INVARIANT MEASURES ON DOUBLE COSET SPACES

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1. Introduction

Let G be a locally compact group with left invariant Haar measure m. Let H be a closed subgroup of G and K a compact subgroup of G. Let R be the equivalence relation in G defined by $(a, b) \in R$ if and only if $a = kbh$ for some k in K and h in H. We call $E = G/R$ the *double coset space* of G modulo K and H. Denote by α the canonical mapping of G onto E. It can be shown that E is a locally compact space and α is continuous and open. Let N be the normalizer of K in G, i.e.

$$N = \{g \in G : gK = Kg\}.$$

There is a naturally defined mapping $\pi : N \times E \to E$ given by

$$\pi(n, \alpha(g)) = n\alpha(g) = \alpha(ng).$$

It can be verified that π is well-defined, continuous and open, and that (N, E, π) is a transformation group.

A positive Radon measure ν on E is said to be *relatively invariant* if ν is not identically zero and if

$$\int f(nx)d\nu(x) = \chi(n) \int f(x)d\nu(x)$$

for every positive continuous function f with compact support and for every $n \in N$. The function χ occurring in this definition is called the *modular function* of ν; it is necessarily a real character on N, i.e., a continuous homomorphism of N into the multiplicative group of positive real numbers. A relatively invariant measure is said to be *invariant* if its modular function is identically 1.

In this paper we shall prove that *a necessary and sufficient condition for the existence of an invariant measure on E is that there exists a non-zero positive Radon measure β on G such that*

$$\int f(ng)d\beta(g) = \int f(g)d\beta(g)$$

and

$$\int f(gh^{-1})d\beta(g) = \delta(h) \int f(g)d\beta(g)$$

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for all continuous function f with compact support and all $n \in N$, $h \in H$ (δ denotes the modular function of a left invariant Haar measure on H). For the special case where K is the identity group and hence E is the homogeneous space G/H, a condition was given by A. Weil (see [5] or Theorem 4.5). Our result is a generalization of his. We shall also give conditions for a relatively invariant measure on E to exist in various special cases. We take E. Hewitt and R. A. Ross [2] as our basic reference on Haar measures and group algebras.

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2. Preliminaries

Let m be a left invariant Haar measure on G with modular function Δ. Let μ be a left invariant Haar measure on H with modular function δ and let λ be the Haar measure on K. Since N is a closed subgroup of G, it also has a left invariant Haar measure which we denote by ω; the modular function of ω is denoted by θ. For a locally compact space Z, the symbol $\mathscr{K}(Z)$ will be used to denote the set of all positive continuous functions on Z with compact support. For $f \in \mathscr{K}(G)$ and $g_1 \in G$, $f_{g_1}(g) = f(g_1 g)$ and $f^{g_1}(g) = f(gg_1^{-1})$.

Lemma 2.1. *If ξ is a real character on K, then ξ is identically* 1.

Proof. Since ξ is a continuous homomorphism, $\xi(K)$ is a compact subgroup of the multiplicative group of positive real numbers. The assertion follows from the fact that the latter group has no non-trivial compact subgroup.

Lemma 2.2. $\int f(k)d\lambda(k) = \int f(nkn^{-1})d\lambda(k)$ *for all $f \in \mathscr{K}(K)$ and $n \in N$.*

Proof. Consider the positive Radon measure λ_n on K defined by

$$\lambda_n(f) = \int f(nkn^{-1})d\lambda(k).$$

For every $t \in K$ we have

$$\lambda_n(f_t) = \int f_t(nkn^{-1})d\lambda(k) = \int f(tnkn^{-1})d\lambda(k);$$

replacing k by $n^{-1}t^{-1}nk$ in the above integral we obtain

$$\lambda_n(f_t) = \int f(nkn^{-1})d\lambda(k) = \lambda_n(f).$$

Thus λ_n is a left invariant Haar measure on K. Since $\lambda_n(1) = \int d\lambda(k) = \lambda(1)$, we conclude that $\lambda_n = \lambda$.

LEMMA 2.3. *There is a well-defined mapping from* $\mathscr{K}(G)$ *into* $\mathscr{K}(E)$ *given by*

$$f \to \bar{f} \text{ where } \bar{f}(\alpha(g)) = \iint f(kgh)d\lambda(k)d\mu(h).$$

This mapping has the following properties:

(1) $\overline{f_1+f_2} = \bar{f}_1+\bar{f}_2$;

(2) $r \geqq 0 \Rightarrow \overline{rf} = r\bar{f}$;

(3) *it is onto*;

(4) *it commutes with the operation by* N, *i.e.*, $\bar{f}_n = (\bar{f})_n$ *for all* $n \in N$.

PROOF. It is easy to see that

$$\alpha(g) = \alpha(g') \Rightarrow \iint f(kgh)dkdh = \iint f(kg'h)dkdh;$$

hence the mapping $f \to \bar{f}$ is well-defined. The assertions (1) and (2) are obvious. Let us now prove (3). Let $F \in \mathscr{K}(E)$. There exists a compact subset D of G such that $\alpha(D) = \text{Supp } F$. Let $f \in \mathscr{K}(G)$ be such that $f(d) > 0$ for all $d \in D$. Define a function f_1 on G by

$$f_1(g) = \begin{cases} \dfrac{f(g)F(\alpha(g))}{\bar{f}(\alpha(g))} & \text{if } \bar{f}(\alpha(g)) \neq 0, \\ 0 & \text{if } \bar{f}(\alpha(g)) = 0. \end{cases}$$

Since $\bar{f}(\alpha(g)) > 0$ for $g \in \alpha^{-1}$ (Supp F) and $F(\alpha(g)) = 0$ for $g \in G-\alpha^{-1}$ Supp F), which is an open set in G, we see that $f_1 \in \mathscr{K}(G)$. Clearly $\bar{f}_1 = F$. Thus (3) is proved.

Finally the assertion (4) is obtained by direct computations:

$$\begin{aligned} \bar{f}_n(\alpha(g)) &= \iint f(nkgh)dkdh = \iint f(nkn^{-1}ngh)dkdh = \iint f(kngh)dkdh \\ &= \bar{f}(\alpha(ng)) = \bar{f}(n\alpha(g)) = (\bar{f})_n(\alpha(g)). \end{aligned}$$

THEOREM 2.1. *If* ν *is a positive Radon measure on* E, *then the positive Radon measure* $\bar{\nu}$ *on* G *defined by*

$$(*) \qquad \int f(g)d\bar{\nu}(g) = \int \bar{f}(x)d\nu(x)$$

has the following properties:

(i′) $\int f_k(g)d\bar{\nu}(g) = \int f(g)d\bar{\nu}(g)$ *for all* $f \in \mathscr{K}(G)$, $k \in K$;

(ii) $\int f^h(g)d\bar{\nu}(g) = \delta(h)\int f(g)d\bar{\nu}(g)$ *for all* $f \in \mathscr{K}(G)$, $h \in H$.

Conversely, if a positive Radon measure $\bar{\nu}$ *on* G *has the properties* (i′) *and* (ii), *then the equation* (*) *defines a positive Radon measure* ν *on* E.

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PROOF. Suppose ν is a positive Radon measure on E, then the $\bar{\nu}$ defined by (*) is clearly a positive Radon measure on G. Since $\bar{f}_k = \bar{f}$, $\bar{\nu}$ satisfies (i′). Since $\bar{f}^h = \delta(h)\bar{f}$, $\bar{\nu}$ also satisfies (ii).

Suppose now that $\bar{\nu}$ is a positive Radon measure on G satisfying (i′) and (ii). To show (*) defines a positive Radon measure on E all we have to do is to show the following implication:

$$\bar{f} = 0 \Rightarrow \int f(g)d\bar{\nu}(g) = 0.$$

Suppose $\bar{f} = 0$. Let $f' \in \mathscr{K}(G)$ be such that $\overline{f'} = 1$ on α (Supp f). We have, from

$$\iint f(kgh)dkdh = 0,$$

that

$$\begin{aligned}
0 &= \iiint f'(g)f(kgh)dkdhd\bar{\nu}(g) \\
&= \iiint f'(gh^{-1})f(kg)\delta(h^{-1})dkdhd\bar{\nu}(g) \quad (\textit{replace } g \textit{ by } gh^{-1} \textit{ and use } \text{(ii)}) \\
&= \iiint f'(gh)f(kg)dkdhd\bar{\nu}(g) \\
&= \iiint f'(k^{-1}gh)f(g)dkdhd\bar{\nu}(g) \qquad (\textit{replace } g \textit{ by } k^{-1}g \textit{ and use } \text{(i')}) \\
&= \iiint f'(kgh)f(g)dkdhd\bar{\nu}(g) \\
&= \int \overline{f'}(\alpha(g))f(g)d\bar{\nu}(g) \\
&= \int f(g)d\bar{\nu}(g).
\end{aligned}$$

This completes the proof.

3. Various conditions

We observe first that the equation (*) in Section 2 establishes a one to one correspondence between Radon measures on E and a subset of Radon measures on G. The measures on G corresponding to relatively invariant measures on E are given by the following theorem.

THEOREM 3.1. *ν is relatively invariant with modular function χ if and only if $\bar{\nu}$ has the following properties*:

(i) $\int f_n(g)d\bar{\nu}(g) = \chi(n)\int f(g)d\bar{\nu}(g)$ *for all* $f \in \mathscr{K}(G)$, $n \in N$;

(ii) $\int f^h(g)d\bar{\nu}(g) = \delta(h)\int f(g)d\bar{\nu}(g)$ *for all* $f \in \mathscr{K}(G)$, $h \in H$.

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Proof. If ν is relatively invariant with modular function χ, then $\bar{\nu}$ satisfies (ii). This follows directly from Theorem 2.1. Since the mapping $f \to \bar{f}$ commutes with the operation by N, we have

$$\bar{\nu}(f_n) = \nu(\overline{f_n}) = \nu((\bar{f})_n) = \chi(n)\nu(\bar{f}) = \chi(n)\bar{\nu}(f).$$

Thus $\bar{\nu}$ also satisfies (i).

Conversely, if $\bar{\nu}$ satisfies (i) and (ii), then since $\chi(k) = 1$ for all $k \in K$, $\bar{\nu}$ satisfies (i′) and (ii) of Section 2. Therefore it makes sense to talk about ν. Since

$$\nu((\bar{f})_n) = \nu(\overline{f_n}) = \bar{\nu}(f_n) = \chi(n)\bar{\nu}(f) = \chi(n)\nu(\bar{f}),$$

ν is relatively invariant with modular function χ.

Remark 3.1. The condition stated in the Introduction follows from Theorem 3.1.

Remark 3.2. The study of relatively invariant measures on E may be reduced to the study of positive Radon measures on G which satisfy (i) and (ii).

Theorem 3.2. *If $\Delta(h) = \delta(h)$ for all $h \in H$, then there exists an invariant measure ν on E with* Supp $\nu = E$.

Proof. If $\Delta(h) = \delta(h)$ for all $h \in H$, then the Haar measure m satisfies (ii) and (i) with $\chi = 1$. Therefore $m = \bar{\nu}$ where ν is an invariant measure on E. Since Supp $m = G$, Supp $\nu = E$.

Corollary 3.1. *If G is unimodular and if H is discrete, then there exists an invariant measure ν on E with* Supp $\nu = E$.

Corollary 3.2. *If H is compact, then there exists an invariant measure ν on E with* Supp $\nu = E$.

Theorem 3.3. *If ξ is a real character on G such that $\Delta(h) = \xi(h)\delta(h)$ for all $h \in H$, then there exists a relatively invariant measure ν on E such that* Supp $\nu = E$ *and $\xi|_N$ is the modular function of ν.*

Proof. Let $\bar{\nu} = \xi^{-1} \cdot m$, i.e.,

$$\int f(g)d\bar{\nu}(g) = \int f(g)\xi^{-1}(g)dm(g),$$

Since

$$\int f_n(g)\xi^{-1}(g)dm(g) = \int f_n(g)\xi^{-1}(n^{-1})\xi^{-1}(ng)dm(g) = \xi(n)\int f(g)\xi^{-1}(g)dm(g)$$

and

$$\int f^h(g)\xi^{-1}(g)dm(g) = \int f^h(g)\xi^{-1h}(g)\xi^{-1}(h)dm(g)$$
$$= \xi^{-1}(h)\Delta(h)\int f(g)\xi^{-1}(g)dm(g) = \delta(h)\int f(g)\xi^{-1}(g)dm(g),$$

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$\bar{\nu}$ satisfies (i) and (ii). Therefore ν is a relatively invariant measure on E with modular function $\xi|_N$. Since Supp $\nu = G$, Supp $\bar{\nu} = E$.

THEOREM 3.4. *If the modular function δ on H can be extended to a real character on G, then there exists a relatively invariant measure ν on E with* Supp $\nu = E$.

PROOF. If ξ is a real character on G such that $\xi|_H = \delta$, then Δ/ξ is a real character on G and $\Delta(h) = \Delta/\xi(h)\delta(h)$ for all $h \in H$. The conclusion then follows from Theorem 3.3.

4. Special cases

THEOREM 4.1. *Suppose N is not locally negligible. If ν is a relatively invariant measure on E with modular function χ such that* Supp $\nu \cap \alpha(N) \neq \emptyset$, *then* Supp $\nu \supset \alpha(N)$ *and* $\Delta(t) = \chi(t)\delta(t)$ *for all* $t \in N \cap H$.

PROOF. We note first that N not locally negligible is equivalent to N open in G. Thus the restriction $m|_N$ of m to N is left invariant and is not 0. Hence $\theta = \Delta|_N$ and we may assume $\omega = m|_N$. Also since N is open every $f \in \mathscr{K}(N)$ may be regarded as a function in $\mathscr{K}(G)$, so that the mapping $f \to \bar{f}$ introduced in Section 2 gives a mapping from $\mathscr{K}(N)$ into $\mathscr{K}(E)$. We remark that the image of this mapping contains $\mathscr{K}(E, \alpha(N))$, i.e., the subset of $\mathscr{K}(E)$ consisting of all functions in $\mathscr{K}(E)$ with support contained in $\alpha(N)$. In fact, if $F \in \mathscr{K}(E, \alpha(N))$, then in the proof of Lemma 2.3, the compact set D can be taken in N. Hence we may suppose the f has support contained in N. It follows that the f_1 is in $\mathscr{K}(N)$. Therefore $F = \bar{f}_1$.

Define a positive Radon measure ω' on N by

$$\omega'(f) = \nu(\bar{f}),\ f \in \mathscr{K}(N).$$

The above remark together with the fact that Supp $\nu \cap \alpha(N) \neq \emptyset$ implies that $\omega' \neq 0$. Now

$$\omega'(f_n) = \nu(\overline{f_n}) = \nu((\bar{f})_n) = \chi(n)\nu(\bar{f}) = \chi(n)\omega'(f).$$

Hence

$$\omega'(\chi f_n) = \chi(n^{-1})\omega'((\chi f)_n) = \chi(n^{-1})\chi(n)\omega'(\chi f) = \omega'(\chi f).$$

Thus $f \to \omega'(\chi f)$ is left invariant. By multiplying a positive constant if necessary, we may therefore assume that $\chi \cdot \omega' = \omega$.

For any $t \in N \cap H$, we have

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$$\begin{aligned}\theta(t)\int f(n)d\omega(n) &= \int f^t(n)d\omega(n) = \int d\nu(\alpha(g))\iint (\chi f^t)(kgh)dkdh\\ &= \int d\nu(\alpha(g))\iint \chi(t)(\chi f)^t(kgh)dkdh\\ &= \chi(t)\int d\nu(\alpha(g))\iint \delta(t)(\chi f)(kgh)dkdh\\ &= \chi(t)\delta(t)\int d\nu(\alpha(g))\iint (\chi f)(kgh)dkdh\\ &= \chi(t)\delta(t)\int f(n)d\omega(n).\end{aligned}$$

Hence $\theta(t) = \chi(t)\delta(t)$. Since $\Delta(t) = \theta(t)$ we obtain $\Delta(t) = \chi(t)\delta(t)$.

Since for any point x in $\operatorname{Supp}\nu \cap \alpha(N)$, we have $Nx = \alpha(N)$, the assertion $\operatorname{Supp}\nu \supset \alpha(N)$ follows from the fact that ν is relatively invariant.

THEOREM 4.2. *If N is not locally negligible and if $H \subset N$, then a necessary and sufficient condition for the existence of a relatively invariant measure ν on E with modular function χ is that there exist a positive Radon measure ν_1 on $\alpha(N)$ and a positive Radon measure ν_2 on $E-\alpha(N)$ such that*

1) *at least one of ν_1 and ν_2 is not identically zero;*
2) *if $\nu_i \neq 0$, then ν_i is relatively invariant with χ as its modular function;*
3) *$\nu|_{\alpha(N)} = \nu_1$ and $\nu|_{E-\alpha(N)} = \nu_2$.*

PROOF. Since $H \subset N$, $\alpha(G-N) = E-\alpha(N)$. Hence E is a disjoint union of $\alpha(N)$ and $\alpha(G-N)$ where both subsets are locally compact. Since $\pi(N\times\alpha(N)) = \alpha(N)$ and $\pi(N\times(E-\alpha(N))) = E-\alpha(N)$, we have two transformation groups $(N, \alpha(N), \pi_1)$ and $(N, E-\alpha(N), \pi_2)$ where π_1 and π_2 are restrictions of π. The verification of our theorem is then straight forward.

THEOREM 4.3. *Suppose $H \subset N$. If ν is a relatively invariant measure on E with modular funtcion χ such that $\nu|_{\alpha(N)} \neq 0$, then $\theta(h) = \chi(h)\delta(h)$ for all $h \in H$. Conversely if χ is a real character on N such that $\theta(h) = \chi(h)\delta(h)$ for all $h \in H$, then there exists a relatively invariant measure ν on E with χ as its modular function such that $\nu|_{\alpha(N)} \neq 0$.*

PROOF. Suppose ν is a relatively invariant measure on E with modular function χ such that $\nu_1 = \nu|_{\alpha(N)} \neq 0$. Since $H \subset N$, we can define a positive Radon measure ω' on N by

$$\omega'(f) = \int d\nu_1(\alpha(n))\iint f(knh)d\lambda(k)d\mu(h), \qquad f \in \mathscr{K}(N).$$

Then a process similar to the one used in the proof of Theorem 4.1 shows that $\chi\cdot\omega'$ is a left invariant Haar measure on N and that $\theta(h) = \chi(h)\delta(h)$ for all $h \in H$.

Suppose now that χ is a real character on N such that $\theta(h) = \chi(h)\delta(h)$ for all $h \in H$. Since $H \subset N$, it can be verified that (cf. Theorem 2.1)

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$$\bar{f} = 0 \Rightarrow \int f(n)\chi(n^{-1})d\omega(n) = 0, \qquad f \in \mathscr{K}(G).$$

Hence $\bar{f} \to \nu(\bar{f}) = \int f(n)\chi(n^{-1})d\omega(n)$ defines a positive Radon measure on E. It is clear from the definition of ν that ν is not identically 0 and that $\nu|_{E-\alpha(N)} = 0$. Hence $\nu|_{\alpha(N)} \neq 0$. The fact that ν is relatively invariant with χ as its modular function is obtained by

$$\begin{aligned}\nu((\bar{f})_t) = \nu(\bar{f_t}) &= \int f(tn)\chi(n^{-1})d\omega(n) = \chi(t)\int f(tn)\chi((tn)^{-1})d\omega(n)\\ &= \chi(t)\int f(n)\chi(n^{-1})d\omega(n) = \chi(t)\nu(\bar{f}).\end{aligned}$$

COROLLARY 4.1. *Suppose $H \subset N$ and let ξ be a real character on G. If $\Delta(h) = \xi(h)\delta(h)$ for all $h \in H$, then $\Delta(h) = \theta(h)$ for all $h \in H$.*

PROOF. By Theorem 3.3, there exists a relatively invariant measure ν on E such that Supp $\nu = E$ and $\xi|_N$ is the modular function of ν. By Theorem 4.3, $\theta(h) = \xi(h)\delta(h)$ for all $h \in H$. Therefore $\Delta(h) = \theta(h)$ for all $h \in H$.

THEOREM 4.4. *Suppose $H \subset N$. If there exists a relatively invariant measure ν on E with $\nu|_{\alpha(N)} \neq 0$, then δ can be extended to a real character on N. Conversely, if δ can be extended to a real character on N, then there exists a relatively invariant measure on E.*

PROOF. Suppose ν is a relatively invariant measure on E with $\nu|_{\alpha(N)} \neq 0$. Let χ be the modular function of ν. Then θ/χ is a real character on N whose restriction to H is, by Theorem 3.4, δ.

Suppose now that χ is a real character on N such that $\chi|_H = \delta$. Then $\theta(h) = \theta/\chi(h)\delta(h)$ for all $h \in H$; hence it follows from Theorem 4.3 that there exists a relatively invariant measure on E.

THEOREM 4.5. (A. Weil) *If K is an invariant subgroup of G, then the following statements are true:*

(1) *If ν is a relatively invariant measure on E with modular function χ, then $\Delta(h) = \chi(h)\delta(h)$ for all $h \in H$. If ν' is also a relatively invariant measure on E with the same modular function χ, then $\nu' = r\nu$ for some positive number r.*

(2) *If χ is a real character on G such that $\Delta(h) = \chi(h)\delta(h)$ for all $h \in H$, then there exists a relatively invariant measure on E with χ as its modular function.*

(3) *Every relatively invariant measure on E has support the whole space E.*

PROOF. Since K is invariant, $N = G$. Hence N is not locally negligible and $H \subset N$. Thus all previous results are applicable. Hence the theorem.

THEOREM 4.6. *If H is an invariant subgroup of N, then there exists an invariant measure on E.*

PROOF. Consider the homogeneous space N/H. Since H is invariant N/H is actually a group. It is evident that the left invariant Haar measure on N/H is an invariant measure on N/H. Hence by Theorem 4.5, $\theta(h) = \delta(h)$ for all $h \in H$. The conclusion then follows from Theorem 4.3.

THEOREM 4.7. *If H is an invariant subgroup of G, then there exists an invariant measure on E.*

PROOF. Using a similar argument as in the proof of Theorem 4.6, we obtain $\Delta(h) = \delta(h)$ for all $h \in H$. Theorem 3.2 can then be applied to complete the proof.

5. Some further remarks

In this last section we point out how an invariant measure on E induces an action of $L^1(N)$ on $L^p(E)$ and to investigate a related operator problem. We also discuss the case when H is compact.

Let ν be an invariant measure on E. Then a mapping similar to the ordinary group algebra convolution can be defined on $L^1(N, \omega) \times L^p(E, \nu)$ to $L^p(E, \nu)$, $(p \geqq 1)$, namely, for $f \in L^1(N)$ and $j \in L^p(E)$ the function f^*j on E is defined by

$$f^*j(x) = \int f(n)j(n^{-1}x)d\omega(n).$$

It can be checked that $(f, j) \to f^*j$ is a bilinear mapping of $L^1(N) \times L^p(E)$ into $L^p(E)$; in fact we have $\|f^*j\|_p \leqq \|f\|_1\|j\|_p$ for all $(f, j) \in L^1(N) \times L^p(E)$.

For every $j \in L^p(E)$ the mapping $T_j : f \to f^*j$ is a bounded linear transformation of $L^1(N)$ into $L^p(E)$. Since

$$f^*_{n'}j(n) = \int f(n'n)j(n^{-1}x)d\omega(n) = \int f(n)j(n^{-1}n'x)d\omega(n) = (f^*j)_{n'}(x),$$

we see that the operator T_j commutes with the operation by N. Thus the set $\{T_j : j \in L^p(E)\}$ constitutes a subset of the set of all bounded linear transformations of $L^1(N)$ into $L^p(E)$ commuting with the operation by N. The latter set is characterized by the theorem below:

THEOREM 5.1. *A bounded linear transformation T of $L^1(N)$ into $L^p(E)$ $(1 \leqq p < \infty)$ commutes with the operation by N if and only if $f^*Tb = T(f^*b)$ for all f, b in $L^1(N)$.*

PROOF. For any $y \in L^q(E)$ where $1/p + 1/q = 1$, the mapping $f \to \int (Tf)yd\nu$ is a bounded linear functional on $L^1(N)$. Hence there exists $z \in L^\infty(N)$ such that

$$\int_E (Tf)yd\nu = \int_N fzd\omega.$$

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Now

$$\int_E T(f*b)(x)y(x)d\nu(x) = \int_N f*b(n')z(n')d\omega(n')$$
$$= \int_N\int_N f(n)b_{n^{-1}}(n')z(n')d\omega(n')d\omega(n)$$
$$= \int_N\int_E f(n)Tb_{n^{-1}}(x)y(x)d\nu(x)d\omega(n),$$

and

$$\int_E (f*Tb)(x)y(x)d\nu(x) = \int_E\int_N f(n)(Tb)(n^{-1}x)y(x)d\omega(n)d\nu(x).$$

By comparing these two expressions we see that T commutes with the operation by N if and only if $T(f*b) = f*Tb$. The proof is completed.

Finally suppose H is compact. Then the canonical mapping $\alpha : G \to E$ is proper, i.e., the inverse image $\alpha^{-1}(A)$ of every compact subset A of E is a compact subset of G. We therefore have a one-to-one mapping from $\mathscr{K}(E)$ into $\mathscr{K}(G)$ given by

$$\bar{\alpha}(\bar{f}) = \bar{f} \circ \alpha.$$

The positive Radon measure m^α on E defined by

$$m^\alpha(\bar{f}) = m(\bar{\alpha}(\bar{f})) = \int \bar{f}(\alpha(g))dm(g)$$

is clearly an invariant measure on E. In fact, m^α is nothing else but the invariant measure ν obtained in Corollary 3.2.

For each $x \in E$ let m_x be the positive Radon measure on $E \times E$ defined by

$$m_x(u) = \int u(\alpha(g), \alpha(g^{-1}t))dm(g), \text{ where } t \in \alpha^{-1}(x).$$

Note that if $s \in \alpha^{-1}(x)$, then $s = kth$ for some $k \in K$, $h \in H$ and we have

$$u(\alpha(g), \alpha(g^{-1}s))dm(g) = \int u(\alpha(g), \alpha(g^{-1}kth))dm(g)$$
$$= \int u(\alpha(kg), \alpha(g^{-1}th))dm(g) = \int u(\alpha(g), \alpha(g^{-1}t))dm(g).$$

Hence m_x is well-defined.

In terms of the measures m_x a multiplication on $\mathscr{K}(E)$ can be defined by the formula below:

$$\bar{f}*\bar{p}(x) = m_x(\bar{f} \otimes \bar{p}) = \int \bar{f}(y)\bar{p}(z)dm_x(y, z).$$

It is not hard to verify that we can extend our considerations from $\mathscr{K}(E)$ to $L^1(E, m^\alpha)$. Then the above defined multiplication together with the other operations defines an algebra structure on $L^1(E, m^\alpha)$. Under the norm defined by m^α, $L^1(E, m^\alpha)$ is actually a Banach algebra. The idea of the

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above considerations is from C. Ionescu Tulcea [4]. We remark that if G is unimodular and if H is taken to be K, then an involution can be introduced in $L^1(E)$. An example of a generalized convolution algebra can be obtained in this way. For the notion of a generalized convolution algebra we refer the reader to [3] and [4]; for this particular example see [4].

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