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CONTINUOUS, SLOPE-PRESERVING MAPS OF SIMPLE CLOSED CURVES

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How many of the continuous maps of a simple closed curve to itself are slope-preserving? For the unit circle S^1 with centre $(0, 0)$, a continuous map σ of S^1 to S^1 is slope-preserving if and only if σ is the identity map $[\sigma(x, y) = (x, y)]$ or σ is the antipodal map $[\sigma(x, y) = (-x, -y)]$. Besides the identity map, more general simple closed curves can also possess an "antipodal" map (cf. Figure 1).

Examples of plane curves with continuous, slope-preserving (antipodal) maps.

FIGURE 1

It is perhaps somewhat unexpected that an arbitrary simple, smooth, closed curve behaves, in this respect, very much like S^1. It is the purpose of this paper to establish:

THEOREM. *There are at most two continuous, slope-preserving maps of a simple, smooth, closed curve, to itself. Each such map σ is a homeomorphism satisfying $\sigma \circ \sigma =$ id.*

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Preliminaries. In this section we consider some of the elementary properties of simple plane curves.

For distinct points p and p' in the Euclidean plane $\mathbf{R}^2$ let $\langle p, p' \rangle$ denote the line through p and p'. Let I denote the unit interval $[0, 1]$ in $\mathbf{R}^1$ and set $I^- = [0, 1)$.

A *simple parameter curve* f is a continuous map of I to $\mathbf{R}^2$ such that $f|I^-$ is one-to-one. We call $f(0)$ the *initial point* and $f(1)$ the *terminal point* of $f(I)$. If $f(0) = f(1)$, we identify 0 and 1 in I and call $f(I)$ a *simple closed curve.*

For an element x of I, a line T_x is the *tangent* to f at x if

$$T_x = \lim_{x' \to x} \langle f(x'), f(x) \rangle.$$

We say that f is a *simple differentiable parameter curve of finite type* or, more briefly, a *simple differentiable parameter curve*, if T_x exists for each $x \in I$ and there is a positive integer n such that $|L \cap f(I)| \leqq n$ for each line L in $\mathbf{R}^2$.

Let f be a simple differentiable parameter curve, let L be a line in $\mathbf{R}^2$, and let $x \in I$ satisfy $f(x) \in L$. As $L \cap f(I)$ is finite, there is a deleted neighbourhood N_x of x in I such that $L \cap f(N_x) = \emptyset$. Now, L separates $\mathbf{R}^2$ into two regions. We say that L *supports* f at x if $f(N_x)$ is entirely contained in one of these regions; otherwise, L *cuts* f at x.

LEMMA 1. ([**3**]). *Let f be a simple differentiable parameter curve, let $x \in I$ and let L_x denote the set of all lines in $\mathbf{R}^2$ containing $f(x)$ and distinct from T_x. Then every $L \in L_x$ supports f at x or, every $L \in L_x$ cuts f at x.*

From this standpoint there are precisely four types of points in $f(I) \subseteq \mathbf{R}^2$. We define the *characteristic* $(\alpha_0(x), \alpha_1(x))$ of a point $f(x)$ by taking $\alpha_0(x) = 1$ [2] if some $L \in L_x$ cuts [supports] f at x and by taking $\alpha_1(x) \in \{1, 2\}$ such that $\alpha_0(x) + \alpha_1(x)$ is odd [even] if T_x cuts [supports] f at x. There are then four types of points: *ordinary*, characteristic (1, 1); *inflection*, characteristic (1, 2); *cusp*, characteristic (2, 1); *beak*, characteristic (2, 2) (cf. Figure 2).

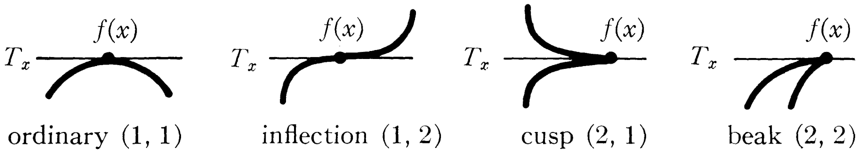

FIGURE 2

In this connection we note:

LEMMA 2. ([**2**]). *A simple differentiable parameter curve contains only finitely many points that are not ordinary.*

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For what follows we assume that f is a simple differentiable parameter curve. We put $C = f(I)$ and refer directly to C as a *simple differentiable curve*. If $p = f(0) \neq f(1) = q$, we also call C a *simple differentiable arc* and denote it also by $A(p, q)$ or A. From this viewpoint a simple closed differentiable curve C $(f(0) = f(1))$ consists of simple differentiable arcs. Indeed, if p and q are distinct points of C then there are simple differentiable arcs $A(p, q)$ and $A(q, p)$ satisfying

$$A(p, q) \cup A(q, p) = C \text{ and } A(p, q) \cap A(q, p) = \{p, q\}.$$

For convenience we often identify $x \in I$ with $p = f(x) \in C$ and also write T_p for the tangent T_x of C at p. As a connected subset of C with only ordinary points has continuous tangents, it follows from Lemma 2 that T_p depends continuously on $p \in C$.

For distinct $p = f(x)$ and $q = f(y)$, we say that *p precedes q* [*q follows p*] in C if $x < y$ in I and we write $p < q$. If $f(0) \neq f(1)$ then, evidently, either $p < q$ or $q < p$ for any distinct $p, q \in C$. If $f(0) = f(1)$ then $p = f(0)$ both precedes and follows each $q \in C\backslash\{p\}$ and $f(0) < q < f(1)$. In either case, we say that C is *oriented* in the direction of increasing $x \in I$. This orientation of C induces, in turn, an orientation of every arc of C. In fact, if p and q are distinct points of C then $A(p, q)$ $[A(q, p)]$ is oriented from p to q [q to p] and, as above, $C = A(p, q) \cup A(q, p)$.

For distinct points q, r in C, let $\overrightarrow{qr}$ denote the vector in $\mathbf{R}^2$ with initial point q and terminal point r. Let $||\overrightarrow{qr}||$ denote the usual length of $\overrightarrow{qr}$ in $\mathbf{R}^2$. Now, let $p \in C$ and let (p_λ) be a sequence of points in C such that $p_\lambda < p$ for each λ and $\lim p_\lambda = p$. We put

$$\mathbf{p} = \lim_{p_\lambda \to p} \frac{\overrightarrow{p_\lambda p}}{||\overrightarrow{p_\lambda p}||}$$

and call $\mathbf{p}$ the *tangent vector* of C at p. For completeness we set

$$\mathbf{p}_0 = \lim\nolimits_{p \to p_0} \mathbf{p}$$

if C is an arc with initial point p_0.

Evidently, $\mathbf{p}$ exists for each $p \in C$ and $\mathbf{p}$ is parallel to T_p. Moreover, the tangent vectors $\mathbf{p}$ of C depend continuously on $p \in C$ provided that C contains neither cusps nor breaks. We shall for brevity call a simple differentiable curve with only ordinary points and inflection points a *simple smooth curve*.

From Lemma 2 it now readily follows that

LEMMA 3. *Let C be a simple differentiable curve, let L be a line in $\mathbf{R}^2$, and let $q \in C$. Then both*

$$\mathscr{P}(L) = \{p \in C | T_p \text{ is parallel to } L\}$$

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and

$$\mathscr{P}(q) = \{p \in C | \mathbf{p} = \mathbf{q} \text{ or } \mathbf{p} = -\mathbf{q}\}$$

are finite sets.

If p_0 is the initial point of C, then we may so enumerate the elements $p_0, p_1, \ldots, p_k$ of $\mathscr{P}(p_0)$ that $p_0 < p_1 < p_2 < \ldots < p_k$. Let p_{k+1} denote the terminal point of C ($p_{k+1} = p_0$ if C is closed). Then

$$C = \bigcup_{i=1}^{k+1} A(p_{i-1}, p_i)$$

and, for each $i = 1, 2, \ldots, k + 1$,

$$\text{int } A(p_{i-1}, p_i) \cap \mathscr{P}(p_0) = \emptyset$$

where int A denotes the interior of A.

The measure of a point and a curve. Let C be a simple differentiable curve with initial point p_0 and terminal point p_{k+1}, where

$$\mathscr{P}(p_0) = \{p_0 < p_1 < p_2 < \ldots < p_k\}.$$

Let the unit circle S^1 in $\mathbf{R}^2$ with centre $(0, 0)$ be assigned the counter-clockwise orientation. For $p \in \text{int } A(p_{i-1}, p_i)$, the vectors $\mathbf{p}$ and $\mathbf{p}_{i-1}$ positioned with initial point $(0, 0)$ meet S^1 at, say, t and t_{i-1}, respectively. Let $\angle\ (\mathbf{p}_{i-1}, \mathbf{p})$ denote the arclength of the smaller of the two arcs of S^1 determined by t_{i-1} and t. Denote the smaller arc by $A(t_{i-1}, t)$ and set

$$\bar{\mu}_{p_{i-1}}(p) = \angle\ (\mathbf{p}_{i-1}, \mathbf{p})$$

if the orientation from t_{i-1} to t in $A(t_{i-1}, t)$ is counter-clockwise; otherwise, set

$$\bar{\mu}_{p_{i-1}}(p) = -\angle\ (\mathbf{p}_{i-1}, \mathbf{p}).$$

Note that, for each $p \in \text{int } A(p_{i-1}, p_i)$ and for each $i = 1, 2, \ldots, k + 1$

$$0 < |\bar{\mu}_{p_{i-1}}(p)| < \pi.$$

Finally, let

$$\mu_{p_0}(p_0) = 0,$$

for $p \in \text{int } A(p_{i-1}, p_i)$ and for each $i = 1, 2, \ldots, k + 1$, let

$$\mu_{p_0}(p) = \mu_{p_0}(p_{i-1}) + \bar{\mu}_{p_{i-1}}(p)$$

and

$$\mu_{p_0}(p_i) = \lim_{p \to p_i, p_{i-1} < p < p_i} \mu_{p_0}(p).$$

Evidently $\bar{\mu}_{p_0}(p)$ is defined only for $p \in \text{int } A(p_0, p_1)$ while $\mu_{p_0}(p)$ is

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defined for all $p \in C$. Moreover, for any $q \in C$, q is the initial point of some oriented arc A of C and if $p \in A$ then $\mu_q(p)$ is defined.

Recall that $p_0[p_{k+1}]$ is the initial [terminal] point of C. We call $|\mu_{p_0}(p_{k+1})|$ the *measure* of C and denote it by $\mu(C)$.

PROPOSITION 4. ([**4**], [**1**]). *Let C be a simple smooth closed curve. Then $\mu(C) = 2\pi$ for any choice of initial point for C.*

We conclude this section with several elementary observations intended as a rationale for arguments to follow.

Let C be a simple differentiable curve with initial point p_0, terminal point p_{k+1}, and $\mathscr{P}(p_0) = \{p_0 < p_1 < p_2 < \ldots < p_k\}$. Let $0 \leqq i \leqq k + 1$ and let $p \in \text{int}\, A(p_{i-1}, p_i)$.

(a) If p is an ordinary point or an inflection point then there is a neighbourhood $N(p)$ of p in int $A(p_{i-1}, p_i)$ such that either $\mu_{p_0}(q) > 0$ for all $q \in N(p)$ or $\mu_{p_0}(q) < 0$ for all $q \in N(p)$.

(b) If p is either a cusp point or a beak point and $|\mu_0(p)| < \pi/2$, then in any neighbourhood $N(p)$ of p in C there exist points q and r such that

$$\mu_{p_0}(q) \cdot \mu_{p_0}(r) < 0.$$

(c) If $A(p_{i-1}, p_i)$ is a smooth arc then either $\mu_{p_0}(q) \geqq 0$ for all $q \in A(p_{i-1}, p_i)$ or $\mu_{p_0}(q) \leqq 0$ for all $q \in A(p_{i-1}, p_i)$.

Simple closed curves with beaks and cusps. Our main result is concerned with simple, closed, smooth curves C and *continuous*, *slope-preserving maps* σ of C to C (that is, continuous maps σ for which $T_{\sigma(p)}$ is parallel to T_p, for each $p \in C$). It is perhaps instructive at this point to indicate just how "smoothness" of a simple closed curve must enter into our consideration.

Example 1. *A simple, closed curve with beaks.* Let C be the curve consisting of the arcs A_1, A_2 described by

$$A_1 = \{(x, (1 - x^2)^{1/2}) | 0 \leqq x \leqq 1\}$$

and

$$A_2 = \{(x, (1 - x^4)^{1/4}) | 0 \leqq x \leqq 1\}.$$

This curve is illustrated in Figure 3. Evidently, C has a beak at $(1, 0)$ and at $(0, 1)$.

We define a map ρ of int A_1 to A_2 by

$$\rho((1 + m^{-2})^{-1/2}, (1 + m^2)^{-1/2}) = ((1 + m^{-4/3})^{-1/4}, (1 + m^{4/3})^{-1/4})$$

for all $m > 0$. (Note that the slope at $((1 + m^{-2})^{-1/2}, (1 + m^2)^{-1/2})$ of int A_1 is $-m$ and equals the slope at $((1 + m^{-4/3})^{-1/4}, (1 + m^{4/3})^{-1/4})$

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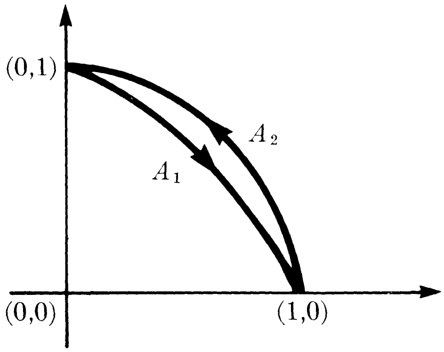

FIGURE 3

of A_2.) Evidently, ρ is continuous. We now define the map σ of C to C by

$$\sigma(p) = \begin{cases} \rho(p) & \text{if } p \in \operatorname{int} A_1 \\ p & \text{if } p \in A_2. \end{cases}$$

Then σ is continuous and slope-preserving. It is, however, neither one-to-one nor onto.

Example 2. *A simple close curve with cusps.* Let C be the curve consisting of the arcs

$$\{(x, -1 - \cos x) | -\pi \leqq x \leqq \pi\}$$

and

$$\{(x, 1 + \cos x) | -\pi \leqq x \leqq \pi\}.$$

This curve is illustrated in Figure 4. It has a cusp at $(-\pi, 0)$ and at $(\pi, 0)$.

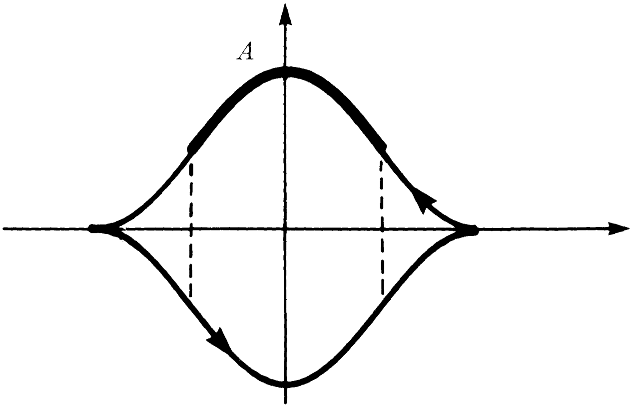

FIGURE 4

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Let A denote the arc consisting of the points of $\{(x, 1 + \cos x) | -\pi/2 \leqq x \leqq \pi/2\}$ and define a map σ of C onto A by

$$\sigma((x, -1 - \cos x)) = \begin{cases} (x - \pi, 1 + \cos(x - \pi)) & \text{if } \pi/2 \leqq x \leqq \pi \\ (-x, 1 + \cos(-x)) & \text{if } -\pi/2 \leqq x \leqq \pi/2 \\ (x + \pi, 1 + \cos(x + \pi)) & \text{if } -\pi \leqq x \leqq -\pi/2 \end{cases}$$

and

$$\sigma((x, 1 + \cos x)) = \begin{cases} (\pi - x, 1 + \cos(\pi - x)) & \text{if } \pi/2 \leqq x \leqq \pi \\ (x, 1 + \cos x) & \text{if } -\pi/2 \leqq x \leqq \pi/2 \\ (-\pi - x, 1 + \cos(-\pi - x)) & \text{if } -\pi \leqq x \leqq -\pi/2. \end{cases}$$

Again, σ is a continuous, slope-preserving map of C to C. Of course, σ is neither one-to-one nor onto C.

The curve C illustrated in Figure 3 has no tangent vector with inclination $\pi/4$, for instance, while the curve of Figure 4 has no tangent vector with inclination $\pi/2$. The curve C illustrated schematically in Figure 5 has tangent vectors with inclination $0 \leqq \theta \leqq 2\pi$, yet it is a straightforward matter to construct a continuous, slope-preserving map of C onto the arc A.

Each of the curves described above is simple differentiable. Indeed, for our purposes only simple differentiable curves need apply. A simple,

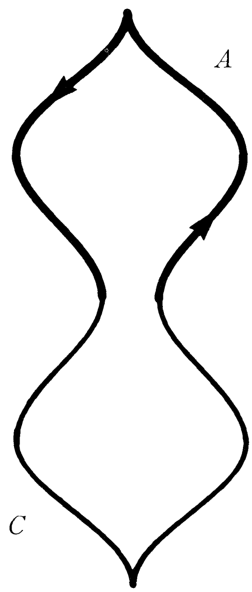

FIGURE 5

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closed curve C containing a proper line segment will evidently give rise to infinitely many continuous, slope-preserving maps of C to itself.

Slope-preserving maps of smooth curves. In this section we intend to prove the result announced in the introduction.

We call a simple differentiable curve *ordinary* if it contains only ordinary points.

LEMMA 5. *Let ρ be a continuous, slope-preserving map of a simple, smooth, closed curve C onto a simple, ordinary curve $A^* = A^*(a_0, a_1)$.*

i) *If $q_0 < q_1 < \ldots < q_k$ are all inflection points of C then $\rho|A(q_{i-1}, q_i)$ is one-to-one, for each $i = 1, 2, \ldots, k$.*

ii) *For each $i = 1, 2, \ldots, k$, there is an open neighbourhood $N(q_i)$ of q_i such that $\rho(N(q_i))$ is a one-sided neighbourhood of $\rho(q_i)$.*

(iii) *If $a_0 \neq a_1$ then every $p \in \rho^{-1}(a_0) \cup \rho^{-1}(a_1)$ is an inflection point of C.*

Proof. It is enough to observe that a point p of C is ordinary if there is an open neighbourhood $N(p)$ of p such that, for $r, s \in N(p)$, $r < p < s$, T_r is not parallel to T_s.

LEMMA 6. *Let ρ be a continuous, slope-preserving map of a simple, smooth, closed curve C onto a simple ordinary curve $A^* = A^*(a_0, a_1)$. Let $p_0 \in \rho^{-1}(a_0)$. Then*

$$\mu_{p_0}(p) = \mu_{a_0}(\rho(p))$$

for all $p \in C$, or

$$\mu_{p_0}(p) = -\mu_{a_0}(\rho(p))$$

for all $p \in C$.

Proof. Let $\mathscr{P}(p_0) = \{p_0 < p_1 < p_2 < \ldots < p_{n-1}\}$ with $p_n = p_0$. We shall show first that

$$\mu_{p_0}(p_i) = \mu_{a_0}(\rho(p_i)) \text{ for each } i = 0, 1, 2, \ldots, n.$$

To this end, let I denote the set of all inflection points of C and let q, q' be distinct, successive, members of I, that is,

$$\text{int}\, A(q, q') \cap I = \emptyset.$$

Then, for any $p \in C$, μ_p is either strictly increasing or strictly decreasing on $A(q, q')$. Now, $\mu_{p_0}(p_{i-1}) = \mu_{p_0}(p_i)$ (that is, $\mu_{p_{i-1}}(p_i) = 0$) if and only if there is an odd number of inflection points in int $A(p_{i-1}, p_i)$.

Let $\mu_{p_{i-1}}(p_i) = 0$. Then there are elements $q_1, q_2, \ldots, q_k$ of I, k odd, satisfying

$$p_{i-1} < q_1 < q_2 < \ldots < q_k < p_i \leqq q_{k+1}.$$

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By Lemma 6 (ii),

$$\rho(A(q_1, q_2)) \subseteq A^*(\rho(p_{i-1}), \rho(q_1)),$$

and, in fact,

$$\rho(A(q_l, q_{l+1})) \subseteq A^*(\rho(p_{i-1}, q_l))$$

for $1 \leqq l \leqq k$ and l odd. Therefore, $\rho(p_i) \in A^*(\rho(p_{i-1}), \rho(q_k))$. As ρ is slope-preserving,

$$A^*(\rho(p_{i-1}), \rho(q_k)) \cap \mathscr{P}(a_0) = \{\rho(p_{i-1})\}$$

and so $\rho(p_i) = \rho(p_{i-1})$ and

$$\mu_{\rho(p_{i-1})}(\rho(p_i)) = 0.$$

Similar considerations show that

$$|\mu_{\rho(p_{i-1})}(\rho(p_i))| = \pi$$

if

$$|\mu_{p_{i-1}}(p_i)| = \pi.$$

Now, A^* is ordinary, so $\mu_{a_0}(a) \geqq 0$ for all $a \in A^*$, or else $\mu_{a_0}(a) \leqq 0$ for all $a \in A^*$. But ρ is a map of C onto A^* so either $\mu_{p_0}(p) \geqq 0$ for all $p \in C$, or else $\mu_{p_0}(p) \leqq 0$ for all $p \in C$. As

$$\mu_{p_0}(p_0) = 0 = \mu_{a_0}(a_0) = \mu_{a_0}(\rho(p_0))$$

our claim is established.

This together with Lemma 5(i) completes the proof.

COROLLARY 7. *Let ρ be a continuous, slope-preserving map of a simple, smooth, closed curve onto a simple, ordinary curve A^*. Then A^* is closed.*

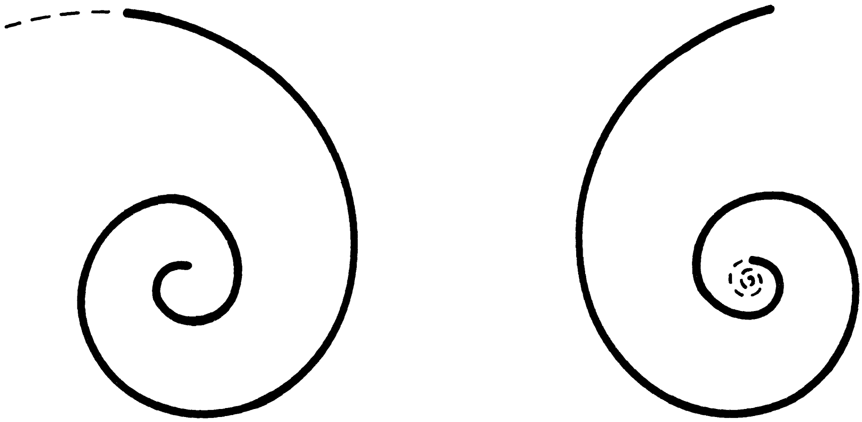
Proof. This follows at once from the fact that

$$\mu_{a_0}(\rho(p_0)) = 0,$$
$$\mu_{a_0}(\rho(p_n)) = 2\pi \quad \text{and}$$
$$a_0 = \rho(p_0) = \rho(p_n).$$

LEMMA 8. *Let A be a simple, smooth arc. Then there exists a continuous, slope-preserving map of A onto an ordinary arc.*

Proof. Let A' be a spiral (logarithmic or hyperbolic, see Figure 6) such that for any $p \in A$, there is at least one $q \in A'$ with T_q parallel to T_p. An appropriate segment of A' will provide the required ordinary arc.

THEOREM 9. *Let σ be a continuous, slope-preserving map of a simple, smooth, closed curve C to itself. Then σ is a homeomorphism and either $\sigma(\mathbf{p}) = \mathbf{p}$ for all $p \in C$, or $\sigma(\mathbf{p}) = -\mathbf{p}$ for all $p \in C$ (where $\sigma(\mathbf{p})$ denotes the tangent vector of C at $\sigma(p)$).*



Logarithmic spiral
$\log r = a\theta$, a constant.

Hyperbolic spiral
$r\theta = a$, a constant.

FIGURE 6

Proof. If $\sigma(C) \neq C$ then $\sigma(C)$ is a simple, smooth arc. There is, then, by Lemma 8, a continuous, slope-preserving map ρ of $\sigma(C)$ onto an ordinary arc A'. Then $\rho \circ \sigma$ is a continuous, slope-preserving map of C onto A', which is impossible (cf. Corollary 7). Therefore, σ is onto.

As $|\mathscr{P}(p)|$ is finite for every $p \in C$, and $T_{\sigma(p)}$ is parallel to T_p it follows that σ must also be one-to-one, whence σ is a homeomorphism.

Finally, each of the sets $\{p \in C|\sigma(\mathbf{p}) = \mathbf{p}\}$ and $\{p \in C|\sigma(\mathbf{p}) = -\mathbf{p}\}$ is closed in C. As C is connected one of these must be empty.

LEMMA 10. *Let σ be a continuous, slope-preserving map of a simple, smooth, closed curve C to itself. Then $\sigma(\sigma(p)) = p$ for each $p \in C$.*

Proof. Let us suppose that there is $p \in C$ such that

$$|\{\sigma^i(p)|i = 0, 1, \ldots, n-1\}| = n \geqq 3$$

while $\sigma^n(p) = p$.

We shall show that $\sigma^i(p) < \sigma^{i+1}(p)$ for each $i = 0, 1, 2, \ldots, n-2$ or $\sigma^i(p) > \sigma^{i+1}(p)$ for each $i = 0, 1, 2, \ldots, n-2$. To see this we need only verify that $\sigma(p) < \sigma^2(p)$ if $p < \sigma(p)$.

Let

$$A(p, \sigma(p)) \cap \mathscr{P}(p) = \{p = p_0 < p_1 < p_2 < \ldots < p_k = \sigma(p)\}.$$

If $\sigma(p_1) < p_k = \sigma(p)$ then $\sigma(p_1) = p_{k-1}$. Therefore,

$$\sigma(p_0) = p_k > \sigma(p_1) = p_{k-1} > \sigma(p_2) = p_{k-2} > \ldots > \sigma(p_k) = p_0 = p$$

so

$$\sigma^2(p) = \sigma(\sigma(p_0)) = \sigma(p_k) = p,$$

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contrary to our supposition. Thus, $\sigma(p_0) < \sigma(p_1)$. From the fact that σ is one-to-one we deduce that

$$\sigma(p) < \sigma(p_1) < \ldots < \sigma(p_k) = \sigma^2(p).$$

Let us then assume that

$$p = \sigma^0(p) < \sigma(p) < \ldots < \sigma^{n-1}(p)$$

and $\sigma^n(p) = p$. Then

$$C = \bigcup_{i=0}^{n-1} A(\sigma^i(p), \sigma^{i+1}(p)),$$

and from

$$2\pi = \mu(C) = \mu_{\sigma^0(p)}(\sigma^n(p)) = \sum_{i=0}^{n-1} \mu_{\sigma^i(p)}(\sigma^{i+1}(p)).$$

Moreover, $\sigma(\mathbf{p}) = \mathbf{p}$ for all $p \in C$ or $\sigma(\mathbf{p}) = -\mathbf{p}$ for all $p \in C$. In either case from

$$\sigma(A(\sigma^{i-1}(p), \sigma^i(p))) = A(\sigma^i(p), \sigma^{i+1}(p))$$

we conclude that

$$\mu_{\sigma^{i-1}(p)}(\sigma^i(p)) = \mu_{\sigma^i(p)}(\sigma^{i+1}(p))$$

for $i = 1, 2, \ldots, n - 1$. Therefore,

$$2\pi = n\mu_{\sigma^0(p)}(\sigma(p)) = n\mu_p(\sigma(p))$$

and

$$\mu_p(\sigma(p)) = 2\pi/n.$$

But $\sigma(\mathbf{p}) = \pm\mathbf{p}$ so $\mu_p(\sigma(p))$ is an integral multiple of π which is impossible unless $n = 1$ or $n = 2$.

Finally, we are ready to complete the proof of our main result. (Note that while Theorem 9 discloses an important feature of the collection of all continuous slope-preserving maps of simple smooth closed curves it does not yet enumerate them.)

THEOREM 11. *Let σ be a continuous, slope-preserving map of a simple, smooth, closed curve C to itself. Then either σ is the identity map of C ($\sigma(p) = p$ for each $p \in C$) or σ is the unique antipodal map of C ($\sigma(\mathbf{p}) = -\mathbf{p}$ for each $p \in C$).*

Proof. Suppose there are points p_0, p_1 of C satisfying $\sigma(p_0) \neq p_0$ yet $\sigma(p_1) = p_1$. Then $\sigma(\mathbf{p}_1) = \mathbf{p}_1$ implies that $\sigma(\mathbf{p}) = \mathbf{p}$ for all $p \in C$. Now from $\sigma(\sigma(p_0)) = p_0$ it follows that

$$\mu_{p_0}(\sigma(p_0)) = k \cdot 2\pi$$

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and, as in the proof of Lemma 10 above,

$$2\pi = \mu(C) = 2(k \cdot 2\pi).$$

As this is impossible we conclude that either σ is the identity map, or else, $\sigma(\mathbf{p}) = -\mathbf{p}$ for all $p \in C$.

Suppose that $\sigma(\mathbf{p}) = -\mathbf{p}$ for all $p \in C$. Let $p_0 \in C$ and suppose that $\sigma(p_0) = p_i$, where $\mathscr{P}(p_0) = \{p_0 < p_1 < p_2 < \ldots < p_{n-1}\}$ and $p_n = p_0$. Evidently, $|\mathscr{P}(p_0)|$ must be even, that is, n is even, and since σ is a homeomorphism $i = n/2$. It follows that σ is unique.

While implicit in the proof of Theorem 11 it is perhaps appropriate to record

COROLLARY 12. *Let σ be a continuous map of a simple, smooth, closed curve C to itself. If $\sigma(\mathbf{p}) = \mathbf{p}$ for each $p \in C$ then $\sigma(p) = p$ for each $p \in C$.*

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