ELEMENTS OF PACKING AND COVERING

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The term 'covering' is known to any student who has seen the Heine-Borel theorem and he soon learns that it denotes a very basic and widely used concept. Quite generally, a family $\{X_\alpha : \alpha \in A\}$ of subsets of X is a covering of the subset Y of X if $Y \subseteq \bigcup_{\alpha \in A} X_\alpha$.

The concept of packing is perhaps no less frequently encountered although the term has only a rather specialized use. In general, a packing is any family of subsets $\{X_\alpha : \alpha \in A\}$ of a set X which are pairwise disjoint. To make this definition more similar to that of covering, we might define $\{X_\alpha\}$ to be a packing of the subset Y of X if $X_\alpha \cap X_\beta \cap Y = \phi$ for $\alpha \neq \beta$. This is intended to suggest only that there is a certain parallel between the ideas of packing and covering but not a duality in any technical sense.

A partition of a set X is simply a family of subsets of X which is both a packing and covering.

As defined above, that is in a purely set-theoretic context, the concepts of packing and covering allow for little speculation. However, just as soon as we place any structure on X and make some restrictions on the families $\{X_\alpha\}$ many interesting questions arise. It is in the case of X, a euclidean space, that problems of packing and covering have received almost exclusive attention (see, however, A.M. Macbeath [1]). Such problems arise in a great variety of ways and only in part do they have mathematical origins. There has, however, been a concentration of effort on one set of problems in particular; namely that of packing and covering with equal spheres in a euclidean space. The mathematical motivation of this was first felt by Minkowski and much of the work in the subject may be considered as resulting from this part of his legacy. Excellent accounts are given in the books by L. Fejes-Toth [2], J.W.S. Cassels [3] and, most recently, C.A. Rogers [4].

In dealing with packings and coverings by equal euclidean spheres, we are concerned with families which consist of translates of the same set, i.e. with families of the form $\{S + x : x \in L\}$ where by $S + x$ we

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denote as usual the set $\{y + x : y \in S\}$. When $\{S + x : x \in L\}$ is a packing, we speak of S as a packing with respect to L or an L-packing and say that L is admissible with respect to S; similarly for coverings. Fixing S we allow L to range over all admissible sets which have a density or, in particular, over all admissible lattices and order these according to the density $\Delta(L)$ of L. $\Delta(L)$ is defined to be $\lim_{t\to\infty} t^{-n} \# (L \cap A(T))$ where $A(t)$ is the cube whose vertices are the 2^n points $(\pm \frac{1}{2} t, \ldots, \pm \frac{1}{2} t)$ and $\#(X)$ means the cardinality of X.

As already noted the class of lattices plays a special rôle. However, from the standpoint purely of packing and covering this rôle is not intrinsically clear. For example, it is well known that the density of the densest lattice packing of spheres in E_3 is attained by sets other than lattices also. Moreover, the density of L as just defined and its admissibility for a fixed S are not affected by the addition to it in the case of covering or deletion from it in the case of packing of arbitrarily many points in any bounded part of the space.

It is in an effort to gain a better understanding of the rôle which lattices, in particular, play in the theory of packing and covering that we initiate the following investigation.

Let X be a locally compact group with left Haar measure μ. For any subset L of X we call a subset C an L-covering if $LC = X$ and P an L-packing if $xP \cap yP = \phi$ for all x and y in L with $x \neq y$. Denote by $\mathcal{C}$ the class of measurable sets which are bounded (i.e. have compact closure) and by $\mathcal{U}$ the class of measurable sets which have non-empty interior.

Definition. For any subset L of X which is such that there is an L-covering in $\mathcal{C}$ we define the upper dispersion $\overline{\text{disp}}(L)$ to be $\inf\{\mu(c) : C \in \mathcal{C},\ C \text{ an } L\text{-covering}\}$.

For any subset L of X which is such that there is an L-packing in $\mathcal{U}$ we define the lower dispersion $\underline{\text{disp}}(L)$ to be $\sup\{\mu(P) : P \in \mathcal{U},\ P \text{ an } L\text{-packing}\}$.

We say that L is equi-dispersed if $\overline{\text{disp}}(L) = \underline{\text{disp}}(L)$ and lattice-like if there is a set in $\mathcal{C} \cap \mathcal{U}$ which is both an L-packing and an L-covering.

It is easy to show that in E_n if $\Delta(L)$ exists then $\underline{\text{disp}}(L) \leq \Delta(L)^{-1} \leq \overline{\text{disp}}(L)$.

In general if $\bar{\Delta}(L) = \limsup t^{-n} \# (L \cap A(t))$ and $\underline{\Delta}(L) = \liminf t^{-n} \# (L \cap A(t))$ i.e. $\bar{\Delta}$ and $\underline{\Delta}$ are respectively the upper and lower densities then

$$\underline{\text{disp}}(L) \leq \ddot{\Delta}(L)^{-1} \leq \Delta(L)^{-1} \leq \overline{\text{disp}}(L)$$

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What of the inequality $\underline{\text{disp}}(L) \leq \overline{\text{disp}}(L)$ in general? It obviously holds for any compact group. Thus, if P is an L-packing then since $P \in \mathcal{U}$, $\mu(P) > 0$ and, since $\#(L)\mu(P) \leq \mu(X)$, $\#(L)$ is finite. For any L-covering C, $\mu(X) = \mu(LC) \leq \#(L)\mu(C)$. Thus $\mu(P) \leq \mu(X)/\#(L) \leq \mu(C)$. We have in addition the following result.

THEOREM 1. *Let* X *be a locally compact group with left invariant Haar measure* μ *and let* L *be a subset of* X *for which both upper and lower dispersions are defined. If* L *has the property* $LL^{-1} = L^{-1}L$ *then* $\overline{\text{disp}}(L) \geq \underline{\text{disp}}(L)$.

Proof. We want to show that for any L-covering C and L-packing P, $\mu(C) \geq \mu(P)$.

Since μ is regular (see for example [5, p. 224]) we can, for any $\epsilon > 0$, find an open set U and a compact set K such that

$$U \supset C, \ \mu(U) < \mu(C) + \epsilon$$

and

$$K \subset P, \ \mu(K) > \mu(P) - \epsilon .$$

Since LU covers K and K is compact, there is a finite set of elements in L say $x_1, \ldots, x_n$ such that $K \subset x_1 U \cup \ldots \cup x_n U$.

The condition that $LL^{-1} = L^{-1}L$ provides that P is an L^{-1}-packing, namely:

$$(L^{-1}L - \{e\}) P \cap P = \phi$$

implies that

$$(LL^{-1} - \{e\}) P \cap P = \phi .$$

It follows that $\{x_i^{-1}(K \cap x_i U) : i = 1, \ldots, n\}$ = $\{x_i^{-1} K \cap U : i = 1, \ldots, n\}$ is a disjoint family of subsets of U. Hence

$$\begin{aligned} \mu(K) &= \mu\{\cup(K \cap x_i U)\} \leq \Sigma \mu(K \cap x_i U) \\ &= \Sigma \mu(x_i^{-1} K \cap U) = \mu\{\cup(x_i^{-1} K \cap U)\} \\ &\leq \mu(U) . \end{aligned}$$

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Thus $\mu(P) - \epsilon \leq \mu(C) + \epsilon$ and this for any $\epsilon > 0$ implies that $\mu(P) \leq \mu(C)$.*

Note that if C and P are respectively an L-covering and an L-packing, then for any $x \in X$, Cx and Px are again an L-covering and an L-packing respectively. Now suppose that X is not unimodular. Since, whenever defined, $\overline{\text{disp}}(L)$ is finite we must in this case have $\overline{\text{disp}}(L) = 0$. Also $\underline{\text{disp}}(L)$, when defined, is positive so that $\underline{\text{disp}}(L) = +\infty$. Thus for any L for which both upper and lower dispersions are defined the inequality in question certainly does not hold. As a consequence, we have:

COROLLARY. *If X contains a subset L for which upper and lower dispersions are defined and $LL^{-1} = L^{-1}L$ then X is unimodular.*

This result in the case when X satisfies the second axiom of countability and L is a subgroup is due to C. L. Siegel [6, Lemma 5].

If X is abelian then the condition $LL^{-1} = L^{-1}L$ in the theorem is always satisfied. Whether or not this condition is removable when X is any unimodular group is an open question.

In regard to the question of whether upper and lower dispersions are attained the next two theorems give affirmative answers under certain special hypotheses.

THEOREM 2. *Let X be a locally compact group and L a discrete subgroup of X. Then $\underline{\text{disp}}(L)$ is defined and it is attained.*

Proof. That L is discrete is equivalent to the existence of an open L-packing so that certainly $\underline{\text{disp}}(L)$ is defined.

We consider first the case in which $\underline{\text{disp}}(L)$ is finite.

For any positive ϵ there exists for each L-packing P in $\mathfrak{U}$ a bounded (indeed compact) subset K of P such that $\mu(K) \geq \mu(P) - \epsilon$; this by the regularity of μ. K is clearly an L-packing, furthermore we can assume that K is in $\mathfrak{U}$ since we can adjoin to it any bounded neighbourhood contained in P. There exists therefore a sequence $\{K_n : n = 1, 2, \dots\}$ of bounded L-packings in $\mathfrak{U}$ such that $\mu(K_n) \geq \underline{\text{disp}}(L) - \frac{1}{n}$.

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*The referee remarks that essentially the same proof yields: Let X be a locally compact group with left invariant Haar measure. Let L be a subset of X and suppose $\overline{\text{disp}}(L)$ and $\underline{\text{disp}}(L^{-1})$ are defined. Then $\overline{\text{disp}}(L) \geq \underline{\text{disp}}(L^{-1})$.

Let P_1 and P_2 be any L-packings; then it is clear that $P = P_1 \cup (P_2 - LP_1)$ is again an L-packing. (In fact for arbitrary L, $P_1 \cup (P_2 - L^{-1}LP_1)$ is an L-packing whenever P_1 and P_2 are.) Suppose further that P_1 and P_2 are bounded and in $\mathfrak{U}$. Then since L is discrete only a finite number of translates xP_1 with $x \in L$ have non-empty intersection with P_2; let these be $x_1P_1, \ldots, x_rP_1$. It follows that $P_2 - LP_1$ is measurable and hence that P is measurable. Since $P \supset P_1$, P is in $\mathfrak{U}$ and since $P \subset P_1 \cup P_2$ it follows that P is bounded. Moreover

$$\begin{aligned}
\mu(P_2 - LP_1) &= \mu(P_2) - \mu[(x_1P_1 \cup \ldots \cup x_rP_1) \cap P_2] \\
&= \mu(P_2) - \Sigma\mu\ (x_iP_1 \cap P_2) \\
&= \mu(P_2) - \Sigma\mu\ (P_1 \cap (x_i^{-1}P_2) \\
&= \mu(P_2) - \mu\ [P_1 \cap (x_1^{-1}P_2 \cup \ldots \cup x_r^{-1}P_2)] \\
&\geq \mu(P_2) - \mu(P_1)
\end{aligned}$$

and since P_1 and $P_2 - LP_1$ are disjoint it follows that $\mu(P) \geq \mu(P_2)$.

Now let $M_1 = K_1$ and $M_n = M_{n-1} \cup (K_n - LM_{n-1})$ $n = 2, 3 \ldots$. The sequence $\{M_n : n = 1, 2, \ldots\}$ is an expanding sequence of bounded L-packings in $\mathfrak{U}$ and $\mu(M_n) \geq \underline{\text{disp}}(L) - \frac{1}{n}$, $n = 1, 2, \ldots$. Since the union of any expanding sequence of L-packings is again an L-packing, $\cup M_n$ is an L-packing. It is furthermore in $\mathfrak{U}$ hence $\mu(\cup M_n) \leq \underline{\text{disp}}(L)$. But $\mu(\cup M_n) \geq \underline{\text{disp}}(L) - \frac{1}{n}$ for $n = 1, 2, \ldots$, hence $\mu(\cup M_n) = \underline{\text{disp}}(L)$.

In the case that $\underline{\text{disp}}(L) = +\infty$ there exists a sequence of L-packings in $\mathfrak{U}$, $\{P_n : n = 1, 2, \ldots\}$ such that $\mu P_n > n$. Arguing as in the finite case we find that we may assume that P_n is bounded $n = 1, 2, \ldots$. The second step is then also applicable: we can assume that the sequence is an expanding one. It then follows that $\cup P_n$ is an L-packing in $\mathfrak{U}$ such that $\mu(\cup P_n) = \underline{\text{disp}}(L)$.

THEOREM 3. _Let_ X _be a locally compact group with left Haar measure_ μ _and_ L _a subsemigroup of_ X _containing the identity with the property that the bounded subsets of_ L _are finite_. _If_ $\overline{\text{disp}}(L)$ _is defined then it is attained._

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Proof. Let $\{C_n : n = 1, 2, \ldots\}$ be a sequence of L-coverings in $\mathcal{C}$ such that $\mu(C_n) \leq \overline{\text{disp}}(L) + \frac{1}{n}$, $n = 1, 2, \ldots$.

For any two L-coverings in $\mathcal{C}$, say D_1 and D_2, we can make the following construction. Let $x_1D_1, \ldots, x_rD_1$ be the finitely many translates xD_1, $x \in L$ which have non-empty intersection with D_2. Put $S_1 = x_1D_1 \cap D_2$ and $S_k = x_kD_1 \cap [D_2 - (S_1 \cup \ldots \cup S_{k-1})]$ for $k = 2, \ldots, r$, then $D_2 = S_1 \cup \ldots \cup S_r$. Let $S = x_1^{-1}S_1 \cup \ldots \cup x_r^{-1}S_r$, then $S \subset D_1$. Furthermore, since L contains the identity, $LS = LLS$; but $LLS \supset LD_2 = X$ hence S is an L-covering. It is moreover in $\mathcal{C}$. Also

$$\mu(S) \leq \Sigma\mu(x_i^{-1}S_i) = \Sigma\mu(S_i) = \mu(D_2)$$

since $S_1, \ldots, S_r$ are pairwise disjoint.

Denoting the set S obtained in this way by $D_1 \circ D_2$ we replace $\{C_n : n = 1, 2, \ldots\}$ by $\{C'_n : n = 1, 2, \ldots\}$ in which $C'_1 = C_1$; $C'_k = C'_{k-1} \circ C_k$, $k = 2, 3, \ldots$. Then $\{C'_n : n = 1, 2, \ldots\}$ is a nested sequence of L-coverings in $\mathcal{C}$ and $\mu(C'_n) \leq \overline{\text{disp}}(L) + \frac{1}{n}$ for $n = 1, 2, \ldots$. Let $C = \cap C'_n$ then C is certainly in $\mathcal{C}$; we claim that C is also an L-covering. For let x be any point in X. There is a positive finite number of translates yC'_n, $y \in L$ say $y_{n1}C'_n, \ldots y_{nr_n}C'_n$ which contain x for each n. The finite sets $\{y_{n1}, \ldots, y_{nr_n}\}$ $n = 1, 2, \ldots$, are nested and none is empty. It follows that they have at least one element in common, call it y. Then $x \in yC'_n$ for all n, hence $x \in yC$. Thus C is an L-covering and $\mu(C) \geq \overline{\text{disp}}(L)$. But $\mu(C) \leq \overline{\text{disp}}(L) + \frac{1}{n}$ for $n = 1, 2, \ldots$, so $\mu(C) = \overline{\text{disp}}(L)$ and the proof is complete.

We remark that while the union of an expanding sequence of L-packings is again an L-packing, the intersection of a nested sequence of L-coverings is in general not an L-covering. For example, let $X = R$, the additive reals, let L be the integers and as L-coverings take $\{R - (-n, n) : n = 1, 2, \ldots\}$; for a second example let L be the rationals and as L-coverings take $\{(-\frac{1}{n}, \frac{1}{n}) : n = 1, 2, \ldots\}$.

We have made no assumptions in the theorems above regarding the cardinality of the topology of X. We conclude by noting that when X satisfies the second axiom of countability, Siegel has shown [6, p. 678] that if L is a discrete subgroup then it has a fundamental domain. Thus, if $\overline{\mathrm{disp}}(L)$ is defined then $\overline{\mathrm{disp}}(L) = \underline{\mathrm{disp}}(L)$ and these are attained on a set in $\mathcal{C} \cap \mathfrak{U}$.

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REFERENCES

1. A.M. Macbeath, Abstract theory of packings and coverings I. Proc. Glasgow Math. Assoc. 4 (1959) 92-95.

2. L. Fejes-Toth, Lagerungen in der Ebene auf der Kugel und im Raum. (Springer, Berlin, 1953).

3. J.W.S. Cassels, An introduction to the geometry of numbers. (Springer, Berlin, 1959).

4. C.A. Rogers, Packing and covering. (Cambridge, 1964).

5. P.R. Halmos, Measure theory. (Van Nostrand, New York, 1950).

6. C.L. Siegel, Discontinuous groups. Annals of Maths. 44 (1943) 674-689.

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