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On the Image of Certain Extension Maps. I

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Abstract. Let X be a smooth complex projective curve of genus $g \geq 1$. Let $\xi \in J^1(X)$ be a line bundle on X of degree 1. Let $W = \mathrm{Ext}^1(\xi^n, \xi^{-1})$ be the space of extensions of ξ^n by ξ^{-1}. There is a rational map $D_\xi\colon G(n, W) \to SU_X(n+1)$, where $G(n, W)$ is the Grassmannian variety of n-linear subspaces of W and $SU_X(n+1)$ is the moduli space of rank $n+1$ semi-stable vector bundles on X with trivial determinant. We prove that if $n = 2$, then D_ξ is everywhere defined and is injective.

1 Introduction

Unless otherwise stated, we shall assume X to be a smooth complex projective curve of positive genus. Let $M(m, d)$ (resp. $SU_X(m)$) denote the moduli space of semi-stable vector bundles on X of rank m and degree d (resp. rank m and trivial determinant). This is a normal projective variety of dimension $m^2(g-1)+1$ (if X has genus $g \geq 2$) whose points are the S-equivalence classes of semi-stable vector bundles of rank m and degree d on X. Seshadri [9] first constructed $M(m, d)$ and one can also find details about its construction in [6]. For vector bundles A and C on X, the space $\mathrm{Ext}^1(C, A) = H^1(X, \mathrm{Hom}(C, A))$ parametrizes the classes of extensions $0 \to A \to B \to C \to 0$ up to isomorphisms of exact sequences acting as the identity on A and C (see [8, Lemma 3.1] or [7, Proposition 3.1]). This induces a rational map $\mathrm{Ext}^1(C, A) \dashrightarrow M(m, d)$, where $m = \mathrm{rank}(A) + \mathrm{rank}(C)$ and $d = \deg(A) + \deg(C)$, and this map is what one refers to as an extension map. Extension maps have been an important tool for studying these moduli spaces, see for instance [4, 7]. It follows from Lemma 2.1 below that the isomorphism class of a bundle with extension class $u \in \mathrm{Ext}^1(C, A^{\oplus n})$ only depends on the subspace of $\mathrm{Ext}^1(C, A)$ spanned by the components of u in the canonical decomposition $\mathrm{Ext}^1(C, A^{\oplus n}) \overset{\rho}{\cong} \mathrm{Ext}^1(C, A)^{\oplus n}$. The rational map $\mathrm{Ext}^1(C, A^{\oplus n}) \dashrightarrow M(m, d)$ discussed above therefore induces a rational map $G(n, W) \dashrightarrow M(m, d)$, where $W = \mathrm{Ext}^1(C, A)$ and $G(n, W)$ is the Grassmannian variety of n-planes in W. The point of this note is to prove that for any line bundle $\xi \in J^1(X)$, if we pick $C = \xi^2$, $A = \xi^{-1}$ and $n = 2$, then the corresponding rational map $D_\xi\colon G(2, W) \to SU_X(3)$ is defined everywhere and is an injection, see Theorem 2.8. We also identify the image of D_ξ as a certain Brill–Noether locus in $SU_X(3)$ and give a geometric criterion for a point of $G(2, W)$ to be a stable bundle. Specifically, the image under D_ξ of a point in $G(2, W)$ is a stable bundle if and only if the corresponding line in $\mathbb{P}(W)$ does not intersect the image of X under the embedding defined by the linear system $|K\xi^3|$, where K is the canonical line bundle of X. The image of D_ξ is the Brill–Noether locus

$$\mathcal{S}_\xi = \{[V] \in SU_X(3) \mid h^0(X, \xi \otimes gr(V)) \geq 2\},$$

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where $gr(V)$ is the associated graded bundle of V.

While this paper was still under revision, we were told that in [2] it is shown that in the non-fixed determinant case, the map considered here is birrational onto the Brill–Noether locus.

2 Extensions and Vector Bundles

We refer to [1, Proposition 2] for details about the bijection between the space $\mathrm{Ext}^1(C, A) = H^1(X, \mathrm{Hom}(C, A))$ and the isomorphism classes of extensions $0 \to A \to B \to C \to 0$. See also [7, §3] for more properties.

Our notation is as follows. Given line bundles ξ, L on X, we represent $\xi \otimes L$ by ξL, in particular, $\xi^m = \xi^{\otimes m}$. Given a divisor D on X, we write $\mathcal{O}(D)$ to represent the corresponding line bundle on X and $\xi(D) = \xi \otimes \mathcal{O}(D)$. The Jacobian of line bundles on X of degree m is denoted by $J^m(X)$. Given an extension $e\colon 0 \to A \to B \to C \to 0$, we denote by $\delta(e)$ the corresponding vector in $\mathrm{Ext}^1(C, A)$ and by $\mathcal{V}(e)$ the vector bundle B. Given $u \in \mathrm{Ext}^1(C, A)$, let e_u be an extension $0 \to A \to B \to C \to 0$ such that $\delta(e_u) = u$.

There is a canonical isomorphism

$$\mathrm{Ext}^1(C, A_1 \oplus A_2) \overset{\rho}{\cong} \mathrm{Ext}^1(C, A_1) \oplus \mathrm{Ext}^1(C, A_2). \tag{1}$$

So, given $u \in \mathrm{Ext}^1(C, A^{\oplus n})$ we can represent it by n vectors $(\rho_1(u), \dots, \rho_n(u))$, $\rho_i(u) \in \mathrm{Ext}^1(C, A)$.

The rational map from the Grassmannian to the moduli space $M(m, d)$ is induced by the following.

Lemma 2.1 *Let $u, u' \in \mathrm{Ext}^1(C, A^{\oplus n})$ be two vectors such that the subspaces*

$$\langle \rho_1(u), \dots, \rho_n(u) \rangle \quad \text{and} \quad \langle \rho_1(u'), \dots, \rho_n(u') \rangle$$

of $\mathrm{Ext}^1(C, A)$ coincide. Then $\mathcal{V}(e_u) \cong \mathcal{V}(e_{u'})$. Moreover, assume that the only endomorphisms of A and C are scalars. Then if every non-zero homomorphism of A into C is an isomorphism or if $h^0(X, A^\vee \otimes \mathcal{V}(e_v)) = n$, we have $\mathcal{V}(e_u) \cong \mathcal{V}(e_{u'})$ if and only if the subspaces $\langle \rho_1(u), \dots, \rho_n(u) \rangle$ and $\langle \rho_1(u'), \dots, \rho_n(u') \rangle$ coincide.

Proof The proof is similar to the proof of [7, Lemma 3.3] and we only give the first part. First notice that if we think of $\langle \rho_1(u), \dots, \rho_n(u) \rangle$ and $\langle \rho_1(u'), \dots, \rho_n(u') \rangle$ as the row spaces of matrices M_u and $M_{u'}$ representing u and u' respectively, then we see that there is $\mu \in GL_n(\mathbb{C})$ such that $M_{u'} = \mu \cdot M_u$. Now, given $\mu \in GL_n(\mathbb{C})$ we can make it act on $A^{\oplus n}$, and therefore (by functoriality) on $\mathrm{Ext}^1(C, A^{\oplus n})$. It is clear from the construction of extensions that for $u \in \mathrm{Ext}^1(C, A^{\oplus n})$ and $\mu \cdot u$ we have $\mathcal{V}(e_u) \cong \mathcal{V}(e_{\mu \cdot u})$ (although, of course, not by an isomorphism of exact sequences which is the identity on $A^{\oplus n}$ and C). ■

Remark 2.2 (i) The isomorphism (1) can be described in terms of extensions (see [3, §2.6, exercise 2, p. 37]). Let $\sigma_i \in \mathrm{Ext}^1(C, A_i)$, $i = 1, 2$, and let $u = \rho^{-1}(\sigma_1, \sigma_2) \in$

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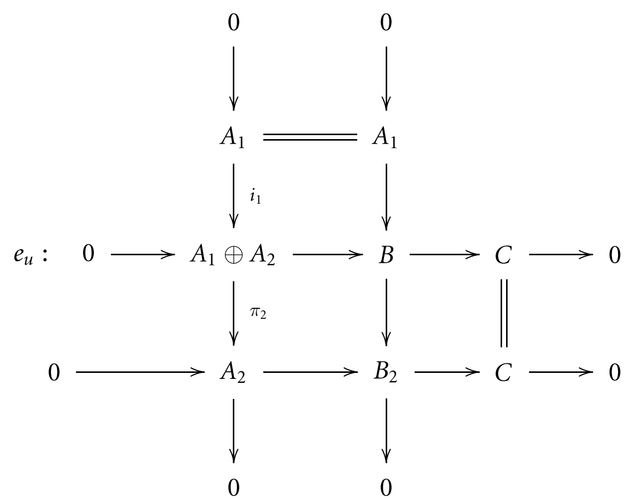

Figure 1

$\mathrm{Ext}^1(C, A_1 \oplus A_2)$. Given the extension $e_u \colon 0 \to A_1 \oplus A_2 \to B \to C \to 0$, one can recover the extensions e_{σ_i}, $i = 1, 2$. For instance, e_{σ_2} is the bottom exact sequence of the commutative diagram in Figure 1, which is obtained by constructing the push out diagram of the canonical projection $A_1 \oplus A_2 \xrightarrow{\pi_2} A_2$ and the injection $A_1 \oplus A_2 \to B$ of e_u, and where i_1 is the natural embedding of A_1 into $A_1 \oplus A_2$.

(ii) Notice from the diagram in Figure 1 that for any $\sigma_2 \in \mathrm{Ext}^1(C, A_2)$, one can define a map $\Psi_{\sigma_2} \colon \mathrm{Ext}^1(C, A_1) \to \mathrm{Ext}^1(\mathcal{V}(e_{\sigma_2}), A_1)$ by taking $\Psi_{\sigma_2}(\sigma_1) = \delta(e)$ for $\sigma_1 \in \mathrm{Ext}^1(C, A_1)$, where e is the vertical extension in the middle of this diagram. Notice also that Ψ_{σ_2} is nothing but the homomorphism $\mathrm{Ext}^1(C, A_1) \to \mathrm{Ext}^1(\mathcal{V}(e_{\sigma_2}), A_1)$ induced by the surjection $s_2 \colon \mathcal{V}(e_{\sigma_2}) \to C$ of e_{σ_2}. In particular $\mathcal{V}(e_{\Psi_{\sigma_2}(\sigma_1)})$ is the fibered product of the surjections $s_i \colon \mathcal{V}(e_{\sigma_i}) \to C$ of e_{σ_i}, $i = 1, 2$.

(iii) We will consider extensions e_u of the form $0 \to A \oplus A \to B \to C \to 0$. For any $\alpha, \beta \in \mathbb{C}$ such that $(\alpha, \beta) \neq (0, 0)$, let $i_{\alpha,\beta} = \alpha i_1 + \beta i_2$ where i_1, i_2 are the two natural embeddings $A \to A \oplus A$. Let $f_{\alpha,\beta}$ be the composite map $A \xrightarrow{i_{\alpha,\beta}} A \oplus A \to B$ and let $E_{\alpha,\beta}$ be the corresponding cokernel. The bundle $E_{\alpha,\beta}$ admits an extension structure $e(\alpha, \beta) \colon 0 \to A \to E_{\alpha,\beta} \to C \to 0$ such that if $u = \rho^{-1}(\sigma_1, \sigma_2)$, then $\delta(e(\alpha, \beta)) = \beta\sigma_1 - \alpha\sigma_2$.

Recall that a vector bundle A on X is semi-stable (resp. stable) if for any proper subbundle $B \subset A$, we have $\mu(B) \leq \mu(A)$ (resp. $\mu(B) < \mu(A)$), where the slope μ is defined by $\mu(B) = \deg(B)/\operatorname{rank}(B)$. Equivalently, A is semi-stable (resp. stable) if $\mu(Q) \geq \mu(A)$ (resp. $>$) for every proper quotient bundle Q of A. Any semi-stable vector bundle has a *Jordan–Hölder filtration*, that is, given a semi-stable vector bundle A, we can always find an increasing filtration $\{0\} \subset A_1 \subset A_2 \subset \cdots \subset A_m = A$ such that A_i/A_{i-1} are stable and $\mu(A_i/A_{i-1}) = \mu(A)$. Such a filtation is not unique, in general, however the associated graded bundle $gr(A) = \bigoplus_{i=1}^{m} A_i/A_{i-1}$ is well defined up to isomorphism. Two semi-stable vector bundles A, B are said to be S-equivalent if $gr(A) = gr(B)$.

Lemma 2.3 *Let $\xi \in J^1(X)$ and let E be a rank r vector bundle of degree 1. Let $0 \neq u \in \operatorname{Ext}^1(E, \xi^{-1})$, then we have*

(i) *If E is stable, then $\mathcal{V}(e_u)$ is semi-stable.*
(ii) *If $\mathcal{V}(e_u)$ is stable, then E is stable.*

Proof (i) Suppose that $\mathcal{V}(e_u)$ is not semi-stable. Then there exists a vector subbundle F of $\mathcal{V}(e_u)$ with $\mu(F) > \mu(\mathcal{V}(e_u))$. By choosing a minimal such F, we can assume that F is stable.

Consider the extension $e_u \colon 0 \to \xi^{-1} \to \mathcal{V}(e_u) \to E \to 0$. Since $\mu(F) > \mu(\xi^{-1})$, then from [7, Lemma 2.1], the composite $F \to E$ is not 0 and $\mu(F) \leq \mu(E) = 1/r$. But $\mu(F) \geq 1/r$. Therefore $F \to E$ is either 0 or an isomorphism and the extension e_u is trivial. This is a contradiction.

(ii) Let Q be a proper quotient bundle of E. Then it is also a quotient of $\mathcal{V}(e_u)$. By the stability of $\mathcal{V}(e_u)$, we have that $\deg Q \geq 1$ and, since $\operatorname{rank}(Q) < \operatorname{rank}(E)$, this implies that $\mu(Q) > \mu(E)$. Therefore $\mathcal{V}(e_u)$ is stable. ■

Lemma 2.4 *Let $x \in X$ and let $h \colon L \to V$ be a homomorphism of a line bundle L into a vector bundle V. Then h factors as a map $L \to L \otimes \mathcal{O}(x) \to V$, where the map $L \to L \otimes \mathcal{O}(x)$ is induced by the canonical section of $\mathcal{O}(x)$ if and only if the fiber map h_x of h at x is zero.*

Proof See [7, Lemma 5.3] ■

Lemma 2.5 *Let $\xi \in J^1(X)$, $\lambda \in J^2(X)$. Consider an extension*

$$0 \to \xi^{-1} \oplus \xi^{-1} \to V \to \lambda \to 0.$$

For $(\alpha, \beta) \in \mathbb{C}^2$ with $(\alpha, \beta) \neq (0,0)$, let $f_{\alpha,\beta} \colon \xi^{-1} \xrightarrow{i_{\alpha,\beta}} \xi^{-1} \oplus \xi^{-1} \to V$ be the embedding defined in Remark 2.2(iii), and let $0 \to \xi^{-1} \xrightarrow{f_{\alpha,\beta}} V \to E_{\alpha,\beta} \to 0$ be the corresponding extension. Then V is stable if and only if $E_{\alpha,\beta}$ is stable for all $(\alpha, \beta) \neq (0,0)$.

Proof If V is stable, then $E_{\alpha,\beta}$ is stable by Lemma 2.3(i). Conversely, suppose that all the vector bundles $E_{\alpha,\beta}$ are stable. By Lemma 2.3(ii), we know that V is semi-stable. So, suppose that L is a line subbundle of V of degree 0. Then for every point $x \in X$ the fiber L_x does not intersect the fiber of $(\xi^{-1} \oplus \xi^{-1})_x$ in V_x: if $L_x = (f_{\alpha,\beta})_x(\xi_x^{-1})$, then the fiber map $L_x \to (E_{\alpha,\beta})_x$ of the map $L \to E_{\alpha,\beta}$ at the point x is the zero map. Using Lemma 2.4 we see that the map $L \to E_{\alpha,\beta}$ factors through a map $L(x) \to E_{\alpha,\beta}$. Since $\mu(L(x)) = 1 > \mu(E_{\alpha,\beta}) = 1/2$, we see that $E_{\alpha,\beta}$ cannot be a stable bundle. So $V \cong \xi^{-1} \oplus \xi^{-1} \oplus L$, a contradiction. Therefore V does not contain line bundles of degree 0.

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Now suppose that U is a rank 2 subbundle of V of degree 0. Then we have an extension $0 \to U \to V \to L \to 0$ with L a line bundle of degree 0. Since $h^0(\xi \otimes L)$ is 1 or 0, we can find $\alpha, \beta \in \mathbb{C}$ such that the composite map

$$\xi^{-1} \xrightarrow{f_{\alpha,\beta}} V \to L$$

is zero. Then there exists a map $E_{\alpha,\beta} \to L$ making the following diagram commutative.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \xi^{-1} & \xrightarrow{f_{\alpha,\beta}} & V & \longrightarrow & E_{\alpha,\beta} & \longrightarrow & 0 \\ & & & & \downarrow & \swarrow & & & \\ & & & & L & & & & \end{array}$$

Since $\mu(E_{\alpha,\beta}) = 1/2 > \mu(L) = 0$, we see that $E_{\alpha,\beta}$ cannot be a stable vector bundle. Therefore V is stable. ■

Lemma 2.6 *Let $\xi \in J^1(X)$, $\lambda \in J^2(X)$ and let $0 \neq u \in \operatorname{Ext}^1(\lambda, \xi^{-1})$.*

(i) *Let K denote the canonical line bundle of X. Then $\mathcal{V}(e_u)$ is non stable if and only if the class of u in $\mathbb{P}H^1(X, \lambda^{-1}\xi^{-1})$ is in the image of X under the natural embedding*

$$\phi_{\xi,\lambda} \colon X \xrightarrow{|K\lambda\xi|} \mathbb{P}H^1(X, \lambda^{-1}\xi^{-1}).$$

If the class of u is $\phi_{\xi,\lambda}(x)$, then $\mathcal{V}(e_u)$ contains a unique maximal line bundle $L \cong \lambda(-x)$.

(ii) *Assume that $\mathcal{V}(e_u)$ is non stable. Let $\eta \in J^1(X)$. The destabilizing line bundle $L \subset \mathcal{V}(e_u)$ induces a linear map $\varphi_L \colon \operatorname{Ext}^1(\mathcal{V}(e_u), \eta^{-1}) \to \operatorname{Ext}^1(L, \eta^{-1})$. Consider the map $\Psi_u \colon \operatorname{Ext}^1(\lambda, \eta^{-1}) \to \operatorname{Ext}^1(\mathcal{V}(e_u), \eta^{-1})$ defined in Remark* 2.2(ii). *Then $\varphi_L \circ \Psi_u \colon \operatorname{Ext}^1(\lambda, \eta^{-1}) \to \operatorname{Ext}^1(L, \eta^{-1})$ is surjective.*

(iii) *Set $\lambda = \xi^2$ and $\eta = \xi$ in* (ii). *Given $w \in \operatorname{Ext}^1(\lambda, \xi^{-1})$, there is an exact sequence $0 \to \mathcal{V}(e_\theta) \to \mathcal{V}(e_{\Psi_u(w)}) \to L^{-1}\xi \to 0$, where $\theta = \varphi_L \circ \Psi_u(w) \in \operatorname{Ext}^1(L, \xi^{-1})$.*

Proof Part (i) is an application of [5, Proposition 1.1]. To prove (ii), it is enough to notice that $\varphi_L \circ \Psi_u$ is the natural map induced by the composition $c \colon L \subset \mathcal{V}(e_u) \to \lambda$, and that this is a non zero map. Moreover, using Lemma 2.4, $L = \lambda(-y)$ for some $y \in X$ and in particular the kernel has dimension 1 and $x = y$ (if $\phi_{\xi,\lambda}(x)$ is the class of u).

For (iii), notice that $\mathcal{V}(e_\theta)$ has determinant $L\xi^{-1}$. From the definition of φ_L, we see that $\mathcal{V}(e_\theta) \subset \mathcal{V}(e_{\Psi_u(w)})$. The claim follows using the fact that $\mathcal{V}(e_{\Psi_u(w)})$ has trivial determinant. ■

Lemma 2.7 *Let $L_0 \in J^0(X)$ and let $\xi \in J^1(X)$. Let $SU_X(2, L_0)$ denote the moduli space of semi-stable rank 2 vector bundles on X with determinant L_0. If $[V] \in SU(2, L_0)$ and $h^0(X, \xi \otimes V) \neq 0$, then there is $\theta \in \operatorname{Ext}^1(\xi L_0, \xi^{-1})$ such that $\mathcal{V}(e_\theta)$ is S-equivalent to V.*

Proof The proof is the same as that of [7, Lemma 5.8]. For instance, if V is non stable, say V is S-equivalent to $L_1 \oplus L_2$ with $L_1, L_2 \in J^0(X)$, then since $h^0(X, \xi \otimes V) \neq 0$ we can assume that $h^0(X, \xi L_1) \neq 0$. Therefore $L_1 = \xi^{-1}(x)$ and $L_2 = L_0\xi(-x)$ for a single point $x \in X$ and we can take θ to be a non zero element of the kernel of the map $\operatorname{Ext}^1(\xi L_0, \xi^{-1}) \to \operatorname{Ext}^1(\xi L_0(-x), \xi^{-1})$ induced by $\xi L_0(-x) \to \xi L_0$. ■

Let $\xi \in J^1(X)$, let $W := H^0(X, K\xi^3)^*$ and let $G(2, W)$ be the Grassmannian variety of 2-dimensional linear subspaces of W. We define the set

$$\mathcal{S}_\xi = \{[V] \in SU_X(3) \mid h^0(X, \xi \otimes gr(V)) \geq 2\},$$

where $gr(V)$ is the associated graded bundle of V. Consider the rational map

$$D_\xi \colon G(2, W) \to SU_X(3)$$

induced by Lemma 2.1

Theorem 2.8 *D_ξ is an injective morphism whose image is the set $\mathcal{S}_\xi$.*

Proof Let us first show that D_ξ is defined at each point of $G(2, W)$. Using Lemma 2.1 we know that each line $l \subset \mathbb{P}(W)$ determines a rank 3 vector bundle V with trivial determinant *i.e.*, given two generators $\sigma_1, \sigma_2 \in W$ of l. Let

$$u = \rho^{-1}(\sigma_1, \sigma_2) \in \operatorname{Ext}^1(\xi^2, \xi^{-1} \oplus \xi^{-1}).$$

Then one takes V to be $\mathcal{V}(e_u)$. Consider the embedding $\phi_{\xi,\xi^2} \colon X \xrightarrow{|K\xi^3|} \mathbb{P}(W)$, since the genus of our curve X is at least 1, there are points in any line $l \subset \mathbb{P}(W)$ that are not contained in $\phi_{\xi,\xi^2}(X)$. Then using Lemma 2.6(i) and Lemma 2.3(i), we see that V is semi-stable, so $D_\xi(l) = [V]$ and $D_\xi(l) \in \mathcal{S}_\xi$ since $h^0(X, \xi \otimes gr(V)) \geq h^0(X, \xi \otimes V)$. In fact, from Lemma 2.5, we see that a line in $\mathbb{P}(W)$ that does not intersect $\phi_{\xi,\xi^2}(X)$, determines a stable vector bundle V with $h^0(X, \xi \otimes V) = 2$. Conversely, if $[V] \in SU_X(3)$ is the class of a stable vector bundle with $h^0(X, \xi \otimes V) \geq 2$, then one can see, using Lemma 2.4, that $h^0(X, \xi \otimes V) = 2$ and that V admits an extension structure

$$0 \to \xi^{-1} \oplus \xi^{-1} \to V \to \xi^2 \to 0.$$

So V is induced by a unique line $l \subset \mathbb{P}(W)$ not intersecting $\phi_{\xi,\xi^2}(X)$.

Given the class of a non stable vector bundle $[V] \in \mathcal{S}_\xi$, we can write $gr(V) = L_1 \oplus E$, where L_1 is a line bundle and E is a rank 2 bundle which is either stable or a direct sum of two line bundles. Using Lemma 2.4, we see that if $h^0(X, \xi \otimes E) \geq 2$ then E cannot be stable because it contains a line bundle of degree 0. So we can assume that $h^0(X, \xi L_1) = 1 \leq h^0(X, \xi \otimes E)$. Let $\lambda = \xi^2$. There is a unique $x \in X$ such that $L := \lambda \otimes \mathcal{O}(-x) = \xi L_1^{-1}$. Let $u \in \operatorname{Ext}^1(\lambda, \xi^{-1})$ be such that its class in $\mathbb{P}\operatorname{Ext}^1(\lambda, \xi^{-1})$ is $\phi_{\xi,\lambda}(x)$. From Lemma 2.7 we know that there is $\theta \in \operatorname{Ext}^1(\xi L_1^{-1}, \xi^{-1})$ such that $\mathcal{V}(e_\theta)$ is S-equivalent to E. Set $\eta = \xi$ in Lemma 2.6(ii). Take $w \in \operatorname{Ext}^1(\lambda, \xi^{-1})$ such that $\varphi_L \circ \Psi_u(w) = \theta$. So, from Lemma 2.6(iii) we see that if l is the line spanned by w and

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u, then $D_\xi(l) = [V]$. Moreover, the natural map $\mathbb{P}\,\mathrm{Ext}^1(\xi L_1^{-1}, \xi^{-1}) \to SU_X(2, L_1^{-1})$ is injective, see [4, Corollary 4.4]. From this and from the fact that the kernel of $\varphi_L \circ \Psi_u$ has dimension 1, one can deduce that there is a unique line $l \subset \mathbb{P}(W)$ such that $x \in l$ and $D_\xi(l) = [V] = [L_1 \oplus E]$.

Consider a line $l_2 \subset \mathbb{P}(W)$ such that $D_\xi(l_2) = [V]$. Let $x_2 \in X$ such that $\phi_{\xi,\lambda}(x_2) \in l_2 \cap \phi_{\xi,\lambda}(X)$ and let $u' \in \mathrm{Ext}^1(\lambda, \xi^{-1})$ such that its class in $\mathbb{P}\,\mathrm{Ext}^1(\lambda, \xi^{-1})$ is $\phi_{\xi,\lambda}(x_2)$.

Let L' be the maximal line subbundle of $\mathcal{V}(e_{u'})$. Let $x_2 \neq p \in l_2$. Let $w \in \mathrm{Ext}^1(\lambda, \xi^{-1})$ such that its class in $\mathbb{P}(W)$ is p. From Lemma 2.6(iii), we see that V is S-equivalent to $L'^{-1}\xi \oplus \mathcal{V}(e_{\theta'})$, where $\theta' = \phi_{L'} \circ \Psi_{u'}(w) \in \mathrm{Ext}^1(L', \xi^{-1})$. Now, we consider the cases E stable and $E = L_2 \oplus L_3$ separately.

If E is stable, then we see that the maximal line subbundle of $\mathcal{V}(e_{u'})$ is L. So, $x = x_2$, that is, there is a unique line $l \subset \mathbb{P}(W)$ such that $D_\xi(l) = [L_1 \oplus E]$.

Now suppose that $E = L_2 \oplus L_3$ and that $x_2 \neq x$. We have $L' \ncong L$, so we can assume that $L_2 \cong L'^{-1}\xi$. Now, $\mathcal{V}(e_{\Psi_u(u')}) \cong \mathcal{V}(e_{\Psi_{u'}(u)})$, so using Lemma 2.6(iii) twice, we see that $\mathcal{V}(e_{\Psi_u(u')})$ is S-equivalent to $L_1 \oplus L_2 \oplus L_3$. That means that the line containing $\phi_{\xi,\lambda}(x)$ and $\phi_{\xi,\lambda}(x_2)$ is mapped to $[V]$ by D_ξ. Therefore $l = l_2$. ■

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