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A VARIATIONAL-LIKE INEQUALITY PROBLEM

J. PARIDA, M. SAHOO AND A. KUMAR

Given a closed and convex set K in $\mathbf{R}^n$ and two continuous maps $F\colon K \to \mathbf{R}^n$ and $\eta\colon K \times K \to \mathbf{R}^n$, the problem considered here is to find $\bar{x} \in K$ such that

$$(\forall x \in K) \quad \langle F(\bar{x}), \eta(x, \bar{x})\rangle \geqslant 0.$$

We call it a variational-like inequality problem, and develop a theory for the existence of a solution. We also show the relationship between the variational-like inequality problem and some mathematical programming problems.

1. INTRODUCTION

Given a closed and convex set K in $\mathbf{R}^n$ and a continuous map $F\colon K \to \mathbf{R}^n$, the problem of finding $\bar{x} \in K$ such that

$$(1.1) \qquad (\forall x \in K) \quad \langle F(\bar{x}), x - \bar{x}\rangle \geqslant 0,$$

is called a variational inequality problem [8]. In recent years, various extensions of (1.1) have been proposed and analysed [3]. We introduce an extension of (1.1) as follows: Let K be a closed and convex set in $\mathbf{R}^n$. Given two continuous maps $F\colon K \to \mathbf{R}^n$ and $\eta\colon K \times K \to \mathbf{R}^n$, find $\bar{x} \in K$ such that

$$(1.2) \qquad (\forall x \in K) \quad \langle F(\bar{x}), \eta(x, \bar{x})\rangle \geqslant 0.$$

We call it a variational-like inequality problem. If $\eta(x, \bar{x}) = x - \bar{x}$, then (1.2) reduces to (1.1).

In this paper, we establish some existence theorems for (1.2) under various conditions on the maps F and η. We also demonstrate the relationship between the variational-like inequality problem (1.2) and some mathematical programming problems. In a subsequent paper, the results of the present paper will be utilised to establish existence theorems for (1.2) where the underlying map F is η-monotone.

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2. Notation and definitions

The real n-dimensional linear space of column vectors $x = (x_1, \ldots, x_n)^T$ is denoted by $\mathbf{R}^n$. For any $x, y \in \mathbf{R}^n$, we denote the usual inner product in $\mathbf{R}^n$ by $\langle x, y\rangle = y^T x$, while $\|x\|$ is any norm of x. For closed convex cone S of $\mathbf{R}^n$, S^* denotes the dual cone of S, and is given by

$$S^* = \{y \in \mathbf{R}^n \colon (\forall x \in S)\langle y, x\rangle \geqslant 0\}.$$

Given a multivalued map $V\colon \mathbf{R}^n \to 2^{\mathbf{R}^n}$, V is said to be upper semicontinuous [**2**, p.111] if $\{x^n\}$ converging to x, and $\{y^n\}$, with $y^n \in V(x^n)$, converging to y, implies $y \in V(x)$.

We introduce a generalisation of monotone functions. The map $F\colon K \to \mathbf{R}^n$ is said to be monotone over K if

$$(\forall x, y \in K) \quad \langle F(y) - F(x), y - x\rangle \geqslant 0.$$

F is said to be η-monotone on K if there exists a continuous map $\eta\colon K \times K \to \mathbf{R}^n$ such that

$$(2.1) \qquad (\forall x, y \in K) \quad \langle F(y), \eta(x, y)\rangle + \langle F(x), \eta(y, x)\rangle \leqslant 0.$$

Note that this definition reduces to the definition of monotone functions if $\eta(y, x) = y - x$.

F is said to be strictly η-monotone over K if the equality holds in (2.1) only when $y = x$.

Hanson [**4**] and Mond and Hanson [**9**] have defined a certain class of differentiable functions, now known as invex functions [**1**], which contains as a subclass the class of differentiable convex functions. We follow the nomenclature and definitions given in [**1**, **4**].

Let $\psi\colon K \to \mathbf{R}^n$ be differentiable. Then ψ is η-convex on K if there exists a continuous map $\eta\colon K \times K \to \mathbf{R}^n$ such that

$$(2.2) \qquad (\forall x, y \in K) \quad \psi(y) - \psi(x) \geqslant \langle \nabla\psi(x), \eta(y, x)\rangle.$$

It is known that if ψ is convex on K, then $\nabla\psi$ is monotone on K. In the same vein, we have here that $\nabla\psi$ is η-monotone whenever ψ is η-convex.

3. Existence theorems

Theorem 3.1. *Let K be a compact and convex set in $\mathbf{R}^n$, and let $F\colon K \to \mathbf{R}^n$ and $\eta\colon K \times K \to \mathbf{R}^n$ be two continuous maps. Suppose that*

$$(3.1) \qquad (\forall x \in K) \quad \langle F(x), \eta(x, x)\rangle = 0,$$

(3.2) *for each fixed $x \in K$, the function $\langle F(x), \eta(y, x)\rangle$ is quasiconvex in $y \in K$.*

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Then

$$(\exists \bar{x} \in K)(\forall x \in K) \quad \langle F(\bar{x}), \eta(x, \bar{x})\rangle \geqslant 0.$$

PROOF: For each $x \in K$, let

$$V(x) = \{s \in K : \langle F(x), \eta(s, x)\rangle = \min_{v \in K} \langle F(x), \eta(v, x)\rangle\}.$$

Since K is compact and $\langle F(x), \eta(v, x)\rangle$ is quasiconvex in v, $V(x)$ is a nonempty, closed and convex subset of K. It is also easy to see that the multivalued map $V: K \to 2^K$ is upper semicontinuous. Now invoking the Kakutani fixed-point theorem [6], we get $\bar{x} \in V(\bar{x})$. Consequently, for all $x \in K$,

$$\langle F(\bar{x}), \eta(x, \bar{x})\rangle \geqslant \langle F(\bar{x}), \eta(\bar{x}, \bar{x})\rangle = 0.$$

This completes the proof of the theorem. ∎

The following result is an extension of Theorem 3.1 to noncompact sets. Consider the set $K_r \subseteq K$ defined by $K_r = \{x : x \in K \text{ and } \|x\| \leqslant r\}$ for real $r > 0$. There always exists an $r_0 > 0$ such that K_r is nonempty whenever $r \geqslant r_0$. From now on we always assume that r satisfies this requirement and K_r is nonempty.

We state this condition:

Condition 3.2. Let K be a closed and convex set in $\mathbf{R}^n$. Let $F: K \to \mathbf{R}^n$ and $\eta: K \times K \to \mathbf{R}^n$ be two continuous maps such that

(i) $(\forall x \in K)\langle F(x), \eta(x, x)\rangle = 0$, and
(ii) for each fixed $x \in K$, the function $\langle F(x), \eta(y, x)\rangle$ is convex in $y \in K$.

We notice that K_r is compact and convex, and hence, by the previous theorem there exists at least one $x_r \in K_r$ such that

$$(\forall x \in K_r) \quad \langle F(x_r), \eta(x, x_r)\rangle \geqslant 0 \tag{3.3}$$

whenever Condition 3.2 is satisfied.

THEOREM 3.3. *Let K, F and η be such that Condition 3.2 is satisfied. A necessary and sufficient condition that there exists a solution to (1.2) is that there exists an $r > 0$ such that a solution $x_r \in K_r$ of (3.3) satisfies the estimate*

$$\|x_r\| < r.$$

PROOF: It is clear that if there exists a solution $\bar{x}$ to (1.2), then $\bar{x}$ is a solution to (3.3) whenever $\|\bar{x}\| < r$. Suppose now that $x_r \in K_r$ is a solution of (3.3) and that $\|x_r\| < r$. Given $x \in K$, we can choose $0 < \lambda < 1$ sufficiently small so that

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$w = \lambda x + (1-\lambda)x_r \in K_r$. Consequently, by the convexity of $\langle F(x_r), \eta(y, x_r)\rangle$, $x_r \in K_r \subseteq K$ satisfies

$$\begin{aligned} 0 &\leqslant \langle F(x_r), \eta(w, x_r)\rangle \\ &\leqslant \lambda\langle F(x_r), \eta(x, x_r)\rangle + (1-\lambda)\langle F(x_r), \eta(x_r, x_r)\rangle \\ &= \lambda\langle F(x_r), \eta(x, x_r)\rangle \end{aligned}$$

for all $x \in K$, which implies that x_r is a solution to (1.2). ∎

We now use Theorem 3.3 to give three other important sufficient conditions for the existence of a solution to (1.2).

THEOREM 3.4. *Let K, F and η be such that Condition 3.2 is satisfied. Then the variational-like inequality problem (1.2) has a solution under each of the following conditions:*

(i) *there is a $u \in K$ and a scalar $r > \|u\|$ such that*

$$\langle F(x), \eta(u, x)\rangle \leqslant 0 \text{ for all } x \text{ with } \|x\| = r; \tag{3.4}$$

(ii) *for some constant $r > 0$, and for each $x \in K$ with $\|x\| = r$, there is a $u \in K$ with $\|u\| < r$ and $\langle F(x), \eta(u, x)\rangle \leqslant 0$;*

(iii) *there exists a nonempty, compact and convex subset C of K with the property that, for every $x \in K \setminus C$, there exists a $u \in C$ such that*

$$\langle F(x), \eta(u, x)\rangle < 0. \tag{3.5}$$

PROOF: (i) Suppose that $x_r \in K_r$ is a solution to (3.3) (which certainly has solutions). Then

$$(\forall x \in K_r) \quad \langle F(x_r), \eta(x, x_r)\rangle \geqslant 0. \tag{3.6}$$

We distinguish two cases. If $\|x_r\| < r$, then Theorem (3.3) yields the desired result. If $\|x_r\| = r$, then it follows from (3.4) and (3.6) that $\langle F(x_r), \eta(u, x_r)\rangle = 0$. Now, let $x \in K$, and choose $0 < \lambda < 1$ small enough so that $w = \lambda x + (1-\lambda)u$ lies in K_r. Then, by the convexity of $\langle F(x_r), \eta(y, x_r)\rangle$,

$$\begin{aligned} 0 &\leqslant \langle F(x_r), \eta(w, x_r)\rangle \\ &\leqslant \lambda\langle F(x_r), \eta(x, x_r)\rangle + (1-\lambda)\langle F(x_r), \eta(u, x_r)\rangle \\ &= \lambda\langle F(x_r), \eta(x, x_r)\rangle \end{aligned}$$

and consequently, x_r is a solution to (1.2).

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(ii) This is little bit more general than the result given in (i) and can be easily proved by arguments similar to that used in the proof of (i).

(iii) Since C is a compact, we can find an $r > 0$ such that $\|x\| < r$ for all $x \in C$. Now, let $x_r \in K_r$ be a solution of (3.3). Then

$$\langle F(x_r),\, \eta(x,\, x_r)\rangle \geqslant 0 \tag{3.7}$$

for all $x \in C$ since $C \subseteq K_r$. If $x_r \notin C$, then it follows from (3.5) that there exists $u \in C$ such that $\langle F(x_r),\, \eta(u,\, x_r)\rangle < 0$, which contradicts (3.7). Therefore, $x_r \in C$, and consequently, $\|x_r\| < r$. From Theorem 3.3, it follows that x_r solves (1.2). ∎

Remark 3.5. From definition of an η-convex function it does not necessarily follow that $\eta(x,\, x) = 0$, but from examples given in [**4**, p.547] and the form of $\eta(x,\, y)$ given in [**1**, p.2], it follows that $\eta(x,\, x) = 0$.

In this contest, we mention that the relation $\langle F(x),\, \eta(x,\, x)\rangle = 0$ in Theorem 3.1 and Condition 3.2 can be replaced by $\eta(x,\, x) = 0$ which is, no doubt, a stronger assumption.

Remark 3.6. If we take $\eta(x,\, y) = x - y$, then the above theorems yield known existence results for the variational inequality (1.1). For example, our Theorems 3.1 and 3.3 reduce to Theorems 3.1 and 4.2, respectively, in [**8**]. Theorems. 3.4 (i) and (ii) reduce to yield the results of Theorems 2.3 and 2.4, respectively, of More' [**10**].

If $F\colon K \to \mathbf{R}^n$ and $\eta\colon K \times K \to \mathbf{R}^n$ are continuous on the closed and convex set K, and if, for some $u \in K$,

$$\lim_{\substack{\|x\|\to\infty \\ x\in K}} \frac{\langle F(x),\, \eta(u,\, x)\rangle}{\|x\|} = -\infty \tag{3.8}$$

then it is easily seen that there exists $r > \|u\|$ such that $\langle F(x),\, \eta(u,\, x)\rangle < 0$ for all $x \in K$ with $\|x\| = r$. This observation leads us to the following existence theorem.

THEOREM 3.8. *Let K, F and η be such that Condition 3.2 is satisfied. If, for some $u \in K$, (3.8) holds, then there exists a solution to (1.2).*

PROOF: The result follows from Theorem 3.4. ∎

Generally, the solution to the variational-like inequaltiy (1.2) may not be unique. There is however a very natural condition which ensures uniqueness.

THEOREM 3.9. *Let $F\colon K \to \mathbf{R}^n$ and $\eta\colon K \times K \to R^n$ be continuous over the closed and convex set K. If F is strictly η-monotone over K, then there can exist at most one solution to the variational-like inequality problem (1.2).*

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PROOF: In fact, if $\bar{x}$ and $\bar{z}$ are two distinct solutions to (1.2), then we have

$$(\forall x \in K) \quad \langle F(\bar{x}), \eta(x, \bar{x})\rangle \geqslant 0,$$
$$(\forall x \in K) \quad \langle F(\bar{z}), \eta(x, \bar{z})\rangle \geqslant 0.$$

Setting $x = \bar{z}$ in the first inequality, $x = \bar{x}$ in the second, and adding the two, we obtain

$$\langle F(\bar{x}), \eta(\bar{z}, \bar{x})\rangle + \langle F(\bar{z}), \eta(\bar{x}, \bar{z})\rangle \geqslant 0$$

which implies that $\bar{x} = \bar{z}$ by the strict η-monotonicity of F. ∎

4. Some problems related to variational-like inequalities

We touch lightly on some mathematical programming problems that are related to variational-like inequalities.

Convex programming. Consider the minimisation problem:

$$\min f(x) \text{ subject to } x \in K,$$

where K is a closed and convex set in $\mathbb{R}^n$ and $f: K \to \mathbb{R}^n$ is continuously differentiable.

PROPOSITION 4.1. *Suppose that f is η-convex over K for some continuous map $\eta: K \times K \to \mathbb{R}^n$. If $\bar{x} \in K$ satisfies*

$$(\forall x \in K) \quad \langle F(\bar{x}), \eta(x, \bar{x})\rangle \geqslant 0, \tag{4.1}$$

where

$$F(x) = \nabla f(x),$$

then

$$f(\bar{x}) = \min_{x \in K} f(x).$$

PROOF: By the η-convexity of f, we have

$$(\forall x \in K) \quad f(x) - f(\bar{x}) \geqslant \langle \nabla f(\bar{x}), \eta(x, \bar{x})\rangle.$$

Now, setting $\nabla f(\bar{x}) = F(\bar{x})$, and using (4.1), we obtain $f(x) \geqslant f(\bar{x})$ for all $x \in K$. ∎

Complementarity problem. Let K be a closed and convex cone in $\mathbb{R}^n$, and let $F: K \to \mathbb{R}^n$ be continuous. Then find $\bar{x} \in R^n$ such that

$$\bar{x} \in K,\ F(\bar{x}) \in K^*,\ \langle F(\bar{x}), \bar{x}\rangle = 0. \tag{4.2}$$

Problem (4.2) is the mathematical form for a variety of problems in optimisation theory, structural mechanics, lubrication theory, elasticity theory, economical equilibrium theory, etcetra, hence its importance. It is known (see, for example [7]) that (4.2) is equivalent to the problem of finding an $\bar{x} \in K$ such that

$$(\forall x \in K) \quad \langle F(\bar{x}), x - \bar{x}\rangle \geqslant 0.$$

But, if $\eta(x, y) = x - y$, then (1.2) reduces to this problem.

Implicit complementarity problem. Let K be a closed and convex cone in $\mathbb{R}^n$, and let F, $g\colon K \to \mathbb{R}^n$ be continuous. Then find $\bar{x} \in \mathbb{R}^n$ such that

$$g(\bar{x}) \in K,\ F(\bar{x}) \in K^*,\ \langle F(\bar{x}), g(\bar{x})\rangle = 0. \tag{4.3}$$

The implicit complementarity problem arose in some special problems in stochastic control, and in the above form, it is considered by Isac [5]. We associate (4.3) with the following variational-like inequality problem: Find $\bar{x} \in K$ such that

$$(\forall x \in K) \quad \langle F(\bar{x}), x - g(\bar{x})\rangle \geqslant 0. \tag{4.4}$$

This problem is a special case of (1.2) when K is a cone and $\eta(x, y) = x - g(y)$. Let y be any vector in K. If $\bar{x}$ is a solution of (4.4), and $g(\bar{x}) \in K$, then replacing x by $y + g(\bar{x})$ in (4.4), we have $\langle F(\bar{x}), y\rangle \geqslant 0$. Therefore, $F(\bar{x}) \in K^*$. Now setting $x = 0$ in (4.4) we obtain $\langle F(\bar{x}), g(\bar{x})\rangle \leqslant 0$, and consequently, $\bar{x}$ is a solution of (4.3). It can also be seen that a solution $\bar{x}$ of (4.3) is a solution of (4.4) if $\bar{x} \in K$.

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Dr J. Parida,
Department of Mathematics
Regional Engineering College
Rourkela 769008,
India

Mr M. Sahoo
Ispat Vidya Mandir
Rourkela 769005,
India

Mr A. Kumar,
Lecturer in Mathematics
Vedvyas College
Rourkela 769041,
India