Abstract

This dissertation is highly motivated by d.r.e. Nondensity Theorem, which is interesting in two perspectives. One is that it contrasts Sacks Density Theorem, and hence shows that the structures of r.e. degrees and d.r.e. degrees are different. The other is to investigate what other properties a maximal degree can have.

In Chapter 1, we briefly review the backgrounds of Recursion Theory which motivate the topics of this dissertation.

In Chapter 2, we introduce the notion of (m,n)-cupping degree. It is closely related to the notion of maximal d.r.e. degree. In fact, a $(2,2)$-cupping degree is maximal d.r.e. degree. We then prove that there exists an isolated $(2,\omega)$-cupping degree by combining strategies for maximality and isolation with some efforts.

Chapter 3 is part of a joint project with Steffen Lempp, Yiqun Liu, Keng Meng Ng, Cheng Peng, and Guohua Wu. In this chapter, we prove that any finite boolean algebra can be embedded into d.r.e. degrees as a final segment. We examine the proof of d.r.e. Nondensity Theorem and make developments to the technique to make it work for our theorem. The goal of the project is to see what lattice can be embedded into d.r.e. degrees as a final segment, as we observe that the technique has potential be developed further to produce other interesting results.

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Patrick Lutz. ***Results on Martin's Conjecture***, University of California, Berkeley, CA, USA, 2021. Supervised by Theodore Slaman. MSC: 03D28, 03D30, 03E60, 03D55. Keywords: computability, Turing degrees, determinacy, Martin's conjecture, Rudin-Keisler order, countable Borel equivalence relations.

Abstract

Martin's conjecture is an attempt to classify the behavior of all definable functions on the Turing degrees under strong set theoretic hypotheses. Very roughly it says that every such function is either eventually constant, eventually equal to the identity function or eventually equal to a transfinite iterate of the Turing jump. It is typically divided into two parts: the first part states that every function is either eventually constant or eventually above the identity function and the second part states that every function which is above the identity is eventually equal to a transfinite iterate of the jump. If true, it would provide an explanation for the unique role of the Turing jump in computability theory and rule out many types of constructions on the Turing degrees.

In this thesis, we will introduce a few tools which we use to prove several cases of Martin's conjecture. It turns out that both these tools and these results on Martin's conjecture have some interesting consequences both for Martin's conjecture and for a few related topics.

The main tool that we introduce is a basis theorem for perfect sets, improving a theorem due to Groszek and Slaman. We also introduce a general framework for proving certain special cases of Martin's conjecture which unifies a few pre-existing proofs. We will use these tools to prove three main results about Martin's conjecture: that it holds for regressive functions on the hyperarithmetic degrees (answering a question of Slaman and Steel), that part 1 holds for order preserving functions on the Turing degrees, and that part 1 holds for a class of functions that we introduce, called measure preserving functions.

This last result has several interesting consequences for the study of Martin's conjecture. In particular, it shows that part 1 of Martin's conjecture is equivalent to a statement about the Rudin-Keisler order on ultrafilters on the Turing degrees. This suggests several possible strategies for working on part 1 of Martin's conjecture, which we will discuss.

The basis theorem that we use to prove these results also has some applications outside of Martin's conjecture. We will use it to prove a few theorems related to Sacks' question about whether it is provable in ZFC that every locally countable partial order of size continuum

embeds into the Turing degrees. We will show that in a certain extension of ZF (which is incompatible with ZFC), this holds for all partial orders of height two, but *not* for all partial orders of height three. Our proof also yields an analogous result for Borel partial orders and Borel embeddings in ZF, which shows that the Borel version of Sacks' question has a negative answer.

We will end the thesis with a list of open questions related to Martin's conjecture, which we hope will stimulate further research.

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Justin Miller, ***Intrinsic density, asymptotic computability, and stochasticity***, University of Notre Dame, Notre Dame, IN, USA, 2021. Supervised by Peter Cholak. MSC: Primary 03D32, 03D30. Keywords: Intrinsic density, asymptotic computation, stochasticity, randomness.

Abstract

There are many computational problems which are generally "easy" to solve but have certain rare examples which are much more difficult to solve. One approach to studying these problems is to ignore the difficult edge cases. Asymptotic computability is one of the formal tools that uses this approach to study these problems. Asymptotically computable sets can be thought of as almost computable sets, however every set is computationally equivalent to an almost computable set. Intrinsic density was introduced as a way to get around this unsettling fact, and which will be our main focus.

Of particular interest for the first half of this dissertation are the intrinsically small sets, the sets of intrinsic density 0. While the bulk of the existing work concerning intrinsic density was focused on these sets, there were still many questions left unanswered. The first half of this dissertation answers some of these questions. We proved some useful closure properties for the intrinsically small sets and applied them to prove separations for the intrinsic variants of asymptotic computability. We also completely separated hyperimmunity and intrinsic smallness in the Turing degrees and resolved some open questions regarding the relativization of intrinsic density.

For the second half of this dissertation, we turned our attention to the study of intermediate intrinsic density. We developed a calculus using noncomputable coding operations to construct examples of sets with intermediate intrinsic density. For almost all $r \in (0,1)$, this construction yielded the first known example of a set with intrinsic density r which cannot compute a set random with respect to the r-Bernoulli measure. Motivated by the fact that intrinsic density coincides with the notion of injection stochasticity, we applied these techniques to study the structure of the more well-known notion of MWC-stochasticity.

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Cheng Peng, ***On Transfinite Levels of the Ershov Hierarchy***, National University of Singapore, Singapore, 2018. Supervised by Yue Yang. MSC: 03D28, 03D55. Keywords: Turing degree, Ershov hierarchy.

Abstract

In this thesis, we study Turing degrees in the context of classical recursion theory. What we are interested in is the partially ordered structures $\mathcal{D}_\alpha$ for ordinals $\alpha < \omega^2$ and $\mathcal{D}_a$ for notations $a \in \mathcal{O}$ with $|a|_o \geq \omega^2$.