous ([7], p. 233). Let us give X an equivalent metric which makes G a group of isometries ([**1**], p. 604). We want to show that $\{O_G(x) | x \in X\}$ is an upper semicontinuous decomposition. To this effect, let $x_0 \in X$ and let $U \subset X$ be an open set such that $O_G(x_0) \subset U$. Since G is compact, $O_G(x_0)$ is closed. Let

$$\epsilon = \operatorname{dist}(O_G(x_0), X - U).$$

Then

$$N_{\epsilon/2}(x_0) \cap (X - U) = \emptyset$$

and also

$$[g(N_{\epsilon/2}(x_0)) = N_{\epsilon/2}(g(x_0))] \cap (X - U) = \emptyset \text{ for any } g \in G.$$

Therefore, the open set

$$V = \bigcup_{g \in G} g(N_{\epsilon/2}(x_0))$$

satisfies

$$O_G(x_0) \subset V \subset U.$$

Furthermore,

$$V = \bigcup_{x \in N_{\epsilon/2}(x_0)} O_G(x).$$

Thus $\{O_G(x) \mid x \in X\}$ is an upper semicontinuous decomposition. Then X/G is Hausdorff and being the continuous image (under π) of a compact metric space, it is metric.

THEOREM 3.2. [**10**] *If there is an open map from the pseudo-arc onto a metric space X then X is also a pseudo-arc.*

As a consequence of the last two theorems we have:

COROLLARY 3.3. *If G is a compact action on the pseudo-arc P then P/G is also a pseudo-arc.*

Now we quote the following useful result.

THEOREM 3.4. ([**3**], p. 179). *Let*

$$D = \{d_i\}_{i=1}^{n} \quad \text{and} \quad E = \{e_i\}_{i=1}^{m}$$

be chains c-covering the pseudo-arcs Q and P, respectively. Let $h:\bar{n} \to \bar{m}$ be an onto chain function. Then there exists a homeomorphism $\phi:Q \to P$ such that $\phi(d_i) \subset e_{h(i)}$.

We are now ready to prove the main result of this section.

THEOREM 3.5. *Any compact action on the pseudo-arc can be made arbitrarily small. More precisely, let P be the pseudo-arc, let G be a compact action on P, and let $\epsilon > 0$. Then there exists a homeomorphism $\phi:P \to P$ such that $\phi G \phi^{-1}$ lies in the ϵ-neighborhood of the identity.*

Proof. Let P and Q be fixed embeddings of the pseudo-arc in the plane. Let G be a compact action on Q. By Corollary 3.3, Q/G is a pseudo-arc. Let $\epsilon > 0$ and let

$$E = \{e_i\}_{i=1}^{m}$$

be an ϵ-chain c-covering P. Let $F = \{f_i\}_{i=1}^{n}$ be any chain c-covering Q/G with $n > m$. Set

$$D = \{d_i\} = \{\pi^{-1}(f_i)\}_{i=1}^{n}$$

where $\pi:Q \to Q/G$ is the natural projection, to obtain a chain c-covering Q with the property that if $x \in Q$ then $O_G(x) \subset d_i$ for some i.

Let $h:\bar{n} \to \bar{m}$ be any onto chain function. By Theorem 3.4, there exists a homeomorphism $\phi:Q \to P$ such that $\phi(d_i) \subset e_{h(i)}$. For each $g \in G$, define a homeomorphism g' on P by $g' = \phi g \phi^{-1}$. Let

$$G_\epsilon = \{g' \mid g \in G\}.$$

Then $G_\epsilon = \phi G \phi^{-1}$. Since it is defined through conjugation, G_ϵ is a group of homeomorphisms topologically and algebraically equivalent to G.

Now, if $x \in P$ then $x \in e_i$ for some i, and thus $\phi^{-1}(x) \in d_j$, with $h(j) = i$. Thus

$$O_G(\phi^{-1}(x)) \subset d_j.$$

This, together with

$$g'(x) = \phi(g\phi^{-1}(x)),$$

gives us

$$\begin{aligned} O_{G_\epsilon}(x) &= \{g'(x) \mid g' \in G_\epsilon\} \\ &= \{\phi(g\phi^{-1}(x)) \mid g \in G\} \\ &= \phi\{g\phi^{-1}(x) \mid g \in G\} \\ &= \phi(O_G(\phi^{-1}(x))) \subset e_i. \end{aligned}$$

Since E is an ϵ-chain,

$$\operatorname{diam} O_{G_\epsilon}(x) < \operatorname{diam} e_i < \epsilon.$$

Notice that if $Q = P$, then ϕ is a homeomorphism of P onto itself and $\phi G \phi^{-1} = G_\epsilon$ is ϵ-close to the identity. This concludes the proof.

Lewis **[8]** proved that for an $n > 1$ there exists a period-n homeomorphism on the pseudo-arc. Since a periodic homeomorphism generates a finite and therefore compact group, we have the following.

COROLLARY 3.6. *There exist arbitrarily small periodic homeomorphisms of every finite order on the pseudo-arc.*

Question 3.7. Is this property characteristic of the pseudo-arc among indecomposable tree-like continua?

4. A different example. In this section we will construct an example of a chainable continuum different from the pseudo-arc admitting arbitrarily small period-2 homeomorphisms.

Let $I = [-1, 1]$. Recall that a metric continuum is chainable if and only if it is the inverse limit of arcs.

Let X be the inverse limit of the inverse sequence

$$\{X_i, f_{i+1}\}_{i=0}^{\infty}$$

of continua with onto bonding maps. For $i > j$, let $f_j^i : X_i \to X_j$ be defined by

$$f_j^i = f_{j+1} \ldots f_{i-1} f_i.$$

Let $\pi_i : X \to X_i$ denote the i-th projection, $i = 0, 1, \ldots$.

We state without proof the following:

LEMMA 4.1. *For each $i \geqq 0$, there exists a positive number α_i such that*

$$\operatorname{diam} \pi_i^{-1}(x) < \alpha_i \quad \textit{for every } x \in X_i.$$

These numbers have the property that $\alpha_i \to 0$ when $i \to \infty$.

Now we prove the following lemma that will be used in the proof of Theorem 4.13.

LEMMA 4.2. *Let n be a positive integer. Let $\{b_i\}_{i=0}^{\infty}$ be a strictly increasing sequence of non-negative integers. Let*

$$h_0: X_{b_0} \to X_{b_0}$$

be the identity map and for each $i > 0$, let

$$h_i: X_{b_i} \to X_{b_i}$$

be a period-n homeomorphism such that

$$h_{i-1} f_{b_{i-1}}^{b_i} = f_{b_{i-1}}^{b_i} h_i.$$

Then the sequence $\{h_i\}_{i=0}^{\infty}$ induces a period-n homeomorphism $h: X \to X$ with the property that

$$\operatorname{diam}(O_h(x)) < \alpha_{b_0} \quad \textit{for all } x \in X.$$

Proof. For $x = (x_1, x_2, \dots) \in X$ define $h(x)$ to be the point $y = (y_1, y_2, \dots) \in X$ such that $y_{b_i} = h_i(x_{b_i})$. Because of the commutativity of the h_i's with the bonding maps, h is well defined. Since each h_i is a period-n homeomorphism, so is h. Since h_0 is the identity,

$$\begin{aligned} h^k(x) &= (f_1^{b_0}(h_0^k(x_{b_0})), f_2^{b_0}(h_0^k(x_{b_0})), \dots, h_0^k(x_{b_0}), \dots, h_1^k(x_{b_1}), \dots) \\ &= (x_1, x_2, \dots, x_{b_0}, \dots, h_1^k(x_{b_1}), \dots), \end{aligned}$$

for all $x \in X$ and $k = 1, \dots, n$. Therefore

$$h^k(x) \in \pi_{b_0}^{-1}(x_{b_0})$$

for $k = 1, \dots, n$ so

$$O_h(X) \subset \pi_{b_0}^{-1}(x_{b_0}).$$

By Lemma 4.1,

$$\operatorname{diam} O_h(X) < \alpha_{b_0}.$$

Definition 4.3. A map $f: I \to I$ is *piecewise linear* if and only if there exists a partition $\{-1 = a_1 < a_2 < \dots < a_n = 1\}$ of I such that the restriction $f|_{[a_i, a_{i+1}]}$ is linear for all $i = 1, \dots, n - 1$. We denote the restriction $f|_{[a_i, a_{i+1}]}$ by $f|_i$. There is always a partition P having a minimal number of elements such that f is linear on each subinterval determined by P. We call this partition the *partition of f* and we denote it by P_f.

Remark. A piecewise linear map f is completely determined by its partition P_f and the set $\{ (a_i, f(a_i)) \,|\, a_i \in P_f\}$. It is easy to prove that the composition of piecewise linear maps is piecewise linear.

Definition 4.4. A map $f:I \to I$ is *p-onto* if and only if it is piecewise linear and there exists an i, $1 \leqq i < n$, such that f_i is onto; i.e., such that $f(a_i) \doteq (-1)^k$ and $f(a_{i+1}) = (-1)^{k+1}$ for some integer k. We call $[a_i, a_{i+1}]$ a *p-onto interval* for f.

LEMMA 4.5. *The composition of p-onto maps is p-onto.*

Proof. Let f and g be p-onto maps with partitions P_f and P_g and p-onto intervals $[a_j, a_{j+1}]$ and $[b_k, b_{k+1}]$, respectively. As remarked above, fg is a piecewise linear map with partition P_{fg} of fg. Let

$$c_1 = g|_k^{-1}(a_j) \quad \text{and} \quad c_2 = g|_k^{-1}(a_{j+1})$$

and assume that $c_1 < c_2$. Then

$$fg(c_1) = f(a_j) \quad \text{and} \quad fg(c_2) = f(a_{j+1}).$$

It is easy to see that c_1, c_2 are consecutive elements of P_{fg}, so $fg|_{[c_1,c_2]}$ is linear and onto and therefore fg is p-onto.

For any $t \in (-1, 1)$, let $\phi_t:I \to I$ be the piecewise linear map defined by

$$P_{\phi_t} = \{-1, t, 1\}, \ \phi_t(-1) = 1, \ \phi_t(t) = t \text{ and } \phi_t(1) = -1.$$

It is easy to check that $\phi_0(x) = -x$ and we denote this map by ϕ.

For all $t \in (-1, 1)$ it is easy to see that ϕ_t is a period-2 homeomorphism.

Definition 4.6. Let $s, t \in (-1, 1)$. A map $f:I \to I$ is *$s-t$ symmetric* if and only if $f\phi_s = \phi_t f$. An $s-s$-symmetric map is called *s-symmetric* and a 0-symmetric map is called *symmetric*.

We prove now our key tool, stated in a greater generality than needed.

THEOREM 4.7. *Let f be a p-onto map and t, s two numbers in $(-1, 1)$. Then, given $\alpha_1 < \alpha_2$, two numbers both in $[-1, s)$ or both in $(s, 1]$, $\beta_1 = \pm 1$ and $\beta_2 = -\beta_1$, there exists a p-onto map g with p-onto interval $[\alpha_1, \alpha_2]$ such that*

$$g(\alpha_1) = \beta_1, \ g(\alpha_2) = \beta_2,$$

and such that fg is an $s-t$-symmetric p-onto map.

Proof. Let

$$P_f = \{-1 = a_1 < a_2 < \ldots < a_n = 1\}$$

be the partition of f and $[a_k, a_{k+1}]$, $1 \leqq k < n$, a p-onto interval for f. Let us assume that

(i) $f|_k(a_k) = -1, f|_k(a_{k+1}) = 1,$

(ii) $\alpha_1, \alpha_2 \in (-1, s)$ and

(iii) $\beta_1 = -1.$

The proof for the other cases is similar (see Figure 4.1).

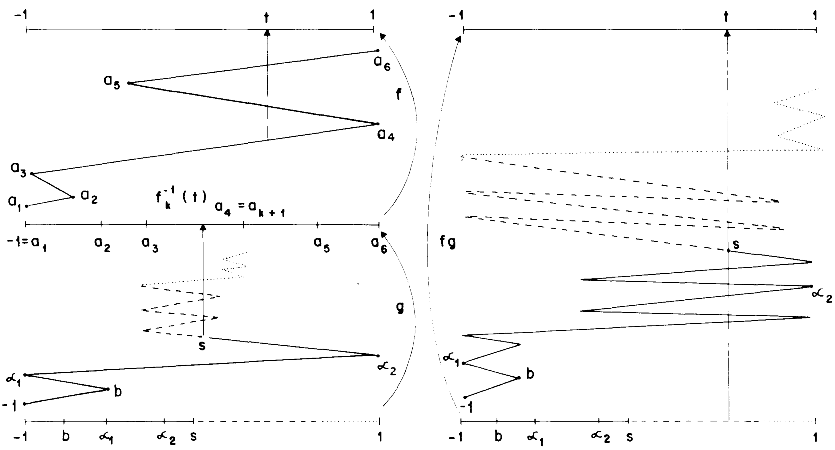

Figure 4.1

Let us define

$$\bar{g}:[-1, s] \to [-1, 1]$$

piecewise linearly by letting

$$p_{\bar{g}} = \{-1 < b < \alpha_1 < \alpha_2 < s\}$$

where $b \in (-1, \alpha_1)$, and by

$$\bar{g}(-1) = -1, \bar{g}(b) = a_2, \bar{g}(\alpha_1) = -1, \bar{g}(\alpha_2) = 1$$

and

$$\bar{g}(s) = f|_k^{-1}(t).$$

Now we want to define

$$\bar{\bar{g}}:[s, 1] \to [-1, 1].$$

If we want fg to be $s-t$-symmetric we must have

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$$\phi_t f\bar{\bar{g}}(x) = f\bar{g}\phi_s(x) \quad \text{for every } x \in [s, 1],$$

or, remembering that ϕ_t is of period 2, we want

$$f\bar{\bar{g}}(x) = \phi_t f\bar{g}\phi_s(x).$$

Now, the fact that

$$f|_k:[a_k, a_{k+1}] \rightarrow [-1, 1]$$

is a homeomorphism allows us to define

$$\bar{\bar{g}}:[s, 1] \rightarrow [a_k, a_{k+1}]$$

by

$$\bar{\bar{g}}(x) = f|_k^{-1}\phi_t f\bar{g}\phi_s(x).$$

Now, simply set

$$g(x) = \begin{cases} \bar{g}(x) \text{ if } x \in [-1, s] \\ \bar{\bar{g}}(x) \text{ if } x \in [s, 1]. \end{cases}$$

Since

$$\begin{aligned} \bar{\bar{g}}(s) &= f|_k^{-1}\phi_t f\bar{g}(s) = f|_k^{-1}\phi_t ff|_k^{-1}(t) \\ &= f|_k^{-1}\phi_t(t) = f|_k^{-1}(t) = \bar{g}(s), \end{aligned}$$

g is a well-defined p-onto map with $[\alpha_1, \alpha_2]$ as p-onto interval. Therefore, by Lemma 4.5 fg is also p-onto.

Let us verify that fg is $s-t$-symmetric. Observe first that from the definition of $\bar{\bar{g}}$ we get

$$\phi_t f|_k\bar{\bar{g}}(x) = f\bar{g}\phi_s(x),$$

so if $x \in [s, 1]$ then $\phi_s(x) \in [-1, s]$ and therefore

$$\begin{aligned} fg\phi_s(x) &= f\bar{g}(\phi_s(x)) = \phi_t f|_k\bar{\bar{g}}(x) \\ &= \phi_t f|_k g(x) = \phi_t fg(x). \end{aligned}$$

Also, if $x \in [-1, s]$ then $\phi_s(x) \in [s, 1]$ and thus

$$\begin{aligned} fg\phi_s(x) &= f\bar{\bar{g}}(\phi_s(x)) = ff|_k^{-1}\phi_t f\bar{g}\phi_s\phi_s(x) \\ &= (ff|_k^{-1})\phi_t f\bar{g}(x) = \phi_t fg(x). \end{aligned}$$

Therefore fg is $s-t$-symmetric and this concludes the proof.

We state now the particular case of the theorem that we need.

COROLLARY 4.8. *Let f be a p-onto map. Then there exists a p-onto map g such that fg is symmetric.*

Definition 4.9. Let $f_1, \ldots, f_n$ be p-onto maps and let $P(f_1, f_2, \ldots,$

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$f_{n-1}, f_n)$ denote the p-onto map guaranteed by Corollary 4.8 for $f_1 f_2 \ldots f_{n-1} f_n$.

Then $f_1 f_2 \ldots f_n P(f_1, f_2, \ldots, f_n)$ is symmetric. The statement $f = P(\phi)$ means only that f is symmetric.

We can start now the construction of the example. Let $\alpha: I \to I$ be the p-onto map with partition $P_\alpha = \{-1, 0, 1\}$ and such that $\alpha(-1) = \alpha(1) = 1$ and $\alpha(0) = -1$. Then $\alpha(x) = \alpha(-x)$, for all $x \in 1$.

Let $\{a_n\}_{n=0}^{\infty}$ be the sequence defined by

$$a_n = \frac{n(n+1)}{2} = \sum_{i=0}^{n} i.$$

We have the trivial

LEMMA 4.10. *For all* $n \geqq 0$,

$$a_n + (n + 1) = a_{n+1}.$$

For all $n \geqq 1$,

$$a_n - n = a_{n-1}.$$

Definition 4.11. An integer $i \geqq 0$ is in *standard form* if and only if $i = a_n + k$ with $0 \leqq k < n + 1$, $n \geqq 0$.

Let $\{X_i, f_{i+1}\}_{i=0}^{\infty}$ be the inverse sequence where

$$X_i = I, \quad i \geqq 0$$

and where $f_i: X_i \to X_{i-1}$ is defined inductively as follows. Write i in standard form, say, $i = a_n + k$. Then define

$$f_2 = \text{any symmetric } p\text{-onto map}$$

$$f_i = f_{a_n + k} = \left\{ \begin{matrix} \alpha & \text{if } k = 0 \\ P(f_{i-n+1} \ldots f_{i-1}) & \text{if } k \neq 0 \end{matrix} \right\}, \; i \neq 2.$$

Notice that f_i is a p-onto map for every $i \geqq 1$.

Our example X is now defined by

$$X = \lim_{\leftarrow} \{X_i, f_{i+1}\}_{i=0}^{\infty}.$$

THEOREM 4.12. *X is a chainable indecomposable continuum distinct from the pseudo-arc.*

Proof. Since $X_i = I$ for all $i \geqq 0$, X is chainable. Since α is a two-to-one map occurring cofinally as a bonding map, the Ingram-Cook [**6**] criterion shows that X is indecomposable. Since each f_i is p-onto, given an arc $J \subset X_i$, there exists an arc $J' \subset X_{i+1}$ that maps homeomorphically onto J under f_{i+1}. Therefore X contains an arc and thus it cannot be homeomorphic to the pseudo-arc.

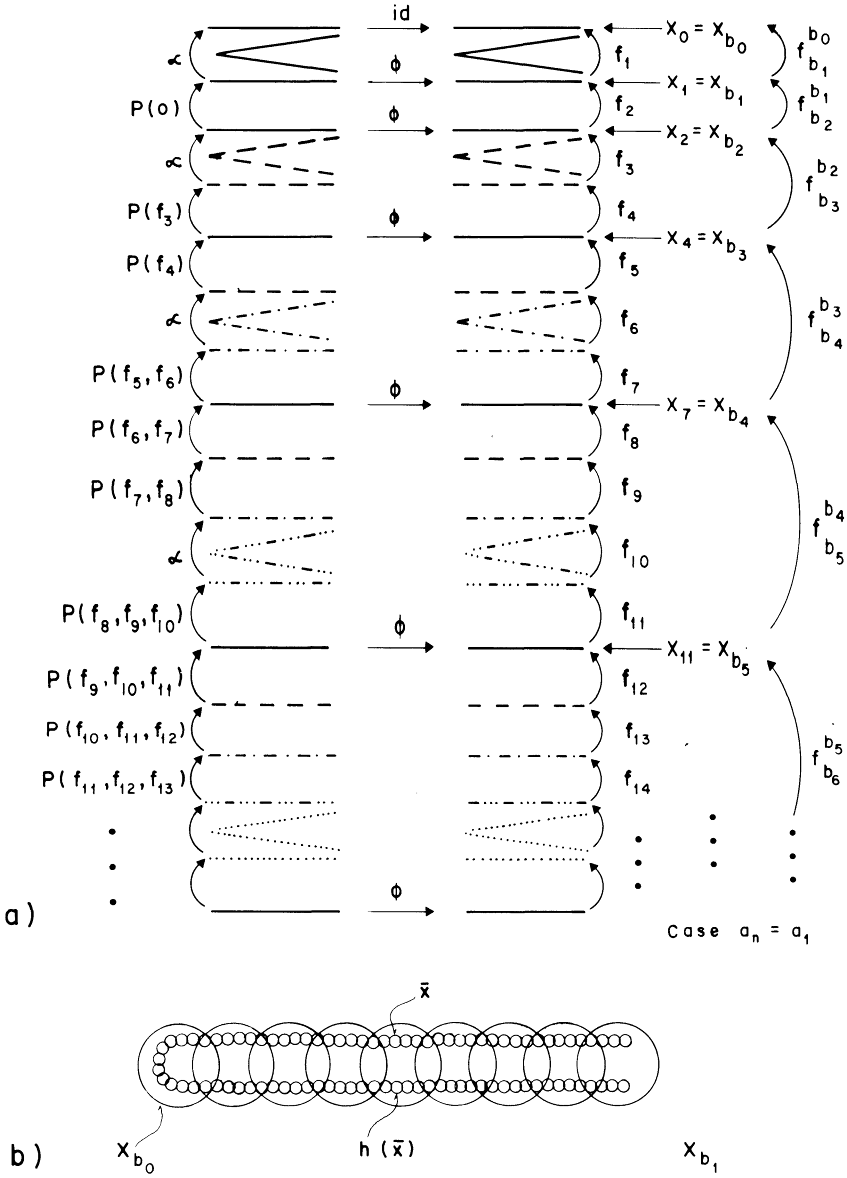

Figure 4.2

Before proceeding to the next theorem let us consider the intuitive idea behind the construction (see Figure 4.2).

We can think of the factor spaces X_i as if they were colored (different colors represented by different patterns in the picture). Each time we introduce a factor space of a new color we insert the map α as the bonding map to the previous space.

Then we introduce a factor space for each color that we already have (including the very last one) in the same order in which they were originally introduced.

Each time a factor space of an old color is introduced we place as the bonding map to the previous space, a map that makes the composition of all the bonding maps up to the previous space of the same color, to commute with the flip ϕ.

Then introduce a new color, repeat all the old colors, introduce another new color and so on.

In this way we end up with a sequence such that each subsequence formed by factor spaces of the same color has the property that its bonding maps (compositions of original bonding maps) commute with flips (see Figure 4.2).

This allows us to define a period-2 homeomorphism on X for each color that we have. Since the bonding map out of the first element of each single-colored subsequence is α, a "v" map, the size of the period-2 homeomorphism defined on such a subsequence is controlled by the "size" of the previous space (see Figure 4.2(b)).

Since each color starts deeper in the sequence, the size of the period-2 homeomorphisms goes to zero.

Now we have the main result of this section.

THEOREM 4.13. *The continuum X admits arbitrarily small period-2 homeomorphisms.*

Proof. Let $\epsilon > 0$. We want to construct a period-2 homeomorphism $h_\epsilon : X \to X$ with the property that

$$\operatorname{diam}(O_{h_\epsilon}(x)) < \epsilon \quad \text{for all } x \in X.$$

To that effect, choose $n \geqq 1$ so large that $\alpha_{a_n - 1} < \epsilon$ (Lemma 4.1). Let $\{b_i\}_{i=0}^{\infty}$ be the sequence defined by

$$b_0 = a_n - 1$$

$$b_i = a_{n+i-2} + n, \quad i > 0.$$

Recall that for $i \geqq 1$,

$$f_{b_{i-1}}^{b_i} : X_{b_i} \to X_{b_{i-1}}$$

is defined by

$$f_{b_{i-1}}^{b_i} = f_{b_{i-1}+1} \ldots f_{b_i - 1} f_{b_i}.$$

By Lemma 4.10, $b_1 = a_{n-1} + n = a_n$, so

$$f_{b_0}^{b_1} = \alpha.$$

If $i > 1$, then $n < (n + i - 2) + 1$ and therefore $b_i = a_{n+i-2} + n$ is in standard form so we have, by the definition of f_{b_i}, that

$$f_{b_i} = f_{a_{n+i-2}+n} = P(f_{b_i-(n+i-2)+1}, \ldots, f_{b_i-1})$$
$$= P(f_{b_{i-1}+1}, \ldots, f_{b_i-1}).$$

This last equality follows from the definition of b_i and Lemma 4.10 as follows:

$$\begin{aligned} b_i - (n + i - 2) + 1 &= (a_{n+i-2} + n) - (n + i - 2) + 1 \\ &= (a_{n+i-2} - (n + i - 2)) + n + 1 \\ &= a_{n+(i-1)-2} + n + 1 \\ &= b_{i-1} + 1. \end{aligned}$$

Then, from definition 4.9, $f_{b_{i-1}+1} \ldots f_{b_i-1}f_{b_i}$ is symmetric so $f^{b_i}_{b_{i-1}}$ is symmetric, for $i > 1$.

Define $h_i:X_{b_i} \to X_{b_i}$ by

$$h_0 = \text{id}$$

$$h_i = \phi, \quad i > 0.$$

Then

$$\begin{aligned} h_0 f^{b_1}_{b_0}(x) &= f^{b_1}_{b_0}(x) = \alpha(x) \\ &= \alpha(-x) = \alpha\phi(x) = f^{b_1}_{b_0}h_1(x) \end{aligned}$$

and since $f^{b_i}_{b_{i-1}}$ is symmetric for $i > 1$ we have

$$h_{i-1}f^{b_i}_{b_{i-1}} = \phi f^{b_i}_{b_{i-1}} = f^{b_i}_{b_{i-1}}\phi = f^{b_i}_{b_{i-1}}h_i.$$

Observe that since $h_i = \phi$, h_i is a period-2 homeomorphism for $i > 1$.

Then, the sequence $\{h_i\}_{i=0}^{\infty}$ satisfies the conditions of Lemma 4.2 so it induces a period-2 homeomorphism $h_\epsilon:X \to X$ with the property that

$$\text{diam}(O_{h_\epsilon}(x)) < \alpha_{b_0} \quad \text{for all } x \in X.$$

But $b_0 = a_n - 1$ so, by our choice of n, $\alpha_{b_0} < \epsilon$. Then

$$\text{diam}(O_{h_\epsilon}(x)) < \epsilon \quad \text{for all } x \in X.$$

Since ϵ was arbitrary, this concludes the proof.

Question 4.14. Does there exist a chainable continuum, other than the pseudo-arc, admitting arbitrarily small period-n homeomorphisms for some $n > 2$?

Remark 4.15. The technique used in the proof of Theorem 4.7 allows variations. Observe then that the use of different variations of this technique may lead to different examples with the desired property.

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