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Abstract

Let ρ be a two-dimensional modulo p representation of the absolute Galois group of a totally real number field. Under the assumptions that ρ has a large image and admits a low-weight crystalline modular deformation we show that any low-weight crystalline deformation of ρ unramified outside a finite set of primes will be modular. We follow the approach of Wiles as generalized by Fujiwara. The main new ingredient is an Ihara-type lemma for the local component at ρ of the middle degree cohomology of a Hilbert modular variety. As an application we relate the algebraic p-part of the value at one of the adjoint L-function associated with a Hilbert modular newform to the cardinality of the corresponding Selmer group.

Contents

1. Introduction

1.1 Statement of the main results

Let F be a totally real number field of degree d, ring of integers $\mathfrak{o}$ and Galois closure $\widetilde{F}$. Denote by J_F the set of all embeddings of F into $\mathbb{R}$. The absolute Galois group of a field L is denoted by $\mathcal{G}_L$.

Let f be a Hilbert modular newform over F of level $\mathfrak{n}$ (an ideal of $\mathfrak{o}$), cohomological weight $k = \sum_{\tau \in J_F} k_\tau \tau$ ($k_\tau \geqslant 2$ of the same parity) and put $w_0 = \max\{k_\tau - 2 \mid \tau \in J_F\}$. For a prime p and an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ one can associate with f and ι_p a p-adic representation (cf. [Tay89, Tay97]):

$$\rho_{f,p} : \mathcal{G}_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p), \tag{1}$$

which is irreducible, totally odd, unramified outside $\mathfrak{n}p$ and characterized by the property that for each prime v not dividing $\mathfrak{n}p$ we have $\mathrm{tr}(\rho_{f,p}(\mathrm{Frob}_v)) = \iota_p(c(f, v))$, where Frob_v denotes a

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geometric Frobenius at v and $c(f, v)$ is the eigenvalue of f for the standard Hecke operator T_v. The embedding ι_p defines a partition $J_F = \coprod_v J_{F_v}$, where v runs over the primes of F dividing p and J_{F_v} denotes the set of embeddings of F_v in $\overline{\mathbb{Q}}_p$. Then $\rho_{f,p}|_{\mathcal{G}_{F_v}}$ is known to be de Rham of Hodge–Tate weights $((w_0 - k_\tau)/2 + 1, (w_0 + k_\tau)/2)_{\tau \in J_{F_v}}$, unless $w_0 = 0$, $\rho_{f,p}$ is residually reducible but not nearly ordinary, d is even and the automorphic representation associated with f is not a discrete series at any finite place (cf. [BR93] and [Kis08]). If $p > w_0 + 2$ is unramified in F and relatively prime to $\mathfrak{n}$, then $\rho_{f,p}|_{\mathcal{G}_{F_v}}$ is crystalline (cf. [Bre99]).

Such a $\rho_{f,p}$ is defined over the ring of integers $\mathcal{O}$ of a finite extension E of $\mathbb{Q}_p$. Denote by κ the residue field of $\mathcal{O}$ and let $\overline{\rho}_{f,p}$ be the semi-simplification of the reduction of $\rho_{f,p}$ modulo a uniformizer ϖ of $\mathcal{O}$. We say that a two-dimensional irreducible p-adic (respectively, modulo p) representation of $\mathcal{G}_F$ is *modular* if it can be obtained by the above construction. The following conjecture is a well-known extension to an arbitrary totally real field F of a conjecture of Fontaine and Mazur [FM97].

Conjecture. A two-dimensional, irreducible, totally odd p-adic representation of $\mathcal{G}_F$ unramified outside a finite set of primes and de Rham at all primes v dividing p with distinct Hodge–Tate weights for each $F_v \hookrightarrow \overline{\mathbb{Q}}_p$, is modular, up to a twist by an integer power of the p-adic cyclotomic character.

We provide some evidence for this conjecture by proving the following modularity lifting theorem.

Theorem A. *Let $\rho : \mathcal{G}_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation. Assume that:*

- $(\mathbf{Mod}_\rho)$ *p is unramified in F, $p - 1 > \sum_{\tau \in J_F}((w_0 + k_\tau)/2)$ and there exists a Hilbert modular newform f of level prime to p and cohomological weight k, such that $\overline{\rho}_{f,p} \cong \rho$; and*
- $(\mathbf{LI}_{\mathrm{Ind}\rho})$ *the image of $\mathcal{G}_{\widetilde{F}}$ by $\otimes \mathrm{Ind}_F^{\mathbb{Q}} \rho = \bigotimes_{\tau \in \mathcal{G}_{\mathbb{Q}}/\mathcal{G}_F} \rho(\tau^{-1} \cdot \tau)$ is irreducible of order divisible by p.*

Then all crystalline deformations of ρ of weights between zero and $p - 2$ which are unramified outside a finite set of primes are modular.

Remark 1.1. We have greatly benefited from the work [Fuj06a] of Fujiwara, although we use a different approach (cf. § 1.2 for a more detailed discussion). Furthermore, the proof of Theorem A relies on Fujiwara's results in the minimal case. Let us mention, however, that if $P_\rho = \varnothing$ (cf. Definition 4.2), then Theorem A is independent of the results of [Fuj06a] (cf. Theorem 5.1).

Remark 1.2. One can show that if F is Galois over $\mathbb{Q}$ and if f is a Hilbert modular newform on F which is not a theta series nor a twist of a base change of a Hilbert modular newform on $E \subsetneq F$, then for all but finitely many primes p, $\rho = \overline{\rho}_{f,p}$ satisfies $(\mathbf{LI}_{\mathrm{Ind}\rho})$ for all $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

Remark 1.3. The level lowering results of Jarvis [Jar99a, Jar99b], Fujiwara [Fuj06b] and Rajaei [Raj01], generalizing classical results of Ribet [Rib90] *et al.* to the case of an arbitrary totally real field F, imply that the newform f in $(\mathbf{Mod}_\rho)$ can be chosen so that $\rho_{f,p}$ is a minimally ramified deformation of ρ in the sense of Definition 4.6.

To a Hilbert modular newform as above, Blasius and Rogawski [BR93] attached, when $w_0 > 0$, a rank-three motive over F with coefficients in $\overline{\mathbb{Q}}$, pure of weight zero and autodual. For all ι_p, its p-adic realization $\mathrm{Ad}^0(\rho_{f,p})$ is given by the adjoint action of $\mathcal{G}_F$ via $\rho_{f,p}$ on the space of two-by-two trace-zero matrices. Denote by $\mathrm{L}(\mathrm{Ad}^0(\rho_{f,p}), s)$ and $\Gamma(\mathrm{Ad}^0(\rho_{f,p}), s)$ the associated L-function and Γ-factor.

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In this setting, Beilinson and Deligne conjecture that the order of vanishing of $\mathrm{L}(\mathrm{Ad}^0(\rho_{f,p}), s)$ at $s = 1$ equals $\dim \mathrm{H}^1_{\mathrm{f}}(F, \mathrm{Ad}^0(\rho_{f,p}) \otimes \mathbb{Q}_p) - \dim \mathrm{H}^0(F, \mathrm{Ad}^0(\rho_{f,p}) \otimes \mathbb{Q}_p)$, where $\mathrm{H}^1_{\mathrm{f}}$ is the Selmer group defined by Bloch and Kato (cf. [DFG04, § 2.1]). By a formula due to Shimura we know that $\mathrm{L}(\mathrm{Ad}^0(\rho_{f,p}), 1)$ is a non-zero multiple of the Petersson inner product of f, hence does not vanish. Since $\rho_{f,p}$ is irreducible, by Schur's lemma $\mathrm{H}^0(F, \mathrm{Ad}^0(\rho_{f,p}) \otimes \mathbb{Q}_p) = 0$. Therefore, in our case, the Beilinson–Deligne conjecture is equivalent to the vanishing of $\mathrm{H}^1_{\mathrm{f}}(F, \mathrm{Ad}^0(\rho_{f,p}) \otimes \mathbb{Q}_p)$.

Let $\mathrm{Tam}(\mathrm{Ad}^0(\rho_{f,p})) \subset \mathcal{O}$ be the Tamagawa ideal introduced by Fontaine and Perrin-Riou (cf. [FP94, §§ I.4.1 and II.5.3.3]).

THEOREM B. *Assume that p is unramified in F and let f be a Hilbert modular newform over F of level prime to p and cohomological weight k satisfying $p - 1 > \sum_{\tau \in J_F}((w_0 + k_\tau)/2)$. If $\rho = \overline{\rho}_{f,p}$ satisfies $(\mathbf{LI}_{\mathrm{Ind}\rho})$ then:*

(i) *the Beilinson–Deligne conjecture holds, $\mathrm{H}^1_{\mathrm{f}}(F, \mathrm{Ad}^0(\rho_{f,p}) \otimes \mathbb{Q}_p) = 0$; and*

(ii) $$\iota_p\bigg(\frac{\Gamma(\mathrm{Ad}^0(\rho_{f,p}), 1)\mathrm{L}(\mathrm{Ad}^0(\rho_{f,p}), 1)}{\Omega_f^J \Omega_f^{J_F \setminus J}}\bigg)\mathcal{O} = \mathrm{Tam}(\mathrm{Ad}^0(\rho_{f,p}))\, \mathrm{Fitt}_{\mathcal{O}}(\mathrm{H}^1_{\mathrm{f}}(F, \mathrm{Ad}^0(\rho_{f,p}) \otimes \mathbb{Q}_p/\mathbb{Z}_p));$$

where $J \subset J_F$ and Ω_f^J, $\Omega_f^{J_F \setminus J}$ are Matsushima–Shimura–Harder periods as in Definition 7.1.

An immediate corollary is that for p as in the theorem, the p-adic valuation of $\Omega_f^J \Omega_f^{J_F \setminus J}$ does not depend on J, nor change when we twist f by a Hecke character.

Theorem B is a first step towards the generalization to an arbitrary totally real field of the work [DFG04] of Diamond, Flach and Guo on the Tamagawa number conjecture for $\mathrm{Ad}^0(\rho_{f,p})$ over $\mathbb{Q}$. When F is not $\mathbb{Q}$, it is an open problem how to identify the periods Ω_f^J used in Theorem B with the motivic periods attached to f used in the formulation of the Tamagawa number conjecture.

1.2 General strategy of the proof

The method we use originates in the work of Wiles [Wil95] and Taylor–Wiles [TW95], later developed by Diamond [Dia97b] and Fujiwara [Fuj06a].

Let ρ be as in Theorem A and let Σ be the finite set of primes of F not dividing p. In § 4.2 we define the notion of a Σ-ramified deformations of ρ. By Mazur [Maz97] and Ramakrishna [Ram93], the functor assigning to a local complete Noetherian $\mathcal{O}$-algebra A with residue field κ, the set of all Σ-ramified deformations of ρ to A, is representable by an $\mathcal{O}$-algebra $\mathcal{R}_\Sigma$, called the universal deformation ring. Since ρ is absolutely irreducible and odd, $\mathcal{R}_\Sigma$ is topologically generated as $\mathcal{O}$-algebra by traces of images of elements of $\mathcal{G}_F$ (cf. [Wil95, pp. 509–510]). Moreover, by the Cebotarev density theorem, it is enough to take traces of images of Frobenius elements outside a finite set of primes.

Let S be a large finite set of primes and let $\mathcal{T}_\Sigma$ be the $\mathcal{O}$-subalgebra of $\prod_f \mathcal{O}$ generated by $(\iota_p(c(f, v)))_{v \notin S}$ where f runs over all Hilbert modular newforms of weight k such that $\rho_{f,p}$ is a Σ-ramified deformation of ρ. The $\mathcal{O}$-algebra $\mathcal{T}_\Sigma$ is local complete Noetherian and reduced. By the above discussion $\mathcal{T}_\Sigma$ does not depend on the choice of S and the natural homomorphism $\mathcal{R}_\Sigma \to \prod_f \mathcal{O}$ factors through a surjective homomorphism of local $\mathcal{O}$-algebras $\pi_\Sigma : \mathcal{R}_\Sigma \to \mathcal{T}_\Sigma$. Then Theorem A amounts to proving that π_Σ is an isomorphism.

We follow Wiles' method consisting of showing first that $\pi_\varnothing$ is an isomorphism (*the minimal case*) and then in proving, by induction on the cardinality of Σ, that π_Σ is an isomorphism

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(*raising the level*). In order to prove that $\mathcal{R}_\Sigma$ is 'not too big' we use Galois cohomology via Proposition 6.5. In order to prove that $\mathcal{T}_\Sigma$ is 'not too small' we realize it *geometrically* as a local component of the Hecke algebra acting on the middle degree cohomology of some Shimura variety and then use this interpretation to study congruences.

It is on that last point that our approach differs from Fujiwara's. Whereas Fujiwara uses some quaternionic Shimura curves or Hida varieties of dimension zero, we use the d-dimensional Hilbert modular variety. The main ingredient in our approach is a result from [Dim05] guaranteeing the torsion freeness of certain local components of the middle degree cohomology of a Hilbert modular variety, which is recalled in the next section.

In the minimal case our modularity result is strictly included in Fujiwara's since we only treat the case $P_\rho = \varnothing$ (cf. Definition 4.2) and furthermore we do not consider the ordinary non-crystalline case. On the other hand, our level raising results are new, thanks to an Ihara-type lemma for the middle degree cohomology of Hilbert modular varieties (cf. Theorem 3.1). Our proof relies substantially on the q-expansion principle, which is available for Hilbert modular varieties.

Finally, let us observe that whereas modularity lifting results similar to Theorem A may be obtained in various ways (cf. [SW99, SW01a, SW01b, Tay06] or [Kis09]), the use of the cohomology of Hilbert modular varieties seems to be inevitable in order to obtain results on the adjoint L-functions and Selmer groups such as Theorem B.

2. Cohomology of Hilbert modular varieties

In this section we state and prove a slightly more general version of a theorem in [Dim05]. We take advantage of this opportunity to correct a wrong assumption in [Dim05], coming from a mistake in [DT04]. We thank the referee for pointing out this error to us.

2.1 Hilbert modular varieties

Denote by $\widehat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$ and by $\mathbb{A} = (F \otimes \widehat{\mathbb{Z}}) \times (F \otimes_{\mathbb{Q}} \mathbb{R})$ the ring of adèles of F. For a prime v, let ϖ_v denote a uniformizer of F_v.

For an open compact subgroup U of $(\mathfrak{o} \otimes \widehat{\mathbb{Z}})^\times$ we denote by $\mathcal{C}_U$ (respectively, $\mathcal{C}_U^+$) the class group $\mathbb{A}^\times / F^\times U (F \otimes_{\mathbb{Q}} \mathbb{R})^\times$ (respectively, the narrow class group $\mathbb{A}^\times / F^\times U (F \otimes_{\mathbb{Q}} \mathbb{R})_+^\times$, where $(F \otimes_{\mathbb{Q}} \mathbb{R})_+^\times$ denotes the open cone of totally positive elements in $(F \otimes_{\mathbb{Q}} \mathbb{R})^\times$).

For an open compact subgroup K of $\mathrm{GL}_2(F \otimes \widehat{\mathbb{Z}})$ we denote by Y_K the Hilbert modular variety of level K with complex points $\mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A}) / K \cdot \mathrm{SO}_2(F \otimes_{\mathbb{Q}} \mathbb{R})(F \otimes_{\mathbb{Q}} \mathbb{R})^\times$. By the strong approximation theorem for GL_2, the group of connected components of Y_K is isomorphic to $\mathcal{C}^+_{\det(K)}$.

We consider the Hilbert modular varieties as analytic varieties, except in the proofs of Theorem 3.1 and Propositions 3.3 and § 5.5 where we use integral models.

For an ideal $\mathfrak{n}$ of $\mathfrak{o}$, we consider the following open compact subgroups of $\mathrm{GL}_2(F \otimes \widehat{\mathbb{Z}})$:

$$K_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathfrak{o} \otimes \widehat{\mathbb{Z}}) \,\middle|\, c \in \mathfrak{n} \right\}, \quad K_1(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathfrak{n}) \,\middle|\, a - 1 \in \mathfrak{n} \right\},$$

$$K_{11}(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_1(\mathfrak{n}) \,\middle|\, d - 1 \in \mathfrak{n} \right\}, \quad \text{and} \quad K(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{11}(\mathfrak{n}) \,\middle|\, b \in \mathfrak{n} \right\}.$$

For $? = 0, 1, 11, \varnothing$ let $Y_?(\mathfrak{n})$ be the Hilbert modular variety of level $K_?(\mathfrak{n})$.

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Consider the following assumption:

(**NT**) $\mathfrak{n}$ does not divide two, nor three, nor $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{d})$.

In [DT04, Lemma 1.4] it is shown that under the assumption (**NT**), for all $x \in \mathrm{GL}_2(F \otimes \widehat{\mathbb{Z}})$, the group $\mathrm{GL}_2(F) \cap xK_1(\mathfrak{n})x^{-1}(F \otimes_{\mathbb{Q}} \mathbb{R})^\times \mathrm{SL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ is torsion free. This is not sufficient to claim that $Y_1(\mathfrak{n})$ is smooth. Here is a corrected statement.

Lemma 2.1. (i) *The variety Y_K is smooth if, and only if, for all $x \in \mathrm{GL}_2(F \otimes \widehat{\mathbb{Z}})$, the quotient of the group $\mathrm{GL}_2(F) \cap xKx^{-1}(F \otimes_{\mathbb{Q}} \mathbb{R})^\times \mathrm{SL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ by its center is torsion free.*

(ii) *If $\mathfrak{n}$ satisfies (**NT**), then $Y_{11}(\mathfrak{n})$ is smooth.*

(iii) *Let $\mathfrak{u}$ be a prime ideal of F above a prime number q such that:*

- *q splits completely in $F(\sqrt{\epsilon} \mid \epsilon \in \mathfrak{o}^\times$, for all $\tau \in J_F,\ \tau(\epsilon) > 0)$; and*
- *$q \equiv -1 \pmod{4\ell}$ for all prime numbers ℓ such that $[F(\zeta_\ell) : F] = 2$.*

Then $Y_0(\mathfrak{u})$ is smooth.

(iv) *If $K' \lhd K$ and Y_K is smooth, then $Y_{K'}$ is smooth and the natural morphism $Y_{K'} \to Y_K$ is étale with group $K/K'(K \cap \overline{F^\times})$.*

Proof. Claims (i) and (iv) are well known, claim (ii) follows easily from [DT04, Lemma 1.4]. We omit the proof of claim (iii) since it is very similar to the proof of Lemma 2.2(i) given below. □

From now on, we only consider compact open subgroups K factoring as a product $\prod_v K_v$ over the primes v of F, such that K_v is maximal for all primes v dividing p and Y_K is smooth. We denote by Σ_K the set of primes v where K_v is not maximal.

For an $\mathcal{O}$-algebra A, we denote by $\mathbb{V}_A$ the sheaf of locally constant sections of

$$\mathrm{GL}_2(F)\backslash(\mathrm{GL}_2(\mathbb{A}) \times V_A)/K \cdot \mathrm{SO}_2(F \otimes_{\mathbb{Q}} \mathbb{R})(F \otimes_{\mathbb{Q}} \mathbb{R})^\times \longrightarrow Y_K, \tag{2}$$

where V_A denotes the algebraic irreducible representation $\bigotimes_{\tau \in J_F}(\det^{(w_0-k_\tau)/2+1} \otimes \mathrm{Sym}^{k_\tau-2} A^2)$ of $\mathrm{GL}_2(A)^{J_F} \cong \mathrm{GL}_2(\mathfrak{o} \otimes A)$ and K acts on the right on V_A via its p-component $\prod_{v|p} K_v$. Note that for $K' \subset K$, there is a natural projection $\mathrm{pr} : Y_{K'} \to Y_K$ and $\mathrm{pr}^* \mathbb{V}_A = \mathbb{V}_A$. For $g \in \mathrm{GL}_2(F \otimes \widehat{\mathbb{Z}}) \cap M_2(\mathfrak{o} \otimes \widehat{\mathbb{Z}})$ we define the Hecke correspondence $[KgK]$ on Y_K by the usual diagram.

$$\begin{array}{ccccc} & Y_{K \cap gKg^{-1}} & \xrightarrow{\cdot g} & Y_{g^{-1}Kg \cap K} & \\ {}^{\mathrm{pr}_1}\swarrow & & & & \searrow^{\mathrm{pr}_2} \\ Y_K & & & & Y_K \end{array} \tag{3}$$

The Hecke correspondences act naturally on the left on the Betti cohomology groups $\mathrm{H}^\bullet(Y_K, \mathbb{V}_A)$ and on those with compact support $\mathrm{H}^\bullet_c(Y_K, \mathbb{V}_A)$ (cf. [Hid88, § 7]). If $K_v \cong \mathrm{GL}_2(\mathfrak{o}_v)$, we define the standard Hecke operators $T_v = [K_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) K_v] = [K_v \left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix}\right) K_v]$ and $S_v = [K_v \left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) K_v] = [\left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) K_v]$.

2.2 Adjoint Hilbert modular varieties

For an open compact subgroup K of $\mathrm{GL}_2(F \otimes \widehat{\mathbb{Z}})$ we define the adjoint Hilbert modular variety of level K:

$$Y_K^{\mathrm{ad}} = \mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})/\mathbb{A}^\times K \cdot \mathrm{SO}_2(F \otimes_{\mathbb{Q}} \mathbb{R}). \tag{4}$$

Again, we have Betti cohomology groups $\mathrm{H}^\bullet(Y_K^{\mathrm{ad}}, \mathbb{V}_A)$ and Hecke action on them. In particular, if $K_v = \mathrm{GL}_2(\mathfrak{o}_v)$, there is a Hecke operator T_v (the operator S_v acts by $\mathrm{N}_{F/\mathbb{Q}}(v)^{w_0}$).

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We call Y_K^{ad} adjoint since it can be rewritten in terms of the adjoint group PGL_2 as follows:

$$Y_K^{\mathrm{ad}} = \mathrm{PGL}_2(F)\backslash \mathrm{PGL}_2(\mathbb{A})/\overline{K}\cdot \mathrm{PSO}_2(F\otimes_{\mathbb{Q}}\mathbb{R}), \tag{5}$$

where $\overline{K}$ is the image of K in $\mathrm{PGL}_2(F\otimes\widehat{\mathbb{Z}})$.

The group of connected components of Y_K is isomorphic to the quotient of $\mathcal{C}^+_{\det(K)}$ by the image of $\mathbb{A}^{\times 2}$, hence it is a 2-group. If $\det(K)=(\mathfrak{o}\otimes\widehat{\mathbb{Z}})^\times$, then the group of connected components of Y_K is isomorphic to the *narrow class group* $\mathcal{C}_F^+$ of F, while the group of connected components of Y_K^{ad} is isomorphic to the *genus group* $\mathcal{C}_F^+/\mathcal{C}_F^2\cong\mathcal{C}_F^+/(\mathcal{C}_F^+)^2$. Each connected component of Y_K^{ad} can be defined more classically using the Hurwitz–Maass extension of the Hilbert modular group.

Lemma 2.2. (i) *Let $\mathfrak{u}$ be a prime ideal of F above a prime number q, such that:*

- *q splits completely in the ray class field of F modulo 4; and*
- *$q\equiv -1 \pmod{4\ell}$ for all prime numbers ℓ such that $[F(\zeta_\ell):F)]=2$.*

Then $Y_0^{\mathrm{ad}}(\mathfrak{u})$ is smooth.

(ii) *If $K'\lhd K$ and Y_K^{ad} is smooth, then $Y_{K'}^{\mathrm{ad}}$ is smooth and the natural morphism $Y_{K'}^{\mathrm{ad}}\to Y_K^{\mathrm{ad}}$ is étale with group $K/K'(K\cap\mathbb{A}^\times)$.*

Proof. We show by contradiction that for all $x\in\mathrm{GL}_2(F\otimes\widehat{\mathbb{Z}})$, the quotient of the group $\mathrm{GL}_2(F)\cap xK_0(\mathfrak{u})x^{-1}\mathbb{A}^\times\,\mathrm{SL}_2(F\otimes_{\mathbb{Q}}\mathbb{R})$ by its center is torsion free. Suppose that we are given an element γ in that group which is torsion of prime order ℓ in the quotient. Consider the (quadratic) extension $F[\gamma]=F[X]/(X^2-\mathrm{tr}\,\gamma X+\det\gamma)$ of F. Since $\gamma_{\mathfrak{u}}\in K_0(\mathfrak{u})F_{\mathfrak{u}}^\times$, it follows that $\mathfrak{u}$ splits in $F[\gamma]/F$.

If ℓ is odd, then necessarily $F[\gamma]=F(\zeta_\ell)$. Our second assumption on q implies then that $\mathfrak{u}$ is inert in $F[\gamma]$, which is a contradiction.

If $\ell=2$, then $\mathrm{tr}\,\gamma=0$ and $\det\gamma\in F^\times\cap(\widehat{\mathbb{Z}}\otimes\mathfrak{o})^\times\mathbb{A}^{\times 2}$. By class field theory, the extension $F(\sqrt{\det\gamma})$ corresponds to a quotient of the class group $\mathcal{C}_{(1+4\widehat{\mathbb{Z}}\otimes\mathfrak{o})^\times}$, hence by our first assumption on q, $\mathfrak{u}$ splits in $F(\sqrt{\det\gamma})$. On the other hand, by the second assumption $\mathfrak{u}$ is inert in $F(\sqrt{-1})$, hence $\mathfrak{u}$ is inert in $F(\sqrt{-\det\gamma})=F[\gamma]$, which is a contradiction.

This proves part (i). The proof of part (ii) is left to the reader. □

2.3 Twisted Hilbert modular varieties and Hecke operators

Let U be an open compact subgroup of $(\mathfrak{o}\otimes\widehat{\mathbb{Z}})^\times$ and let K be an open compact subgroup of $\mathrm{GL}_2(F\otimes\widehat{\mathbb{Z}})$ such that $K_{11}(\mathfrak{n})\subset K\subset K_0(\mathfrak{n})$, for some ideal $\mathfrak{n}\subset\mathfrak{o}$. Assuming that U and K decompose as a product over all primes v, so does the group

$$K'=\{x\in K\mid \det(x)\in U\}. \tag{6}$$

We define the twisted Hecke operators $T'_v=[K'_v\left(\begin{smallmatrix}1&0\\0&\varpi_v\end{smallmatrix}\right)K'_v]$ and $S'_v=[K'_v\left(\begin{smallmatrix}\varpi_v&0\\0&\varpi_v\end{smallmatrix}\right)K'_v]$, for $v\nmid\mathfrak{n}$, and $U'_v=[K'_v\left(\begin{smallmatrix}1&0\\0&\varpi_v\end{smallmatrix}\right)K'_v]$, for $v\mid\mathfrak{n}$.

Note that if $v\notin\Sigma_{K'}$, then T'_v, S'_v and U'_v coincide with the standard Hecke operators. In general, they depend on the choice of ϖ_v in the following way: if we replace ϖ_v by ϖ'_v, then T'_v and U'_v are multiplied by the invertible Hecke operator $U_\delta:=[K'_v\left(\begin{smallmatrix}1&0\\0&\delta\end{smallmatrix}\right)K'_v]=\left(\begin{smallmatrix}1&0\\0&\delta\end{smallmatrix}\right)K'_v$, with $\delta=\varpi'_v/\varpi_v\in\mathfrak{o}_v^\times$, whereas S'_v is multiplied by its square.

For a Hecke character ψ of $\mathcal{C}_{K'\cap\mathbb{A}^\times}$, we denote by $[\psi]$ the ψ-isotypic part for the action of the Hecke operators $S_v\mathrm{N}_{F/\mathbb{Q}}(v)^{-w_0}$, $v\notin\Sigma_{K'}$.

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For a character ν of $(\mathfrak{o}\otimes\widehat{\mathbb{Z}})^\times$, trivial on U, we denote by $[\nu]$ the ν-isotypic part for the action of the Hecke operators U_δ for $\delta\in\mathfrak{o}_v^\times$.

2.4 Freeness results

Consider the maximal ideal $\mathfrak{m}_\rho=(\varpi, T_v-\mathrm{tr}(\rho(\mathrm{Frob}_v)), S_v-\det(\rho(\mathrm{Frob}_v))\mathrm{N}_{F/\mathbb{Q}}(v)^{-1})$ of the abstract Hecke algebra $\mathbb{T}^S=\mathcal{O}[T_v,S_v\mid v\notin S]$, where S is a finite set of primes containing $\Sigma_K\cup\{v\mid p\}$.

Theorem 2.3. *Let $K=\prod_v K_v\subset\mathrm{GL}_2(F\otimes\widehat{\mathbb{Z}})$ be an open compact subgroup, maximal at primes v dividing p and such that Y_K is smooth. Under the assumptions $(\mathbf{Mod}_\rho)$ and $(\mathbf{LI}_{\mathrm{Ind}\,\rho})$:*

(i) $\mathrm{H}_c^\bullet(Y_K,\mathbb{V}_\mathcal{O})_{\mathfrak{m}_\rho}=\mathrm{H}^\bullet(Y_K,\mathbb{V}_\mathcal{O})_{\mathfrak{m}_\rho}=\mathrm{H}^d(Y_K,\mathbb{V}_\mathcal{O})_{\mathfrak{m}_\rho}$ *is a free $\mathcal{O}$-module of finite rank;*

(ii) $\mathrm{H}^\bullet(Y_K,\mathbb{V}_{E/\mathcal{O}})_{\mathfrak{m}_\rho}=\mathrm{H}^d(Y_K,\mathbb{V}_{E/\mathcal{O}})_{\mathfrak{m}_\rho}$ *is a divisible $\mathcal{O}$-module of finite corank and the Pontryagin pairing $\mathrm{H}^d(Y_K,\mathbb{V}_\mathcal{O})_{\mathfrak{m}_\rho}\times\mathrm{H}^d(Y_K,\mathbb{V}_{E/\mathcal{O}})_{\mathfrak{m}_\rho}\to E/\mathcal{O}$ is a perfect duality.*

Moreover, if Y_K^{ad} is smooth, then parts (i) and (ii) remain valid when we replace Y_K by Y_K^{ad}.

Proof. For $K=K_1(\mathfrak{n})$ the theorem is proved in [Dim05, Theorems 4.4 and 6.6], except for the following issues.

- The assumption $(\mathbf{LI}_{\mathrm{Ind}\,\rho})$ in [Dim05, § 3.5] is formulated as follows: the restriction of ρ to $\mathcal{G}_{\widetilde{F}}$ is irreducible of order divisible by p, and is not a twist by a character of any of its other $d-1$ internal conjugates. This is clearly implied by $(\mathbf{LI}_{\mathrm{Ind}\,\rho})$. Conversely, if the assumption from [Dim05, § 3.5] holds, then by [Dim05, Lemma 6.5] every irreducible $\mathcal{G}_{\widetilde{F}}$-representation annihilated by the characteristic polynomial of $(\otimes\mathrm{Ind}_F^\mathbb{Q}\rho)_{|\mathcal{G}_{\widetilde{F}}}$ is isomorphic to $(\otimes\mathrm{Ind}_F^\mathbb{Q}\rho)_{|\mathcal{G}_{\widetilde{F}}}$, so in particular $(\otimes\mathrm{Ind}_F^\mathbb{Q}\rho)_{|\mathcal{G}_{\widetilde{F}}}$ is irreducible. Therefore, these assumptions are equivalent.
- Theorem 4.4 is proved under the assumption $(\mathbf{MW})$. However, this assumption is only used through [Dim05, Lemma 4.2] and under the assumption $(\mathbf{LI}_{\mathrm{Ind}\,\rho})$ we can apply the stronger [Dim05, Lemma 6.5], hence the results of [Dim05, Theorems 4.4] remain valid.
- The part of $(\mathbf{Mod}_\rho)$ assuming that ρ is modular is only used through the knowledge of its weights for the tame inertia. Actually, the proof only uses the fact that the highest weight $\sum_{\tau\in J_F}((w_0+k_\tau)/2)$ occurs with multiplicity one in the tame inertia action of $\otimes\mathrm{Ind}_F^\mathbb{Q}\rho$. This fact is a consequence from [Dim05, Corollary 2.7(ii)] and the theory of Fontaine–Laffaille, if we assume that $p-1$ is bigger than $\sum_{\tau\in J_F}((w_0+k_\tau)/2)$. In contrast to the claim made in [Dim05], assuming that $p-1$ is bigger than $\sum_{\tau\in J_F}(k_\tau-1)$, which is the difference between the highest and the lowest weights, is *not* sufficient for both the above argument and for Faltings' comparison theorem.

Let us now explain how these results extend to more general level structures. Observe first that a conjugate of K has a normal subgroup of the form $K(\mathfrak{n})$ for some ideal $\mathfrak{n}\subset\mathfrak{o}$. Hence, a conjugate of K contains $K_{11}(\mathfrak{n})\cap K_0(\mathfrak{n}^2)$ as a normal subgroup. Therefore, Y_K admits a finite étale cover isomorphic to $Y_{K_{11}(\mathfrak{n})\cap K_0(\mathfrak{n}^2)}$, and the latter has a finite abelian cover $Y_{11}^1(\mathfrak{n}^2):=\coprod_\mathfrak{c} M_1^1(\mathfrak{c},\mathfrak{n}^2)$, where $\mathfrak{c}$ runs over a set of representatives of $\mathcal{C}^+_{(1+\widehat{\mathbb{Z}}\otimes\mathfrak{n}^2)^\times}$ and $M_1^1(\mathfrak{c},\mathfrak{n}^2)$ are the fine moduli spaces defined in [Dim05, § 1.4]. The following morphisms of Hilbert modular varieties are étale:

$$Y_{11}^1(\mathfrak{n}^2)\longrightarrow Y_{11}(\mathfrak{n}^2)\longrightarrow Y_{K_{11}(\mathfrak{n})\cap K_0(\mathfrak{n}^2)}\longrightarrow Y_K\longrightarrow Y_K^{\mathrm{ad}}. \tag{7}$$

Recall that each $M_1^1(\mathfrak{c},\mathfrak{n}^2)$ is a fine moduli space admitting an arithmetic model endowed with a universal Hilbert–Blumenthal abelian variety. In [Dim05, DT04] one proves various geometric results concerning $M_1^1(\mathfrak{c},\mathfrak{n})$, such as the existence of minimal compactifications, the existence of proper smooth toroidal compactifications over $\mathbb{Z}_p$ and the extension of certain vector bundles to these compactifications, the construction of a Berstein–Gelfand–Gelfand complex for distribution algebras over $\mathcal{O}$, having as consequence the degeneracy at E_1 of the Hodge to de Rham spectral sequence. By applying those constructions to each component of $Y^1_{11}(\mathfrak{n}^2)$, it follows that the highest weight $\sum_{\tau\in J_F}((w_0+k_\tau)/2)$ of $\otimes \operatorname{Ind}_F^{\mathbb{Q}}\rho$ does not occur in $\mathrm{H}^i(Y^1_{11}(\mathfrak{n}^2)_{\overline{\mathbb{Q}}},\mathbb{V}_\kappa)$ for $i<d$. By [Dim05, Theorem 6.6] $\mathrm{H}^i(Y^1_{11}(\mathfrak{n}^2)_{\overline{\mathbb{Q}}},\mathbb{V}_\kappa)_{\mathfrak{m}_\rho}$ vanishes for $i<d$ (it is important observe that the Hodge to de Rham spectral sequence is $\mathbb{T}^S$-equivariant; we refer to [Dim05, § 2.4] for a geometric definition of the Hecke correspondences).

If $Y_{K''}\to Y_{K'}$ is an étale morphism of smooth Hilbert modular varieties with group Δ, the corresponding Hoschild–Serre spectral sequence is Hecke equivariant and yields

$$\mathrm{E}_2^{j,i}=\mathrm{H}^j(\Delta,\mathrm{H}^i(Y_{K''},\mathbb{V}_\kappa)_{\mathfrak{m}_\rho})\Rightarrow \mathrm{H}^{i+j}(Y_{K'},\mathbb{V}_\kappa)_{\mathfrak{m}_\rho}. \tag{8}$$

Starting from the vanishing of $\mathrm{H}^i(Y^1_{11}(\mathfrak{n}^2),\mathbb{V}_\kappa)_{\mathfrak{m}_\rho}$ for $i<d$, then applying (8) to the morphisms of (7) yields the vanishing of $\mathrm{H}^i(Y_K,\mathbb{V}_\kappa)_{\mathfrak{m}_\rho}$ and $\mathrm{H}^i(Y_K^{\mathrm{ad}},\mathbb{V}_\kappa)_{\mathfrak{m}_\rho})$ for $i<d$. The theorem then follows by exactly the same arguments as in [Dim05, Theorems 4.4 and 6.6]. □

Proposition 2.4. *Suppose that we are given an étale morphism of smooth Hilbert modular varieties $Y_K\to Y_{K'}$ with group Δ. Assume that Δ is an abelian p-group and that $\mathcal{O}$ is large enough to contain the values of all of its characters. Then, under the assumptions* ($\mathbf{Mod}_\rho$) *and* ($\mathbf{LI}_{\operatorname{Ind}\rho}$), $\mathrm{H}^d(Y_K,\mathbb{V}_\mathcal{O})_{\mathfrak{m}_\rho}$ *is a free $\mathcal{O}[\Delta]$-module and* $\mathrm{H}^d(Y_K,\mathbb{V}_\mathcal{O})_{\mathfrak{m}_\rho}\otimes_{\mathcal{O}[\Delta]}\mathcal{O}\cong\mathrm{H}^d(Y_{K'},\mathbb{V}_\mathcal{O})_{\mathfrak{m}_\rho}$ *as* $\mathbb{T}^S$*-modules.*

Proof. By Theorem 2.3(i) $\mathrm{H}^d(Y_K,\mathbb{V}_\mathcal{O})_{\mathfrak{m}_\rho}$ is free over $\mathcal{O}$, hence by Nakayama's lemma the desired freeness over $\mathcal{O}[\Delta]$ is equivalent to the freeness of $\mathrm{H}^d(Y_K,\mathbb{V}_\mathcal{O})_{\mathfrak{m}_\rho}\otimes_\mathcal{O}\kappa$ over $\Lambda:=\kappa[\Delta]$.

Since Λ is a local Artinial ring, freeness is equivalent to flatness. Hence, we have to show that $\operatorname{Tor}_i^\Lambda(\mathrm{H}^d(Y_K,\mathbb{V}_\kappa)_{\mathfrak{m}_\rho},\kappa)=0$ for $i>0$ and $\mathrm{H}^d(Y_K,\mathbb{V}_\kappa)_{\mathfrak{m}_\rho}\otimes_\Lambda\kappa\cong\mathrm{H}^d(Y_{K'},\mathbb{V}_\kappa)_{\mathfrak{m}_\rho}$.

We reproduce here Fujiwara's *perfect complex* argument (cf. [Fuj06a, Lemma 8.16]) following the presentation of Mokrane and Tilouine (cf. [MT02, § 10]).

Let $\mathcal{C}^\bullet$ be the Godement resolution of the sheaf $\mathbb{V}_\kappa$ on the (complex) variety Y_K. It has a natural action of Λ and there is a hypertor spectral sequence:

$$\mathrm{E}_2^{i,j}=\operatorname{Tor}_{-i}^\Lambda(\mathrm{H}^j(\mathcal{C}^\bullet),\kappa)\Rightarrow\mathrm{H}^{i+j}(\mathcal{C}^\bullet\otimes_\Lambda\kappa).$$

By definition, $\mathrm{H}^j(\mathcal{C}^\bullet)=\mathrm{H}^j(Y_K,\mathbb{V}_\kappa)$. Since $Y_K\to Y_{K'}$ is étale with group Δ, it is a standard property of Godement's resolution that $\mathrm{H}^j(\mathcal{C}^\bullet\otimes_\Lambda\kappa)=\mathrm{H}^j(Y_{K'},\mathbb{V}_\kappa)$ (cf. [Fuj06a, Lemma 8.18]). Hence, the spectral sequence becomes

$$\mathrm{E}_2^{i,j}=\operatorname{Tor}_{-i}^\Lambda(\mathrm{H}^j(Y_K,\mathbb{V}_\kappa),\kappa)\Rightarrow\mathrm{H}^{i+j}(Y_{K'},\mathbb{V}_\kappa).$$

Since the Hecke operators are defined as correspondences, the spectral sequence is $\mathbb{T}^S$-equivariant and we can localize it at $\mathfrak{m}_\rho$. By Theorem 2.3(i), we have $\mathrm{H}^d(Y_K,\mathbb{V}_\kappa)_{\mathfrak{m}_\rho}=0$, unless $j=d$. Therefore, the $\mathfrak{m}_\rho$-localization of the spectral sequence degenerates at E_2, and gives

$$\operatorname{Tor}_{-i}^\Lambda(\mathrm{H}^d(Y_K,\mathbb{V}_\kappa)_{\mathfrak{m}_\rho},\kappa)\cong\mathrm{H}^{i+d}(Y_{K'},\mathbb{V}_\kappa)_{\mathfrak{m}_\rho}.$$

Another application of Theorem 2.3(i) yields $\mathrm{H}^{i+d}(Y_{K'},\mathbb{V}_\kappa)_{\mathfrak{m}_\rho}=0$, unless $i=0$.

Hence, $\operatorname{Tor}_{-i}^\Lambda(\mathrm{H}^d(Y_K,\mathbb{V}_\kappa)_{\mathfrak{m}_\rho},\kappa)=0$, unless $i=0$ in which case

$$\mathrm{H}^d(Y_K,\mathbb{V}_\kappa)_{\mathfrak{m}_\rho}\otimes_\Lambda\kappa=\operatorname{Tor}_0^\Lambda(\mathrm{H}^d(Y_K,\mathbb{V}_\kappa)_{\mathfrak{m}_\rho},\kappa)\cong\mathrm{H}^d(Y_{K'},\mathbb{V}_\kappa)_{\mathfrak{m}_\rho}$$

as desired. □

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2.5 Poincaré duality

In this section we endow the middle degree cohomology of a Hilbert modular variety with various pairings coming from the Poincaré duality.

We define a sheaf $\mathbb{V}_{\mathcal{O}}^{\vee}$ on Y_K by replacing the $\mathrm{GL}_2(\mathcal{O})^{J_F}$-representation $V_{\mathcal{O}}$, in the definition of $\mathbb{V}_{\mathcal{O}}$ in § 2.1, by its dual

$$V_{\mathcal{O}}^{\vee} = \bigotimes_{\tau \in J_F} \det{}^{(-w_0-k_\tau)/2+1} \otimes \mathrm{Sym}^{k_\tau-2}(\mathcal{O}^2).$$

The cup product followed by the trace map induces a pairing:

$$[\ ,\ ] : \mathrm{H}_c^d(Y_K, \mathbb{V}_{\mathcal{O}}) \times \mathrm{H}^d(Y_K, \mathbb{V}_{\mathcal{O}}^{\vee}) \to \mathrm{H}_c^{2d}(Y_K, \mathcal{O}) \to \mathcal{O}, \tag{9}$$

which becomes perfect after extending scalars to E. The dual of the Hecke operator $[KxK]$ under this pairing is the Hecke operator $[Kx^{-1}K]$ (cf. [Fuj06a, § 3.4]). In particular, for $v \notin S$, the dual of T_v (respectively, S_v) is $T_v S_v^{-1}$ (respectively, S_v^{-1}). We modify the pairing (9) in a standard way, in order to make it Hecke equivariant.

First, the involution $x \mapsto x^* = (\det x)^{-1} x$ of GL_2 induces a natural isomorphism $\mathrm{H}^d(Y_K, \mathbb{V}_{\mathcal{O}}^{\vee}) \cong \mathrm{H}^d(Y_{K^*}, \mathbb{V}_{\mathcal{O}})$. Assume next that $\iota K^* \iota^{-1} = K$, where $\iota = \left(\begin{smallmatrix} 0 & -1 \\ \mathfrak{n} & 0 \end{smallmatrix}\right)$ for some ideal $\mathfrak{n}$ of $\mathfrak{o}$ prime to p. Then $\iota^* \mathbb{V}_{\mathcal{O}} \cong \mathbb{V}_{\mathcal{O}}$ and there is a natural isomorphism: $\mathrm{H}^d(Y_{K^*}, \mathbb{V}_{\mathcal{O}}) \cong \mathrm{H}^d(Y_{\iota K^* \iota^{-1}}, \mathbb{V}_{\mathcal{O}}) = \mathrm{H}^d(Y_K, \mathbb{V}_{\mathcal{O}})$. Since for all x diagonal $\iota x^* \iota^{-1} = \det(x^*)(x^*)^{-1} = x^{-1}$ we have the following commutative diagram.

$$\begin{array}{ccccccc}
\mathrm{H}_!^d(Y_K, \mathbb{V}_{\mathcal{O}}^{\vee}) & \xrightarrow{\ *\ } & \mathrm{H}^d(Y_{K^*}, \mathbb{V}_{\mathcal{O}}) & \xrightarrow{[\iota K^*]} & \mathrm{H}^d(Y_{\iota K^* \iota^{-1}}, \mathbb{V}_{\mathcal{O}}) & = & \mathrm{H}^d(Y_K, \mathbb{V}_{\mathcal{O}}) \\
\downarrow [Kx^{-1}K] & & \downarrow [K^*(x^{-1})^* K^*] & & & & \downarrow [KxK] \\
\mathrm{H}^d(Y_K, \mathbb{V}_{\mathcal{O}}^{\vee}) & \xrightarrow{\ *\ } & \mathrm{H}^d(Y_{K^*}, \mathbb{V}_{\mathcal{O}}) & \xrightarrow{[\iota K^*]} & \mathrm{H}^d(Y_{\iota K^* \iota^{-1}}, \mathbb{V}_{\mathcal{O}}) & = & \mathrm{H}^d(Y_K, \mathbb{V}_{\mathcal{O}})
\end{array} \tag{10}$$

By composing the pairing (9) with the first line in the diagram we obtain a new pairing:

$$\langle\ ,\ \rangle = [\ ,\ \iota(^*)] : \mathrm{H}_c^d(Y_K, \mathbb{V}_{\mathcal{O}}) \times \mathrm{H}^d(Y_K, \mathbb{V}_{\mathcal{O}}) \to \mathcal{O}, \tag{11}$$

that we call the *modified* Poincaré pairing. It has the advantage of being equivariant for all of the Hecke operators $[KxK]$ with x diagonal (this is not a restrictive assumption as long as we are concerned with commutative Hecke algebras). In particular, the pairing (11) is $\mathbb{T}^S$-linear, and under the assumptions of Theorem 2.3(i) its $\mathfrak{m}_\rho$-localization yields a perfect duality of free $\mathcal{O}$-modules:

$$\langle\ ,\ \rangle : \mathrm{H}^d(Y_K, \mathbb{V}_{\mathcal{O}})_{\mathfrak{m}_\rho} \times \mathrm{H}^d(Y_K, \mathbb{V}_{\mathcal{O}})_{\mathfrak{m}_\rho} \to \mathcal{O}. \tag{12}$$

We now introduce a variant of this pairing for cohomology groups with fixed central character. Let ψ be a character of $\mathcal{C}_{K \cap \mathbb{A}^\times}$. Consider the sheaf $\mathbb{V}_{\mathcal{O}}^{\psi}$ of locally constant sections of

$$\mathrm{GL}_2(F) \backslash (\mathrm{GL}_2(\mathbb{A}) \times V_{\mathcal{O}}) / \mathbb{A}^{\times(p)} K\, \mathrm{SO}_2(F \otimes_{\mathbb{Q}} \mathbb{R}) \longrightarrow Y_K^{\mathrm{ad}}, \tag{13}$$

where the prime to p idèles $\mathbb{A}^{\times(p)}$ act on $V_{\mathcal{O}}$ via $\psi|\cdot|^{-w_0}/|\cdot|_\infty^{-w_0}$. Since ψ is trivial on $K \cap \mathbb{A}^\times$, this is compatible with the action of K on $V_{\mathcal{O}}$. The cup product followed by the trace map induces a pairing:

$$[\ ,\ ] : \mathrm{H}_c^d(Y_K^{\mathrm{ad}}, \mathbb{V}_{\mathcal{O}}^{\psi}) \times \mathrm{H}^d(Y_K^{\mathrm{ad}}, (\mathbb{V}_{\mathcal{O}}^{\psi})^{\vee}) \to \mathrm{H}_c^{2d}(Y_K^{\mathrm{ad}}, \mathcal{O}) \to \mathcal{O}, \tag{14}$$

and again, the action of the Hecke operator $[KxK]$ is dual to the action of $[Kx^{-1}K]$. Note that the involution $x \mapsto x^*$ sends the sheaf $(\mathbb{V}_{\mathcal{O}}^{\psi})^{\vee}$ to $\mathbb{V}_{\mathcal{O}}^{\psi}$. Similarly to (11) we define the $\mathbb{T}^S$-linear

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modified Poincaré pairing:

$$\langle\ ,\ \rangle = [\ ,\ \iota(^*)] : \mathrm{H}^d_c(Y_K^{\mathrm{ad}}, \mathbb{V}_{\mathcal{O}}^{\psi}) \times \mathrm{H}^d(Y_K^{\mathrm{ad}}, \mathbb{V}_{\mathcal{O}}^{\psi}) \to \mathcal{O}. \tag{15}$$

Finally, under the assumptions of Theorem 2.3(i) there is a natural isomorphism $\mathrm{H}^d(Y_K, \mathbb{V}_{\mathcal{O}})[\psi]_{\mathfrak{m}_\rho} \cong \mathrm{H}^d(Y_K^{\mathrm{ad}}, \mathbb{V}_{\mathcal{O}}^{\psi})_{\mathfrak{m}_\rho}$ and a perfect duality of free $\mathcal{O}$-modules:

$$\langle\ ,\ \rangle : \mathrm{H}^d(Y_K, \mathbb{V}_{\mathcal{O}})[\psi]_{\mathfrak{m}_\rho} \times \mathrm{H}^d(Y_K, \mathbb{V}_{\mathcal{O}})[\psi]_{\mathfrak{m}_\rho} \to \mathcal{O}. \tag{16}$$

3. Ihara's lemma for Hilbert modular varieties

Recall our running assumptions that K factors as a product $\prod_v K_v$ over the primes v of F, that K_v is maximal for all primes v dividing p and that Y_K is smooth.

Let $\mathfrak{q}$ be a prime not dividing p and let S be a finite set of primes containing those dividing $p\mathfrak{q}$ and the set of primes Σ_K where K is not maximal.

Consider the maximal ideal $\mathfrak{m}_\rho = (\varpi, T_v - \mathrm{tr}(\rho(\mathrm{Frob}_v)), S_v - \det(\rho(\mathrm{Frob}_v))\mathrm{N}_{F/\mathbb{Q}}(v)^{-1})$ of the abstract Hecke algebra $\mathbb{T}^S = \mathcal{O}[T_v, S_v \mid v \notin S]$. The Betti cohomology groups $\mathrm{H}^d(Y_K, \mathbb{V}_{\mathcal{O}})$ defined in § 2.1 are modules over $\mathbb{T}^S$.

3.1 Main theorem

Fix a finite index subgroup U of $\mathfrak{o}_{\mathfrak{q}}^\times$, and suppose that $K_{\mathfrak{q}} = \{x \in \mathrm{GL}_2(\mathfrak{o}_{\mathfrak{q}}) \mid \det(x) \in U\}$. In § 2.3 we defined Hecke operators $T'_{\mathfrak{q}}$, $S'_{\mathfrak{q}}$ (respectively, $U'_{\mathfrak{q}}$) acting on $\mathrm{H}^d(Y_K, \mathbb{V}_A)$ (respectively, on $\mathrm{H}^d(Y_{K\cap K_0(\mathfrak{q})}, \mathbb{V}_A)$).

Consider the degeneracy maps $\mathrm{pr}_1, \mathrm{pr}_2 : Y_{K\cap K_0(\mathfrak{q})} \to Y_K$ used in the definition of the Hecke correspondence $T'_{\mathfrak{q}}$.

Theorem 3.1. *Assume that* $(\mathbf{Mod}_\rho)$ *and* $(\mathbf{LI}_{\mathrm{Ind}\,\rho})$ *hold. Then the* $\mathfrak{m}_\rho$*-localization of the* $\mathbb{T}^S$*-linear homomorphism:*

$$\mathrm{pr}_1^* + \mathrm{pr}_2^* : \mathrm{H}^d(Y_K, \mathbb{V}_{\mathcal{O}})^{\oplus 2} \to \mathrm{H}^d(Y_{K\cap K_0(\mathfrak{q})}, \mathbb{V}_{\mathcal{O}})$$

is injective with flat cokernel.

Proof. Our proof is geometric and relies on the existence of smooth models $\mathcal{Y}_K$ (respectively, $\mathcal{Y}_{K\cap K_0(\mathfrak{q})}$) of Y_K (respectively, $Y_{K\cap K_0(\mathfrak{q})}$) over an unramified extension of $\mathbb{Z}_p$ and on the existence of smooth toroidal compactifications thereof. One should be careful to observe that $K \cap K_0(\mathfrak{q})$ is maximal at primes dividing p. By the Betti-étale comparison isomorphism the cohomology groups

$$W := \mathrm{H}^d(Y_{K,\overline{\mathbb{Q}}}, \mathbb{V}_\kappa)_{\mathfrak{m}_\rho} \quad \text{and} \quad W_0(\mathfrak{q}) := \mathrm{H}^d(Y_{K\cap K_0(\mathfrak{q}),\overline{\mathbb{Q}}}, \mathbb{V}_\kappa)_{\mathfrak{m}_\rho},$$

are endowed with a structure of $\mathbb{T}^S[\mathcal{G}_{\mathbb{Q}}]$ modules. The theorem is equivalent to the injectivity of $\mathbb{T}^S[\mathcal{G}_{\mathbb{Q}}]$-linear homomorphism:

$$\mathrm{pr}_1^* + \mathrm{pr}_2^* : W^{\oplus 2} \to W_0(\mathfrak{q}).$$

The image of $\mathbb{T}^S_{\mathfrak{m}_\rho}$ in $\mathrm{End}_\kappa(W)$ is a local Artinian ring and $(\mathfrak{m}_\rho^i W)_{i\geqslant 0}$ is a finite decreasing filtration of W by $\mathbb{T}^S[\mathcal{G}_{\mathbb{Q}}]$-modules. By the torsion freeness result in Theorem 2.3(i), both W and the graded pieces $\mathfrak{m}_\rho^i W/\mathfrak{m}_\rho^{i+1} W$ are quotients of two $\mathbb{T}^S[\mathcal{G}_{\mathbb{Q}}]$-stable $\mathcal{O}$-lattices in $\mathrm{H}^d(Y_{K,\overline{\mathbb{Q}}}, \mathbb{V}_{\mathcal{O}})_{\mathfrak{m}_\rho} \otimes_{\mathcal{O}} E$. By a theorem of Brylinski and Labesse [BL84], it follows that the characteristic polynomial of $\otimes \mathrm{Ind}_F^{\mathbb{Q}} \rho$ annihilates the $\kappa[\mathcal{G}_{\mathbb{Q}}]$-module $\mathfrak{m}_\rho^i W/\mathfrak{m}_\rho^{i+1} W$ (cf. also [Dia98, Lemma 3]). It follows then from $(\mathbf{LI}_{\mathrm{Ind}\,\rho})$ and [Dim05, Lemma 6.5] that every $\mathcal{G}_{\widetilde{F}}$-irreducible subquotient of W is isomorphic to $\otimes \mathrm{Ind}_F^{\mathbb{Q}} \rho$. The same arguments apply also to $W_0(\mathfrak{q})$. Therefore, we can check the above injectivity by checking it on the last graded pieces of the corresponding Fontaine–Laffaille modules.

By Faltings' étale-crystalline comparison theorem and the degeneracy of the Hodge to de Rham spectral sequence (cf. [Dim05, Theorem 5.13]) the claim would follow from the following lemma (although this part of the argument relies on the existence of toroidal compactifications of $\mathcal{Y}_K$ and $\mathcal{Y}_{K\cap K_0(\mathfrak{q})}$, by Köcher's principle we can omit them as long as we are concerned with global sections of the invertible bundle $\underline{\omega}^k \otimes \underline{\nu}^{-w_0/2}$; cf. [Dim05, §§ 1.5 and 1.7]). □

Lemma 3.2. *The following homomorphism is injective*

$$\mathrm{pr}_1^* + \mathrm{pr}_2^* : \mathrm{H}^0(\mathcal{Y}_{K/\kappa}, \underline{\omega}^k \otimes \underline{\nu}^{-w_0/2})^{\oplus 2} \to \mathrm{H}^0(\mathcal{Y}_{K\cap K_0(\mathfrak{q})/\kappa}, \underline{\omega}^k \otimes \underline{\nu}^{-w_0/2}).$$

Proof. Let (g', g) be an element of the kernel: $\mathrm{pr}_1^*(g') = -\,\mathrm{pr}_2^*(g)$.

Since the homomorphism is $U'_{\mathfrak{q}}$-equivariant for the $U'_{\mathfrak{q}}$-action on the left-hand side given by the matrix $\left(\begin{smallmatrix} T'_{\mathfrak{q}} & 1 \\ -S'_{\mathfrak{q}}\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) & 0 \end{smallmatrix}\right)$, we may assume that (g', g) is an eigenvector for $U'_{\mathfrak{q}}$. Similarly may assume that g' is an eigenvector for $S'_{\mathfrak{q}}$. This implies that g' is a multiple of g, hence $\mathrm{pr}_2^*(g) = -\mathrm{pr}_1^*(g')$ is a multiple of $\mathrm{pr}_1^*(g)$. On the other hand, $\mathrm{pr}_1^*(g)$ has the same q-expansion as g, whereas the q-expansions of $\mathrm{pr}_2^*(g)$ and g are related as follows: for every $x \in F \otimes \widehat{\mathbb{Z}}$,

$$c(\mathrm{pr}_2^*(g), x) = \begin{cases} c(g, x\varpi_{\mathfrak{q}}^{-1}) & \text{if } x_{\mathfrak{q}}\varpi_{\mathfrak{q}}^{-1} \in \mathfrak{o}_{\mathfrak{q}}, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

It follows that $c(g, x) = 0$ for all x, which in virtue of the q-expansion principle implies $g = 0$. The proof of Theorem 3.1 is now complete. □

3.2 More cohomological results

Fix a finite index subgroup U of $\mathfrak{o}_{\mathfrak{q}}^\times$, and suppose that $K_{\mathfrak{q}} = \{x \in K_1(\mathfrak{q}^{c-1}) \mid \det(x) \in U\}$, for some integer $c \geqslant 1$. Consider the degeneracy maps

$$\begin{aligned} \mathrm{pr}_1, \mathrm{pr}_2 &: Y_{K\cap K_1(\mathfrak{q}^c)} \to Y_{K\cap K_0(\mathfrak{q}^c)} \to Y_K \quad \text{and} \\ \mathrm{pr}_3, \mathrm{pr}_4 &: Y_{K\cap K_1(\mathfrak{q}^c)\cap K_0(\mathfrak{q}^{c+1})} \to Y_{K\cap K_1(\mathfrak{q}^c)}, \end{aligned} \tag{18}$$

used in the definition of the Hecke correspondence $U'_{\mathfrak{q}}$ in § 2.3.

Proposition 3.3. *Assume that* $(\mathbf{Mod}_\rho)$ *and* $(\mathbf{LI}_{\mathrm{Ind}\,\rho})$ *hold. Then the* $\mathfrak{m}_\rho$*-localization of the* $\mathbb{T}^S$*-linear sequence:*

$$0 \to \mathrm{H}^d(Y_K, \mathbb{V}_{\mathcal{O}}) \xrightarrow{(\mathrm{pr}_1^*, -\mathrm{pr}_2^*)} \mathrm{H}^d(Y_{K\cap K_1(\mathfrak{q}^c)}, \mathbb{V}_{\mathcal{O}})^{\oplus 2} \xrightarrow{\mathrm{pr}_3^* + \mathrm{pr}_4^*} \mathrm{H}^d(Y_{K\cap K_1(\mathfrak{q}^c)\cap K_0(\mathfrak{q}^{c+1})}, \mathbb{V}_{\mathcal{O}})$$

is exact and the last arrow has flat cokernel.

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Proof. We follow closely Fujiwara's argument [Fuj06a, Proposition 5.13], except for the last part of it where we use a geometric argument instead (Fujiwara uses open compact subgroups which do not satisfy our running assumption to be maximal at primes dividing p).

It is enough to prove the exactness after tensoring with κ, which by Theorem 2.3(i) amounts to replacing $\mathbb{V}_{\mathcal{O}}$ by $\mathbb{V}_{\kappa}$. Put $K_0 = K$, $K_1 = K \cap K_1(\mathfrak{q}^c)$,

$$K_2 = \begin{pmatrix} \varpi_{\mathfrak{q}} & 0 \\ 0 & 1 \end{pmatrix} (K \cap K_1(\mathfrak{q}^c)) \begin{pmatrix} \varpi_{\mathfrak{q}}^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and}$$

$$K_3 = \begin{pmatrix} \varpi_{\mathfrak{q}} & 0 \\ 0 & 1 \end{pmatrix} (K \cap K_1(\mathfrak{q}^c) \cap K_0(\mathfrak{q}^{c+1})) \begin{pmatrix} \varpi_{\mathfrak{q}}^{-1} & 0 \\ 0 & 1 \end{pmatrix} = K \cap K_1(\mathfrak{q}^c) \cap K^0(\mathfrak{q}),$$

where $K^0(\mathfrak{q}) = \left(\begin{smallmatrix} \varpi_{\mathfrak{q}} & 0 \\ 0 & 1 \end{smallmatrix}\right) K_0(\mathfrak{q}) \left(\begin{smallmatrix} \varpi_{\mathfrak{q}}^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is the opposite parahoric subgroup.

For $i = 0, 1, 2, 3$ put $Y_i = Y_{K_i}$. By the above computations it is equivalent then to prove the exactness of the sequence:

$$0 \to \mathrm{H}^d(Y_0, \mathbb{V}_{\kappa})_{\mathfrak{m}_\rho} \xrightarrow{(\mathrm{pr}_1'^{\,*}, -\mathrm{pr}_2'^{\,*})} \mathrm{H}^d(Y_1, \mathbb{V}_{\kappa})_{\mathfrak{m}_\rho} \oplus \mathrm{H}^d(Y_2, \mathbb{V}_{\kappa})_{\mathfrak{m}_\rho} \xrightarrow{\mathrm{pr}_3'^{\,*} + \mathrm{pr}_4'^{\,*}} \mathrm{H}^d(Y_3, \mathbb{V}_{\kappa})_{\mathfrak{m}_\rho},$$

where

$$\begin{array}{ccccc} & \overset{\mathrm{pr}_3'}{\swarrow} & Y_3 & \overset{\mathrm{pr}_4'}{\searrow} & \\ Y_1 & & \downarrow \mathrm{pr} & & Y_2 \\ & \underset{\mathrm{pr}_1'}{\searrow} & Y_0 & \underset{\mathrm{pr}_2'}{\swarrow} & \end{array}$$

and the projections are induced by the inclusion of the open compact subgroups.

Taking models of the Y_i $(0 \leqslant i \leqslant 3)$ over $\mathbb{Q}$ and using Betti-étale comparison isomorphisms turns the above sequence into a sequence of $\mathbb{T}^S[\mathcal{G}_{\mathbb{Q}}]$-modules $W_i := \mathrm{H}^d(Y_{i,\overline{\mathbb{Q}}}, \mathbb{V}_{\kappa})_{\mathfrak{m}_\rho}$. As in the proof of Theorem 3.1, the condition $(\mathbf{LI}_{\mathrm{Ind}\,\rho})$ implies that every $\mathcal{G}_{\widetilde{F}}$-irreducible subquotient of W_i $(0 \leqslant i \leqslant 3)$ is isomorphic to $\otimes \mathrm{Ind}_F^{\mathbb{Q}} \rho$. Therefore, it is enough to check the exactness on the last graded pieces of the Fontaine–Laffaille modules. This is the object of the following result. □

Lemma 3.4. *The following sequence is exact:*

$$0 \to \mathrm{H}^0(\mathcal{Y}_{0/\kappa}, \underline{\omega}^k \otimes \underline{\nu}^{-w_0/2}) \xrightarrow{(\mathrm{pr}_1'^{\,*}, -\mathrm{pr}_2'^{\,*})}$$
$$\to \mathrm{H}^0(\mathcal{Y}_{1/\kappa}, \underline{\omega}^k \otimes \underline{\nu}^{-w_0/2}) \oplus \mathrm{H}^0(\mathcal{Y}_{2/\kappa}, \underline{\omega}^k \otimes \underline{\nu}^{-w_0/2}) \xrightarrow{\mathrm{pr}_3'^{\,*} + \mathrm{pr}_4'^{\,*}} \mathrm{H}^0(\mathcal{Y}_{3/\kappa}, \underline{\omega}^k \otimes \underline{\nu}^{-w_0/2}). \quad (19)$$

Proof. We adapt the analytic argument of [Fuj06a, Lemma 5.14] in order to show that the coproduct $\mathcal{Y}_1 \coprod_{\mathcal{Y}_3} \mathcal{Y}_2$ is isomorphic to $\mathcal{Y}_0$ as κ-schemes.

For $0 \leqslant i \leqslant 3$, there exists a fine moduli scheme $\mathcal{Y}_i^1$ such that $\mathcal{Y}_i^1 \to \mathcal{Y}_i$ is a finite étale with group

$$\Delta_i = \frac{F_+^\times \cap \det(K_i)}{(F^\times \cap K_i)^2},$$

where $\Delta_1 = \Delta_2 = \Delta_3 \twoheadrightarrow \Delta_0$ (recall that by definition $\mathcal{Y}_i^1$ has the same number of connected components as $\mathcal{Y}_i$). Since $\mathcal{Y}_i^1 \to \mathcal{Y}_0^1$ is Δ_i-equivariant (where the action on $\mathcal{Y}_0^1$ is via the surjection $\Delta_i \twoheadrightarrow \Delta_0$), we have $\mathcal{Y}_i \coprod_{\mathcal{Y}_i^1} \mathcal{Y}_0^1 \cong \mathcal{Y}_0$ for all i. Hence, it is enough to show that $\mathcal{Y}_1^1 \coprod_{\mathcal{Y}_3^1} \mathcal{Y}_2^1 \cong \mathcal{Y}_0^1$.

We show this claim using the following functorial description of the $\mathcal{Y}_i^1$:

(i) $\mathcal{Y}_0^1$ classifies polarized Hilbert–Blumenthal abelian varieties A with $\mu_{\mathfrak{q}^{c-1}}$-level structure P and some additional level structures that we ignore since they are the same for $\mathcal{Y}_i^1$ for all $0 \leqslant i \leqslant 3$;

(ii) $\mathcal{Y}_1^1$ classifies polarized Hilbert–Blumenthal abelian varieties A with $\mu_{\mathfrak{q}^c}$-level structure Q;

(iii) $\mathcal{Y}_2^1$ classifies polarized Hilbert–Blumenthal abelian varieties A with a $\mu_{\mathfrak{q}}$-subgroup C and a $\mu_{\mathfrak{q}^c}$-level structure $\overline{Q}$ in A/C;

(iv) $\mathcal{Y}_3^1$ classifies polarized Hilbert–Blumenthal abelian varieties A with $\mu_{\mathfrak{q}^c}$-level structure Q and a $\mu_{\mathfrak{q}}$-subgroup C disjoint from the group generated by Q.

The morphisms pr_j' in the diagram above come from forgetful functors described as follows:

(i) $\mathrm{pr}_1'(A, Q) = (A, Q^{\mathfrak{q}})$, where $Q^{\mathfrak{q}}$ is the $\mu_{\mathfrak{q}^{c-1}}$-level structure deduced from Q obtained by composing with the dual $\mu_{\mathfrak{q}^{c-1}} \hookrightarrow \mu_{\mathfrak{q}^c}$ of the natural projection $\mathfrak{o}/\mathfrak{q}^c \to \mathfrak{o}/\mathfrak{q}^{c-1}$;

(ii) $\mathrm{pr}_2'(A, \overline{Q}, C) = (A, \overline{Q}^{\mathfrak{q}})$ where it is important to observe that $\overline{Q}^{\mathfrak{q}}$ is a well-defined $\mu_{\mathfrak{q}^{c-1}}$-level structure on A (not only in A/C);

(iii) $\mathrm{pr}_3'(A, Q, C) = (A, Q)$;

(iv) $\mathrm{pr}_4'(A, Q, C) = (A, Q \mod C, C)$, where $Q \mod C$ is a $\mu_{\mathfrak{q}^c}$-level structure on A/C, since C is disjoint from the group generated by Q.

We have $\mathrm{pr}_1' \circ \mathrm{pr}_3'(A, Q, C) = (A, Q^{\mathfrak{q}}) = (A, (Q \mod C)^{\mathfrak{q}}) = \mathrm{pr}_2' \circ \mathrm{pr}_4'(A, Q, C)$.

We have to show that given any two homomorphisms $h_1 : \mathcal{Y}_1^1 \to \mathcal{X}$ and $h_2 : \mathcal{Y}_2^1 \to \mathcal{X}$ such that $h_1 \circ \mathrm{pr}_3' = h_2 \circ \mathrm{pr}_4'$, there exists a unique homomorphism $h_0 : \mathcal{Y}_0^1 \to \mathcal{X}$ such that $h_1 = h_0 \circ \mathrm{pr}_1'$ and $h_2 = h_0 \circ \mathrm{pr}_2'$. By the functorial description of the $\mathcal{Y}_0^i$ and the pr_j' the claim is reduced to a simple lemma from group theory saying that, if K_0 is generated by K_1 and K_2, then the coproduct $K_0/K_1 \coprod_{K_0/K_3} K_0/K_2 = K_0/K_1 \coprod_{K_0} K_0/K_2$ is a singleton.

Hence, $\mathcal{Y}_1 \coprod_{\mathcal{Y}_3} \mathcal{Y}_2 \cong \mathcal{Y}_0$ yielding an exact sequence of sheaves over $\mathcal{Y}_0$:

$$0 \to \underline{\omega}^k \otimes \underline{\nu}^{-w_0/2} \to \mathrm{pr}_1'{}_* \mathrm{pr}_1'{}^* \underline{\omega}^k \otimes \underline{\nu}^{-w_0/2} \oplus \mathrm{pr}_2'{}_* \mathrm{pr}_2'{}^* \underline{\omega}^k \otimes \underline{\nu}^{-w_0/2} \to \mathrm{pr}_* \mathrm{pr}^* \underline{\omega}^k \otimes \underline{\nu}^{-w_0/2}.$$

Since the functor of global sections is left-exact, this implies the lemma. □

4. Twisting

Let $\rho : \mathcal{G}_F \to \mathrm{GL}_2(\kappa)$ be a totally odd, absolutely irreducible representation.

4.1 Local twist types

For a prime v of F, we identify $\mathcal{G}_{F_v}$ with a decomposition subgroup of $\mathcal{G}_F$ and denote by I_v its inertia subgroup. Let ρ_v be the restriction of ρ to $\mathcal{G}_{F_v}$. We normalize the local class field theory isomorphism so that the uniformizer ϖ_v corresponds to the geometric Frobenius.

Over a totally real field F, twists of minimal conductor exist locally, but not necessarily globally. This observation motivates the following definition, due to Fujiwara.

DEFINITION 4.1. Let v be a prime of F not dividing p. A local twist type character for ρ_v is a character $\nu_v : \mathcal{G}_{F_v} \to \kappa^\times$ such that $\rho_v \otimes \nu_v{}^{-1}$ has minimal conductor amongst all twists of ρ_v by characters of $\mathcal{G}_{F_v}$. For any prime v we choose once and for all a local twist-type character ν_v and use the same notation for the character of $F_v^\times$ coming from local class field theory. For simplicity, we choose ϖ_v and ν_v, so that $\nu_v(\varpi_v)=1$. Denote by ν the character $\prod_v \nu_v$ of $(\mathfrak{o}\otimes\widehat{\mathbb{Z}})^\times$.

Definition 4.2. Let Σ_ρ be the set of primes v not dividing p such that $\rho_v\otimes{\nu_v}^{-1}$ is ramified.

Let S_ρ be the set of primes $v\in\Sigma_\rho$ such that ρ_v is reducible.

Let P_ρ be the set of primes $v\in\Sigma_\rho$ such that ρ_v is irreducible but $\rho_v|_{I_v}$ is reducible, and $\mathrm{N}_{F/\mathbb{Q}}(v)\equiv -1 \pmod p$.

Note that Σ_ρ, S_ρ and P_ρ do not change when we twist ρ by a character.

4.2 Minimally ramified deformations

For a character μ taking values in $\kappa^\times$, we denote by $\widetilde{\mu}$ its Teichmüller lift.

Let A be a local complete Noetherian $\mathcal{O}$-algebra with residue field κ and $\widetilde{\rho}_v:\mathcal{G}_{F_v}\to \mathrm{GL}_2(A)$ be a lifting of ρ_v. For $F=\mathbb{Q}$, the following definition coincides with the notion introduced in [Dia97a].

Definition 4.3. We say that $\widetilde{\rho}_v$ is a minimally ramified if $\det\widetilde{\rho}_v|_{I_v}=\widetilde{\det\rho_v}|_{I_v}$ and, in addition:

- if $v\notin\Sigma_\rho$, then $\widetilde{\rho}_v\otimes\widetilde{\nu}_v^{-1}$ is unramified;
- if $v\in S_\rho$, then $(\widetilde{\rho}_v\otimes\widetilde{\nu}_v^{-1})^{I_v}\neq 0$;
- if $v\in P_\rho$ and $(\rho_v\otimes\mu_v^{-1})^{I_v}\neq 0$ for some character $\mu_v:I_v\to\kappa^\times$, then $(\widetilde{\rho}_v\otimes\widetilde{\mu}_v^{-1})^{I_v}\neq 0$.

Remark 4.4. (i) If $\widetilde{\rho}_v$ is a minimally ramified lifting of ρ_v, then $\widetilde{\rho}_v\otimes\widetilde{\mu}$ is a minimally ramified lifting of $\rho_v\otimes\mu$ for all characters $\mu:\mathcal{G}_{F_v}\to\kappa^\times$.

(ii) If $\widetilde{\rho}_v$ is a minimally ramified lifting of ρ_v, then the Artin conductors of $\widetilde{\rho}_v$ and ρ_v coincide and $\det\widetilde{\rho}_v|_{I_v}$ is the Teichmüller lift of $\det\rho_v|_{I_v}$. The converse holds if ρ_v has minimal conductor among its twists and $v\notin P_\rho$ (cf. [Dia97a, Remark 3.5]).

Let $\chi_p:\mathcal{G}_F\to\mathbb{Z}_p^\times$ be the p-adic cyclotomic character.

Definition 4.5. Let $\phi:\mathcal{G}_F\to\mathcal{O}^\times$ be a finite p-power order character of conductor prime to p. Define $\psi:\mathcal{G}_F\to\mathcal{O}^\times$ as the unique character such that $\psi\phi^{-2}$ is the Teichmüller lift of $(\chi_p^{w_0+1} \bmod p)\cdot\det\rho$.

Definition 4.6. Let Σ be a finite set of primes of F not dividing p. Let A be a local complete Noetherian $\mathcal{O}$-algebra with residue field κ. We say that a deformation $\widetilde{\rho}:\mathcal{G}_F\to\mathrm{GL}_2(A)$ of ρ to A is Σ-ramified, if the following three conditions hold:

- $\widetilde{\rho}\otimes\phi^{-1}$ is minimally ramified at all primes $v\notin\Sigma$, $v\nmid p$ (cf. Definition 4.3);
- $\widetilde{\rho}$ is crystalline at each prime v dividing p with Hodge–Tate weights $((w_0-k_\tau)/2+1,(w_0+k_\tau)/2)_{\tau\in J_{F_v}}$;
- $\det\widetilde{\rho}=\chi_p^{-w_0-1}\psi$.

A $\varnothing$-ramified deformation is called minimally ramified.

Note that if $\rho_{f,p}$ is a Σ-ramified deformation of ρ, then the central character of f has to be $\psi|\cdot|^{-w_0}$. Since p is odd, every p-power character of $\mathcal{G}_F$ has a square root, hence the determinant of any finitely ramified low-weight crystalline deformation of ρ is of the form $\chi_p^{-w_0-1}\psi$, for some ψ as above.

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4.3 Auxiliary level structures

Under the assumption ($\mathbf{LI}_{\mathrm{Ind}\,\rho}$), which implies in particular that the restriction of ρ to the absolute Galois group of any totally real extension of F is absolutely irreducible, a standard argument (cf. [Jar99b, § 12]) using the Cebotarev density theorem implies that there exist infinitely many primes $\mathfrak{u}$ of F as in Lemma 2.2(i), such that:

(i) $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{u}) \not\equiv 1 \pmod{p}$ and

(ii) ϕ and ρ are unramified at $\mathfrak{u}$; and $\mathrm{tr}(\rho(\mathrm{Frob}_{\mathfrak{u}}))^2 \not\equiv \psi(\mathfrak{u})\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{u})^{w_0}(\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{u})+1)^2 \pmod{\varpi}$.

In particular, this implies that $\mathrm{L}_{\mathfrak{u}}(\mathrm{Ad}^0(\rho), 1) \in \kappa^\times$. Let us fix such a prime $\mathfrak{u}$ and denote by $\alpha_{\mathfrak{u}}$ and $\beta_{\mathfrak{u}}$ the eigenvalues of $\rho(\mathrm{Frob}_{\mathfrak{u}})$.

Lemma 4.7. *The natural projection $\mathcal{T}_{P_\rho \cup \{\mathfrak{u}\}} \to \mathcal{T}_{P_\rho}$ is an isomorphism.*

Proof. This amounts to proving that if f is a newform of weight k, central character $\psi|\cdot|^{-w_0}$ and level prime to p, and if $\rho_{f,p}$ is a deformation of ρ, then the local component $\pi_{\mathfrak{u}}$ of the associated automorphic representation π is unramified. Since $\rho_{\mathfrak{u}}$ is unramified, if $\pi_{\mathfrak{u}}$ is ramified, then necessarily the valuation of its conductor is one or two. Since $\pi_{\mathfrak{u}}$ has unramified central character this implies that $\dim \pi_{\mathfrak{u}}^{K_0(\mathfrak{u})} = 1$ or $\dim \pi_{\mathfrak{u}}^{K_0(\mathfrak{u}^2)} = 1$. In the first case $\pi_{\mathfrak{u}}$ is a special representation, hence $\alpha_{\mathfrak{u}} \equiv \beta_{\mathfrak{u}}\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{u})^{\pm 1} \pmod{\varpi}$. In the second case $\pi_{\mathfrak{u}}$ is either a ramified principal series, in which case $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{u}) \equiv 1 \pmod{p}$, or a supercuspidal representation, in which case $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{u}) \equiv -1 \pmod{p}$ and $\mathrm{tr}(\rho(\mathrm{Frob}_{\mathfrak{u}})) \equiv 0 \pmod{\varpi}$. In both cases this contradicts our assumptions. □

By Lemmas 2.1(iii) and 2.2(i), for all $K \subset K_0(\mathfrak{u})$, Y_K and Y_K^{ad} are smooth. However, by Lemma 4.7 the additional level at $\mathfrak{u}$ does not modify the local components of the Hecke algebras and cohomology modules that we consider, hence we omit it in our notation.

4.4 Level structures and Hecke operators associated with ρ

The cohomology of the Hilbert modular varieties for the level structures that we introduce in this section play an important role in the study of modular deformations of ρ.

For v not dividing p denote by c_v be the valuation of the Artin conductor of $\rho_v \otimes {\nu_v}^{-1}$ and by d_v the dimension of $(\rho_v \otimes \nu^{-1})^{I_v}$ (cf. Definition 4.1). Put $c_v = d_v = 0$ if v divides p. Define

$$\begin{aligned} K'_v &= \ker(K_1(v^{c_v}) \xrightarrow{\det} \mathfrak{o}_v^\times \xrightarrow{\widetilde{\nu}_v\phi} \mathcal{O}^\times), \quad \text{and} \\ K''_v &= \ker(K_1(v^{c_v}) \cap K_0(v^{c_v+d_v}) \xrightarrow{\det} \mathfrak{o}_v^\times \xrightarrow{\widetilde{\nu}_v\phi} \mathcal{O}^\times). \end{aligned} \tag{20}$$

For all but finitely many primes v, we have $\nu_v|_{\mathfrak{o}_v^\times} = \phi|_{\mathfrak{o}_v^\times} = 1$.

For a prime $\mathfrak{u}$ as in Lemma 2.2(i) and a finite set of primes Σ of F not dividing p we put $\mathfrak{n}_\Sigma = \mathfrak{u}\prod_{v\in\Sigma} v^{c_v+d_v}\prod_{v\notin\Sigma} v^{c_v}$ and

$$K_\Sigma = K_0(\mathfrak{u}) \cap \prod_{v\in\Sigma} K''_v \prod_{v\notin\Sigma} K'_v \subset K_0(\mathfrak{n}_\Sigma) \quad \text{and} \quad K_\rho = K_\varnothing. \tag{21}$$

As in § 2.3 we define Hecke operators $U_\delta := \left(\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right) K'_v$ and $S_\delta := \left(\begin{smallmatrix} \delta & 0 \\ 0 & \delta \end{smallmatrix}\right) K'_v$, for all v where $\delta \in \mathfrak{o}_v^\times$; $T'_v = [K'_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) K'_v]$ and $S'_v = [K'_v \left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) K'_v]$ for $v \notin \Sigma$ such that $c_v = 0$; $U'_v = [K'_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) K'_v]$ for $v \notin \Sigma$ such that $c_v > 0$; $U''_v = [K''_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) K''_v]$ for $v \in \Sigma$.

Let Q be a finite set of primes $\mathfrak{q}$ of F such that $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) \equiv 1 \pmod{p}$. Put

$$K_{0,Q} = K_\rho \cap \prod_{\mathfrak{q}\in Q} K_0(\mathfrak{q}) \quad \text{and} \quad K^Q = K_\rho \cap \prod_{\mathfrak{q}\in Q} K_{\mathfrak{q}}^Q \tag{22}$$

where $K_{\mathfrak{q}}^Q$ is the kernel of the composition of $K_0(\mathfrak{q}) \to (\mathfrak{o}/\mathfrak{q})^\times$, $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \mapsto a_{\mathfrak{q}}/d_{\mathfrak{q}}$ with the natural projection from $(\mathfrak{o}/\mathfrak{q})^\times$ to its p-Sylow $\Delta_{\mathfrak{q}}$.

For $\mathfrak{q}\in Q$ and $\delta \in \mathfrak{o}_{\mathfrak{q}}^\times$, the operator $S_\delta := [\left(\begin{smallmatrix} \delta & 0 \\ 0 & \delta\end{smallmatrix}\right) K_{\mathfrak{q}}^Q]$ is trivial, the operator $U_\delta := [K_{\mathfrak{q}}^Q \left(\begin{smallmatrix} 1 & 0 \\ 0 & \delta\end{smallmatrix}\right) K_{\mathfrak{q}}^Q]$ depends only on the image of δ in $\Delta_{\mathfrak{q}}$, and the operator $U_{\mathfrak{q}} := [K_{\mathfrak{q}}^Q \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{q}}\end{smallmatrix}\right) K_{\mathfrak{q}}^Q]$ depends on the choice of $\varpi_{\mathfrak{q}}$ as described in § 2.3.

4.5 Decomposing the central action

Since our aim is to study automorphic forms with fixed central character, we only consider open subgroups $K \subset K_\rho$ such that $K \cap \mathbb{A}^\times = K_\rho \cap \mathbb{A}^\times$. Consider the idèle class group

$$\mathcal{C}_\rho := \mathcal{C}_{K_\rho \cap \mathbb{A}^\times}. \tag{23}$$

The natural inclusions induce the following commutative diagram, where all morphisms are étale for the indicated (abelian) groups.

$$\begin{array}{ccc} Y_\rho & \xrightarrow{F^\times K_1(\mathfrak{n}_\varnothing)/F^\times K_\rho} & Y_1(\mathfrak{n}_\varnothing) \\ \Big\downarrow{\scriptstyle \mathcal{C}_\rho} & & \Big\downarrow{\scriptstyle \mathcal{C}_{(1+\widehat{\mathbb{Z}}\otimes \mathfrak{n}_\varnothing)^\times}} \\ Y_\rho^{\mathrm{ad}} & \xrightarrow{\mathbb{A}^\times K_1(\mathfrak{n}_\varnothing)/\mathbb{A}^\times K_\rho} & Y_{\mathfrak{n}_\varnothing}^{\mathrm{ad}} \end{array} \tag{24}$$

If $v \in \Sigma_\rho$ the p-Sylow subgroup of $(\mathfrak{o}/v)^\times$ injects naturally in $\mathbb{A}^\times K_1(\mathfrak{n}_\varnothing)/\mathbb{A}^\times K_\rho$ (*a fortiori* in $F^\times K_1(\mathfrak{n}_\varnothing)/F^\times K_\rho$), hence acts freely on Y_ρ^{ad} and Y_ρ. It follows that the étale morphism $Y_\rho \to Y_{\mathfrak{n}_\varnothing}^{\mathrm{ad}}$ factors through an étale morphism $Y_\rho \to Y_\rho^\Delta$ with group the p-group

$$\Delta_\rho^\phi = (p\text{-Sylow of } \mathcal{C}_\rho) \times \prod_{v\in\Sigma_\rho} (\mathfrak{o}_v^\times/\ker(\phi_v)). \tag{25}$$

Recall that $[\psi]$ denotes the ψ-isotypic part for the action of the Hecke operators $S'_v \mathrm{N}_{F/\mathbb{Q}}(v)^{-w_0}$, $v \notin \Sigma_\rho$, where ψ is seen as a finite-order Hecke character of $\mathcal{C}_\rho$, and that $[\phi\widetilde{\nu}]$ denotes the intersection of the $\phi_v\widetilde{\nu}_v$-isotypic parts for the action of the Hecke operators U_δ for $\delta \in \mathfrak{o}_v^\times$ (cf. Definitions 4.1 and 4.5).

For $v \notin \Sigma_\rho$ we have $U_\delta^2 = S_\delta$ and since p is odd, the ϕ_v-action at those v is determined by the action of the central character.

Hence, the $[\psi, \widetilde{\nu}\phi]$ part is the intersection of the $[\phi^2, \phi]$ part for the action of the p-group Δ_ρ^ϕ with the $[\psi\phi^{-2}, \widetilde{\nu}]$-isotypic part for the action of a prime to p order group. This geometric description of the Hecke action of Δ_ρ^ϕ will play an important role in the proof of Proposition 5.9.

5. Modularity of the minimally ramified deformations

Let $\rho : \mathcal{G}_F \to \mathrm{GL}_2(\kappa)$ be a continuous representation satisfying $(\mathbf{LI}_{\mathrm{Ind}\rho})$ and $(\mathbf{Mod}_\rho)$.

The main aim of this section is to prove the following.

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Theorem 5.1. *Suppose that $P_\rho = \varnothing$. Then all minimally ramified deformations of ρ are modular.*

In the notation of § 1.2, the above theorem amounts to prove that $\pi : \mathcal{R} \twoheadrightarrow \mathcal{T}$ is an isomorphism (since $\Sigma = \varnothing$ in the entire section, we omit the subscripts). Our proof uses a stronger version, due to Fujiwara [Fuj06a, § 2], of a method invented by Wiles [Wil95] and Taylor–Wiles [TW95] and known as a Taylor–Wiles system (a similar formalism has been found independently by Diamond [Dia97b]).

The construction of a Taylor–Wiles system occupies the entire section. It includes namely a geometric realization of $\mathcal{T}$ as a Hecke algebra acting on the local component $\mathcal{M}$ at ρ of the middle degree cohomology of a Hilbert modular variety. The torsion freeness of $\mathcal{M}$ is a crucial ingredient (cf. Theorem 2.3(i)). Lemmas 5.4, 5.6 and 5.7 are proved using standard fact about automorphic representations and local Langlands correspondence for GL(2), whereas Propositions 5.5, 5.8 and 5.9 use finer geometric arguments.

Note that Fujiwara's formalism is not essential for us since we know that $\mathcal{M}$ is free over $\mathcal{T}_{P_\rho}$ and $\mathcal{T}_{P_\rho}$ is Gorenstein. This fact is an important ingredient in the proof of Theorem A, and is shown in Proposition 5.5 *without* assuming $P_\rho = \varnothing$. Actually, we only assume $P_\rho = \varnothing$ in § 5.6.

5.1 The formalism of Taylor–Wiles systems, following Fujiwara

Definition 5.2. Let $\mathcal{Q}$ be a family of finite sets of primes $\mathfrak{q}$ of F such that $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) \equiv 1 \pmod{p}$. A Taylor–Wiles system for $\mathcal{Q}$ is a family $\{\mathcal{R}, \mathcal{M}, (\mathcal{R}_Q, \mathcal{M}^Q)_{Q\in\mathcal{Q}}\}$ such that:

- (TW1) $\mathcal{R}_Q$ is a local complete $\mathcal{O}[\Delta_Q]$-algebra, where $\Delta_Q = \prod_{\mathfrak{q}\in Q} \Delta_\mathfrak{q}$ and $\Delta_\mathfrak{q}$ is the p-Sylow of $(\mathfrak{o}/\mathfrak{q})^\times$;
- (TW2) $\mathcal{R}$ is a local complete $\mathcal{O}$-algebra and there is an isomorphism of local complete $\mathcal{O}$-algebras $\mathcal{R}_Q \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \cong \mathcal{R}$;
- (TW3) $\mathcal{M}$ is a non-zero $\mathcal{R}$-module, and $\mathcal{M}^Q$ is an $\mathcal{R}_Q$-module, free of finite rank over $\mathcal{O}[\Delta_Q]$ and such that $\mathcal{M}^Q \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O}$ is isomorphic to $\mathcal{M}$ as $\mathcal{R}$-module.

We denote by $\mathcal{T}$ the image of $\mathcal{R} \to \mathrm{End}_\mathcal{O}(\mathcal{M})$.

When $\mathcal{Q} = \{Q_m | m \in \mathbb{N}\}$, we write $\mathcal{R}_m, \mathcal{M}_m, \dots$ instead of $\mathcal{R}_{Q_m}, \mathcal{M}^{Q_m}, \dots$.

Theorem 5.3 (Fujiwara [Fuj06a, § 2]). *Let $\{\mathcal{R}, (\mathcal{R}_m, \mathcal{M}_m)_{m\in\mathbb{N}}\}$ be a Taylor–Wiles system. Assume that for all m:*

- (i) *for all $\mathfrak{q} \in Q_m$, $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) \equiv 1 \pmod{p^m}$;*
- (ii) *$\mathcal{R}_m$ can be generated by $\#Q_m = r$ elements as a local complete $\mathcal{O}$-algebra.*

Then, the natural surjection $\mathcal{R} \twoheadrightarrow \mathcal{T}$ is an isomorphism. Moreover, these algebras are a flat and complete intersection of relative dimension zero over $\mathcal{O}$ and $\mathcal{M}$ is free over $\mathcal{T}$.

5.2 The rings $\mathcal{R}_Q$

Let Q be a finite set of auxiliary primes $\mathfrak{q}$ of F satisfying:

- (i) $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) \equiv 1 \pmod{p}$; and
- (ii) ϕ and ρ are unramified at $\mathfrak{q}$, and $\rho(\mathrm{Frob}_\mathfrak{q})$ has two distinct eigenvalues $\alpha_\mathfrak{q}$ and $\beta_\mathfrak{q}$ in κ.

For such a Q we can associate by § 1.2 a universal deformation ring $\mathcal{R}_Q$, endowed with a canonical surjection $\mathcal{R}_Q \twoheadrightarrow \mathcal{R}_\varnothing =: \mathcal{R}$. By a result of Faltings (cf. [TW95, Appendix]) $\mathcal{R}_Q$ is a $\mathcal{O}[\Delta_Q]$-algebra and $\mathcal{R}_Q \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \cong \mathcal{R}$. Thus, conditions (TW1) and (TW2) hold.

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More generally, for any set of primes P disjoint from Q, $\mathcal{R}_{Q\cup P}$ is a $\mathcal{O}[\Delta_Q]$-algebra and

$$\mathcal{R}_{Q\cup P}\otimes_{\mathcal{O}[\Delta_Q]}\mathcal{O}\cong\mathcal{R}_P. \tag{26}$$

In particular, $\mathcal{R}_{Q\cup P_\rho}$ is a $\mathcal{O}[\Delta_Q]$-algebra.

5.3 The module $\mathcal{M}$

Denote by Y_ρ the Hilbert modular varieties of level K_ρ defined in § 4.4.

Let S be a finite set of primes containing $\Sigma_\rho\cup\{v\mid p\}\cup\{\mathfrak{u}\}\cup\{v\mid\phi\widetilde{\nu}_v \text{ ramified}\}=\Sigma_{K_\rho}\cup\{v\mid p\}$. Denote by $\mathfrak{m}_\rho$ the maximal ideal of $\mathbb{T}^S=\mathcal{O}[T_v,S_v\mid v\notin S]$ corresponding to ρ. With the notation introduced in §§ 4.4 and 4.5, we fix an eigenvalue $\alpha_\mathfrak{u}$ of $\rho(\mathrm{Frob}_\mathfrak{u})$ and consider the $\mathcal{O}$-module:

$$\mathcal{M}:=\mathrm{H}^d(Y_\rho,\mathbb{V}_\mathcal{O})[\psi,\widetilde{\nu}\phi]_{(\mathfrak{m}_\rho,U_\mathfrak{u}-\alpha_\mathfrak{u})}. \tag{27}$$

Let $\mathbb{T}'$ be the image of $\mathbb{T}^S$ in the ring of $\mathcal{O}$-linear endomorphisms of $\mathcal{M}$.

By Theorem 2.3(i) the $\mathfrak{m}_\rho$-localization of the $\mathbb{T}^S$-module $\mathrm{H}^d(Y_\rho,\mathbb{V}_\mathcal{O})$ is free over $\mathcal{O}$. Hence, $\mathcal{M}$ is free over $\mathcal{O}$ as a direct factor of free $\mathcal{O}$-module.

Moreover, $\mathcal{M}$ is non-zero by $(\mathbf{Mod}_\rho)$ and Remark 1.3. For any newform f contributing to $\mathcal{M}$, consider the maximal ideal

$$\mathfrak{m}_f=(\varpi,T'_v-\iota_p(c(f,v)),S'_v-\iota_p(\psi(v))\mathrm{N}_{F/\mathbb{Q}}(v)^{w_0},U'_{v'}-\iota_p(c(f,v'));v\notin\Sigma_\rho,v'\in\Sigma_\rho)$$

of $\mathbb{T}^{\mathrm{full}}=\mathcal{O}[T'_v,S'_v;v\notin\Sigma_\rho][U'_{v'};v'\in\Sigma_\rho]$. Note that $\mathfrak{m}_f\cap\mathbb{T}^S=\mathfrak{m}_\rho$.

Let $\mathbb{T}$ (respectively, $\mathfrak{m}$) be the image of $\mathbb{T}^{\mathrm{full}}$ (respectively, $\mathfrak{m}_f$) in the ring of $\mathcal{O}$-linear endomorphisms of $\mathrm{H}^d(Y_\rho,\mathbb{V}_\mathcal{O})[\psi,\widetilde{\nu}\phi]$.

Lemma 5.4. *We have the following results:*

(i) *there is a unique isomorphism of $\mathbb{T}^S$-algebras $\mathcal{T}_{P_\rho}\xrightarrow{\sim}\mathbb{T}'$;*

(ii) *$\mathcal{M}\otimes\mathbb{C}$ is free of rank 2^d over $\mathcal{T}_{P_\rho}\otimes\mathbb{C}$; and*

(iii) *the natural injective algebra homomorphism $\mathbb{T}'\hookrightarrow\mathbb{T}_\mathfrak{m}$ is an isomorphism.*

Proof. (i) By Lemma 4.7, we have $\mathcal{T}_{P_\rho\cup\{\mathfrak{u}\}}\cong\mathcal{T}_{P_\rho}$. Since $\mathcal{O}$-algebras $\mathcal{T}_{P_\rho\cup\{\mathfrak{u}\}}$ and $\mathbb{T}'$ are torsion free (the first by definition, the second because $\mathcal{M}$ is free over $\mathcal{O}$), it is enough to show that there is a unique isomorphism of $\mathbb{T}^S\otimes\mathbb{C}$-algebras between $\mathcal{T}_{P_\rho\cup\{\mathfrak{u}\}}\otimes\mathbb{C}$ and $\mathbb{T}'\otimes\mathbb{C}$ (tensors being over $\mathcal{O}$ for some fixed embedding $\mathcal{O}\hookrightarrow\mathbb{C}$).

Consider a (cuspidal) automorphic representation π generated by a holomorphic newform f of weight k, central character $\psi|\cdot|^{-w_0}$ and prime to p conductor. By definition, π contributes to $\mathcal{T}_{P_\rho\cup\{\mathfrak{u}\}}\otimes\mathbb{C}$ if, and only if, for all primes $v\nmid p\mathfrak{u}$, $v\notin P_\rho$, $\phi^{-1}\otimes\rho_{f,p}|_{\mathcal{G}_{F_v}}$ is a minimally ramified deformation of ρ_v.

For $v\notin P_\rho$, $v\neq\mathfrak{u}$, Remark 4.4 shows that $\phi^{-1}\otimes\rho_{f,p}|_{\mathcal{G}_{F_v}}$ is a minimally ramified deformation of ρ_v if, and only if, $(\phi\widetilde{\nu}_v)^{-1}\otimes\rho_{f,p}|_{\mathcal{G}_{F_v}}$ has conductor c_v. By Carayol's theorem [Car86] on the compatibility between the local and the global Langlands correspondences this is equivalent to $(\pi_v\otimes(\phi\widetilde{\nu}_v)^{-1})^{K_1(v^{c_v})}\cong\pi_v^{K'_v}[\phi\widetilde{\nu}_v]\neq 0$.

If $v\in P_\rho$ then $\dim(\rho_{f,p}\otimes(\phi\widetilde{\nu}_v)^{-1})^{I_v}=\dim(\rho\otimes\nu_v^{-1})^{I_v}=0$, hence $(\phi\widetilde{\nu}_v)^{-1}\otimes\rho_{f,p}|_{\mathcal{G}_{F_v}}$ has conductor c_v and so $(\pi_v\otimes(\phi\widetilde{\nu}_v)^{-1})^{K_1(v^{c_v})}\neq 0$.

Finally, the argument of Lemma 4.7 shows that $\pi_\mathfrak{u}$ is unramified, hence $\pi_\mathfrak{u}^{K_0(\mathfrak{u})}$ is two dimensional and contains a unique eigenline for $U_\mathfrak{u}$ with eigenvalue $\widetilde{\alpha}_\mathfrak{u}$ congruent to $\alpha_\mathfrak{u}$ modulo ϖ.

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Therefore, π contributes to $\mathcal{M} \otimes \mathbb{C}$. By the Matsushima–Shimura–Harder isomorphism, this is equivalent to π contributing to $\mathbb{T}' \otimes \mathbb{C}$. Conversely, if π contributes to $\mathbb{T}' \otimes \mathbb{C}$, the same arguments show that π contributes to $\mathcal{T}_{P_\rho \cup \{\mathfrak{u}\}} \otimes \mathbb{C}$.

(ii) Let π be an automorphic representation contributing to $\mathbb{T}' \otimes \mathbb{C}$. As a byproduct of the computations in part (i) we have $\dim \pi_{\mathfrak{u}}^{K_0(\mathfrak{u})}[U_{\mathfrak{u}} - \widetilde{\alpha}_{\mathfrak{u}}] = 1$ and $\dim \pi_v^{K'_v}[\phi\widetilde{\nu}_v] = 1$ for all $v \neq \mathfrak{u}$. By the Matsushima–Shimura–Harder isomorphism, the $[f]$-part of $\mathcal{M} \otimes \mathbb{C}$ is 2^d-dimensional.

(iii) We have to show that for all $v \in S$ the image T'_v (or U'_v) in $\mathrm{End}_{\mathcal{O}}(\mathcal{M})$ belong to $\mathbb{T}'$. The argument uses local Langlands correspondence and the fact that $\mathcal{M}$ is torsion free. As observed in § 1.2 there exists a P_ρ-deformation $\widetilde{\rho}$ of ρ with coefficients in $\mathcal{T}_{P_\rho}$ and by part (i) there is a unique isomorphism of $\mathbb{T}^S$ algebras $\mathcal{T}_{P_\rho} \cong \mathbb{T}'$. It remains to prove that the resulting homomorphism $\mathcal{T}_{P_\rho} \to \mathbb{T}_{\mathfrak{m}}$ is surjective.

If $v \notin \Sigma_\rho$, then the eigenvalue of T'_v on $\pi_v^{K'_v}[\phi\widetilde{\nu}_v]$ equals the eigenvalue of T_v on $(\pi_v \otimes (\phi\widetilde{\nu}_v)^{-1})^{K_1(v^{c_v})}$. Recall that $\nu_v(\varpi_v) = 1$. Hence, the action of T'_v on $\mathcal{M}$ is given by $\mathrm{tr}(\widetilde{\rho} \otimes (\phi\widetilde{\nu}_v)^{-1})(\mathrm{Frob}_v) \in \mathcal{T}_{P_\rho}$.

If $v \in S_\rho$, then the eigenvalue of U'_v on $\pi_v^{K'_v}[\phi\widetilde{\nu}_v]$ equals the eigenvalue of U_v on $(\pi_v \otimes (\phi\widetilde{\nu}_v)^{-1})^{K_1(v^{c_v})}$. Hence, the action of U'_v on $\mathcal{M}$ is given by the eigenvalue of $(\widetilde{\rho} \otimes (\phi\widetilde{\nu}_v)^{-1})(\mathrm{Frob}_v)$ on the line $(\widetilde{\rho} \otimes (\phi\widetilde{\nu}_v)^{-1})^{I_v}$ hence belongs to $\mathcal{T}_{P_\rho}$.

If $v \in \Sigma_\rho \backslash S_\rho$, then $U'_v = 0$. This completes the proof. □

Proposition 5.5. *The local component $\mathcal{M}$ is free of rank 2^d over $\mathcal{T}_{P_\rho}$ and $\mathcal{T}_{P_\rho}$ is Gorenstein.*

Proof. Put $W = \mathrm{H}^d(Y_\rho, \mathbb{V}_\kappa)[\psi, \widetilde{\nu}\phi]_{(\mathfrak{m}_\rho, U_{\mathfrak{u}} - \alpha_{\mathfrak{u}})}$. By Lemma 5.4 and [Dim05, Lemma 6.8], it is enough to show that $W[\mathfrak{m}] = \mathcal{M} \otimes_{\mathbb{T}_{\mathfrak{m}}} \kappa$ is a κ-vector space of dimension at most 2^d.

As in the proof of Theorem 3.1, the condition ($\mathbf{LI}_{\mathrm{Ind}\,\rho}$) implies that every $\mathcal{G}_{\widetilde{F}}$-irreducible subquotient of $W[\mathfrak{m}] \subset W[\mathfrak{m}_\rho]$ is isomorphic to $\otimes \,\mathrm{Ind}_F^{\mathbb{Q}} \rho$. Therefore, it is enough to check that the last graded piece of the Fontaine–Laffaille module attached to $W[\mathfrak{m}]$ has dimension at most one. Again as in the proofs of Theorems 2.3 and 3.1, this amounts to showing that

$$\dim \mathrm{H}^0(\mathcal{Y}_{\rho/\kappa}, \underline{\omega}^k \otimes \underline{\nu}^{-w_0/2})[\psi, \nu, \mathfrak{m}] \leqslant 1. \tag{28}$$

By the q-expansion principle, a Hilbert modular form in $\mathrm{H}^0(\mathcal{Y}_{\rho/\kappa}, \underline{\omega}^k \otimes \underline{\nu}^{-w_0/2})$ is uniquely determined by the coefficients of its q-expansion. The coefficients are indexed by $(F \otimes \widehat{\mathbb{Z}})^\times / \prod_v \ker(\nu_v)$, hence a form in $\mathrm{H}^0(\mathcal{Y}_{\rho/\kappa}, \underline{\omega}^k \otimes \underline{\nu}^{-w_0/2})[\nu]$ is uniquely determined by the subset of its coefficients indexed by $(F \otimes \widehat{\mathbb{Z}})^\times / (\mathfrak{o} \otimes \widehat{\mathbb{Z}})^\times$ which can be identified with the set of ideals of F, and is it a standard fact that coefficients at non-integral ideals vanish.

Finally, the coefficients of a form in $\mathrm{H}^0(\mathcal{Y}_{\rho/\kappa}, \underline{\omega}^k \otimes \underline{\nu}^{-w_0/2})[\nu][\psi, \mathfrak{m}]$ are uniquely determined, since they are related to the eigenvalues of T'_v, S'_v and U'_v, and those are fixed in the $[\psi, \mathfrak{m}]$-part. □

5.4 The modules $\mathcal{M}^Q$

Denote by $Y_{0,Q}$ (respectively, Y^Q) the Hilbert modular varieties of level $K_{0,Q}$ (respectively, K^Q) introduced in § 4.4. The natural homomorphism $Y^Q \to Y_{0,Q}$ induced by the inclusion $K^Q \subset K_{0,Q}$, is étale with group Δ_Q.

Assume that S contains $\Sigma_\rho \cup \{v \mid p\} \cup Q \cup \{\mathfrak{u}\} \cup \{v \mid \phi\widetilde{\nu}_v \text{ ramified}\} = \Sigma_{K^Q} \cup \{v \mid p\}$.

Let $\mathbb{T}'_{0,Q}$ be the image of the Hecke algebra $\mathbb{T}^S$ in the ring of $\mathcal{O}$-linear endomorphisms of

$$\mathcal{M}_{0,Q} := \mathrm{H}^d(Y_{0,Q}, \mathbb{V}_{\mathcal{O}})[\psi, \widetilde{\nu}\phi]_{(\mathfrak{m}_\rho, U_{\mathfrak{u}} - \alpha_{\mathfrak{u}}, U_{\mathfrak{q}} - \alpha_{\mathfrak{q}}; \mathfrak{q} \in Q)}. \tag{29}$$

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Let $\mathbb{T}'_Q$ be the image of the Hecke algebra $\mathbb{T}^S[\Delta_Q]$ in the ring of $\mathcal{O}$-linear endomorphisms of

$$\mathcal{M}^Q := \mathrm{H}^d(Y^Q, \mathbb{V}_{\mathcal{O}})[\psi, \widetilde{\nu}\phi]_{(\mathfrak{m}_\rho, U_{\mathfrak{u}}-\alpha_{\mathfrak{u}}, U_{\mathfrak{q}}-\alpha_{\mathfrak{q}}; \mathfrak{q}\in Q)}. \tag{30}$$

The group Δ_Q acts on $\mathrm{H}^d(Y^Q, \mathbb{V}_{\mathcal{O}})$ via the Hecke operators U_δ, $\delta \in \mathfrak{o}_{\mathfrak{q}}^\times$, $\mathfrak{q} \in Q$ defined in § 4.4. Note that whereas $U_{\mathfrak{q}} \in \mathrm{End}_{\mathcal{O}}(\mathrm{H}^d(Y^Q, \mathbb{V}_{\mathcal{O}})[\psi, \widetilde{\nu}\phi]_{\mathfrak{m}_\rho})$ depends on the choice of a uniformizer, the ideal $(\varpi, U_{\mathfrak{q}} - \alpha_{\mathfrak{q}})$ does not, so $\mathcal{M}^Q$ does not.

Again by Theorem 2.3(i) the modules $\mathcal{M}_{0,Q}$ and $\mathcal{M}^Q$ are free over $\mathcal{O}$, hence $\mathbb{T}'_{0,Q}$ and $\mathbb{T}'_Q$ are torsion free.

By Lemma 5.4, for all $\mathfrak{q} \in Q$, the Hecke operators $T_{\mathfrak{q}}$ and $S_{\mathfrak{q}}$ belong to $\mathcal{T}_{P_\rho} \xrightarrow{\sim} \mathbb{T}'$, hence act on $\mathcal{M}$. By § 5.2 and Hensel's lemma the polynomial $X^2 - T_{\mathfrak{q}}X + S_{\mathfrak{q}}\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) \in \mathcal{T}_{P_\rho}[X]$ has a unique root $\widetilde{\alpha}_{\mathfrak{q}} \in \mathcal{T}_{P_\rho}$ (respectively, $\widetilde{\beta}_{\mathfrak{q}} \in \mathcal{T}_{P_\rho}$) above $\alpha_{\mathfrak{q}}$ (respectively, $\beta_{\mathfrak{q}}$).

Lemma 5.6. *There exists a unique isomorphism of $\mathbb{T}^S$-algebras $\mathbb{T}'_{0,Q} \xrightarrow{\sim} \mathbb{T}'$.*

Proof. As in Lemma 5.4(i) it is enough to show that there is an isomorphism of $\mathbb{T}^S$-algebras $\mathbb{T}'_{0,Q} \otimes \mathbb{C} \xrightarrow{\sim} \mathbb{T}' \otimes \mathbb{C}$.

The local component at $\mathfrak{q}$ of an automorphic representation π contributing to $\mathbb{T}'_{0,Q} \otimes \mathbb{C}$ (or $\mathcal{M}_{0,\mathfrak{q}} \otimes \mathbb{C}$) admits invariants by $K_0(\mathfrak{q})$ and cannot be special (since $\alpha_{\mathfrak{q}} \neq \beta_{\mathfrak{q}}\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q})^{\pm 1}$ by our assumptions in § 5.2); hence, it is necessarily an unramified principal series and so contributes to $\mathcal{M} \otimes \mathbb{C}$ and $\mathbb{T}' \otimes \mathbb{C}$. Moreover, π contributes with the same multiplicity both in $\mathcal{M}_{0,\mathfrak{q}} \otimes \mathbb{C}$ and $\mathcal{M} \otimes \mathbb{C}$. The proof of this fact is very similar to the proof of Lemma 5.4(ii), once we note that for every such π, $\pi_{\mathfrak{q}}^{K_0(\mathfrak{q})}$ is two dimensional and contains a unique eigenline for $U_{\mathfrak{q}}$ with eigenvalue congruent to $\alpha_{\mathfrak{q}}$ modulo ϖ. □

Lemma 5.7. *There is a unique isomorphism of $\mathbb{T}^S[\Delta_Q]$-algebras $\mathcal{T}_{P_\rho \cup Q} \xrightarrow{\sim} \mathbb{T}'_Q$.*

Proof. Both $\mathcal{T}_{P_\rho \cup Q}$ and $\mathbb{T}'_Q$ are defined as images of $\mathbb{T}^S[\Delta_Q]$ hence the uniqueness. For the existence, as in Lemma 5.4(i), it is enough to show that there is an isomorphism of $\mathbb{T}^S[\Delta_Q]$-algebras between $\mathcal{T}_{P_\rho \cup Q} \otimes \mathbb{C}$ and $\mathbb{T}'_Q \otimes \mathbb{C}$.

Consider a (cuspidal) automorphic representation π generated by a holomorphic newform f of weight k, central character $\psi|\cdot|^{-w_0}$ and prime to p conductor.

If π contributes $\mathbb{T}'_Q \otimes \mathbb{C}$, then it necessarily contributes to $\mathcal{T}_{P_\rho \cup Q} \otimes \mathbb{C}$, since by the proof of Lemma 5.4(i) $\rho_{f,p}$ satisfies all of the deformation conditions at primes outside Q, and there is no deformation conditions at primes in Q.

Conversely, suppose that π contributes to $\mathcal{T}_{P_\rho \cup Q} \otimes \mathbb{C}$. By [TW95, Appendix], $\rho_{f,p}|_{\mathcal{G}_{F_{\mathfrak{q}}}}$ is decomposable and $\rho_{f,p}|_{I_{\mathfrak{q}}} \cong \chi \oplus \chi^{-1}$ where χ factors through the natural surjective homomorphism $I_{\mathfrak{q}} \to \mathfrak{o}_{\mathfrak{q}}^\times \to (\mathfrak{o}/\mathfrak{q})^\times \to \Delta_{\mathfrak{q}}$. By the local Langlands correspondence $\pi_{\mathfrak{q}}$ is a principal series induced from two characters whose restriction to $\mathfrak{o}_{\mathfrak{q}}^\times$ are χ and χ^{-1}. It follows that

$$\pi_{\mathfrak{q}}^{K_{\mathfrak{q}}} = \begin{cases} \pi_{\mathfrak{q}}^{K_0(\mathfrak{q})} & \text{if } \chi \text{ is trivial,} \\ (\pi_{\mathfrak{q}} \otimes \chi)^{K_1(\mathfrak{q})} \oplus (\pi_{\mathfrak{q}} \otimes \chi^{-1})^{K_1(\mathfrak{q})} & \text{if } \chi \text{ is non-trivial.} \end{cases} \tag{31}$$

In both cases $\pi_{\mathfrak{q}}^{K_{\mathfrak{q}}}$ is two dimensional and splits under the action of $U_{\mathfrak{q}}$ as a direct sum of two lines, one with eigenvalue $\widetilde{\alpha}_{\mathfrak{q}}$ congruent to $\alpha_{\mathfrak{q}}$ modulo ϖ and one with eigenvalue $\widetilde{\beta}_{\mathfrak{q}}$ congruent to $\beta_{\mathfrak{q}}$ modulo ϖ. Hence, $\pi_{\mathfrak{q}}^{K_{\mathfrak{q}}}[U_{\mathfrak{q}} - \alpha_{\mathfrak{q}}] \neq 0$. Note that whereas $U_{\mathfrak{q}}$ and the eigenvalue depend on the choice of a uniformizer, the decomposition does not.

Also, note that by local Langlands correspondence, the $\Delta_{\mathfrak{q}}$-action on $\mathbb{T}'_Q \otimes \mathbb{C}$ coming from the Hecke action of $K_0(\mathfrak{q})$ on $\pi_{\mathfrak{q}}^{K_{\mathfrak{q}}}$, corresponds to the $\Delta_{\mathfrak{q}}$-action on $\mathcal{T}_{P_\rho \cup Q} \otimes \mathbb{C}$ coming from the $I_{\mathfrak{q}}$-action on $\rho_{f,p}$.

The above discussion at primes in Q together with the arguments of Lemma 5.4(i) at the primes outside Q imply that π contributes to $\mathcal{M}^Q \otimes \mathbb{C}$, hence to $\mathbb{T}'_Q \otimes \mathbb{C}$. □

5.5 The condition (TW3)

Proposition 5.8. *There is a $\mathbb{T}^S$-linear isomorphism $\mathcal{M} \xrightarrow{\sim} \mathcal{M}_{0,Q}$ such that the $U_{\mathfrak{q}}$-action on $\mathcal{M}_{0,Q}$ corresponds to the $\widetilde{\alpha}_{\mathfrak{q}}$-action on $\mathcal{M}$.*

Proof. We may assume that $Q = \{\mathfrak{q}\}$ and prove the lemma with K_ρ replaced by $K_{0,Q\backslash\{\mathfrak{q}\}}$ in the definitions of Y_ρ, $\mathbb{T}'$ and $\mathcal{M}$. Consider the $\mathbb{T}^S$-linear homomorphism:

$$\mathcal{M} \to \mathcal{M}^2, \quad x \mapsto (x, -\widetilde{\beta}_{\mathfrak{q}} \cdot x).$$

Let $U_{\mathfrak{q}}$ be the $\mathbb{T}^S$-linear endomorphism of $\mathcal{M}^2$ given by the matrix $\begin{pmatrix} T_{\mathfrak{q}} & 1 \\ -\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) S_{\mathfrak{q}} & 0 \end{pmatrix}$ acting on the left. Since its eigenvalues $\widetilde{\alpha}_{\mathfrak{q}}$ and $\widetilde{\beta}_{\mathfrak{q}}$ are distinct modulo ϖ, it induces an isomorphism:

$$\mathcal{M} \xrightarrow{\sim} (\mathcal{M}^2)_{(U_{\mathfrak{q}} - \alpha_{\mathfrak{q}})}.$$

Consider the natural degeneracy maps $\mathrm{pr}_1, \mathrm{pr}_2 : Y_{0,\mathfrak{q}} \to Y_\rho$ used in the definition of the Hecke correspondence $T_{\mathfrak{q}}$ in § 2.1. The $\mathbb{T}^S$-linear homomorphism $\mathrm{pr}_1^* + \mathrm{pr}_2^* : \mathrm{H}^d(Y_\rho, \mathbb{V}_{\mathcal{O}})^2 \to \mathrm{H}^d(Y_{0,\mathfrak{q}}, \mathbb{V}_{\mathcal{O}})$ yields (after taking $[\psi, \widetilde{\nu}\phi]$ parts and localizing at $\mathfrak{m}_\rho$):

$$\xi : \mathrm{H}^d(Y_\rho, \mathbb{V}_{\mathcal{O}})[\psi, \widetilde{\nu}\phi]^2_{\mathfrak{m}_\rho} \to \mathrm{H}^d(Y_{0,\mathfrak{q}}, \mathbb{V}_{\mathcal{O}})[\psi, \widetilde{\nu}\phi]_{\mathfrak{m}_\rho}.$$

From the definition of $U_{\mathfrak{q}}$ acting on $\mathcal{M}^2$ we see that ξ is $U_{\mathfrak{q}}$-linear. It is also $U_{\mathfrak{u}}$-linear, hence after localization at $(\varpi, U_{\mathfrak{q}} - \alpha_{\mathfrak{q}}, U_{\mathfrak{u}} - \alpha_{\mathfrak{u}})$ induces

$$\xi' : (\mathcal{M}^2)_{(U_{\mathfrak{q}} - \alpha_{\mathfrak{q}})} \to \mathcal{M}_{0,\mathfrak{q}}.$$

It is enough to show then that ξ' is an isomorphism.

By Lemma 5.6 and its proof, we see that $\xi' \otimes \mathbb{C}$ is an isomorphism. It remains to prove that ξ (hence, ξ') is injective with flat cokernel.

Let $\widehat{\xi}$ be the dual of ξ with respect to the modified Poincaré pairing defined in § 2.5. The matrix of $\widehat{\xi} \circ \xi : (\mathcal{M} \otimes \kappa)^2 \to (\mathcal{M} \otimes \kappa)^2$ is given by $\begin{pmatrix} 1+\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) & T_{\mathfrak{q}} \\ S_{\mathfrak{q}}^{-1} T_{\mathfrak{q}} & 1+\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) \end{pmatrix}$. It is invertible by our assumptions on $\mathfrak{q}$. Therefore, ξ is injective with flat cokernel. □

By § 5.2, $\mathcal{R}_{P_\rho \cup Q}$ is a $\mathcal{O}[\Delta_Q]$-algebra. Hence, the surjective homomorphism of local $\mathcal{O}$-algebras $\pi_\Sigma : \mathcal{R}_{P_\rho \cup Q} \to \mathcal{T}_{P_\rho \cup Q}$ defined in § 1.2 endows $\mathcal{T}_{P_\rho \cup Q}$ with $\mathcal{O}[\Delta_Q]$-algebra structure.

Proposition 5.9. *The local component $\mathcal{M}^Q$ is a free $\mathcal{O}[\Delta_Q]$-module and $\mathcal{M}^Q \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \cong \mathcal{M}_{0,Q}$ as $\mathbb{T}^S$-modules.*

Proof. By Theorem 2.4(i) $\mathrm{H}^d(Y^Q, \mathbb{V}_{\mathcal{O}})_{\mathfrak{m}_\rho}$ is free over $\mathcal{O}[\Delta_Q]$ and the $\mathbb{T}^S$-module of its Δ_Q-coinvariants is isomorphic to $\mathrm{H}^d(Y_{0,Q}, \mathbb{V}_{\mathcal{O}})_{\mathfrak{m}_\rho}$. If the class group $\mathcal{C}_\rho$ defined in § 4.5 has order prime to p (in particular, ϕ is trivial), then the claim follows simply by taking the $[\psi, \widetilde{\nu}]$-part. In fact, the $[\psi, \widetilde{\nu}]$-part, for the action of a prime to p order group, of a free $\mathcal{O}[\Delta_Q]$-module is a free $\mathcal{O}[\Delta_Q]$-direct factor.

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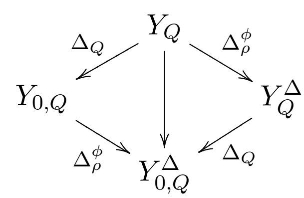
In the general case, denote by Δ_ρ^ϕ the p-Sylow subgroup of $\mathcal{C}_\rho \times \prod_{v\in\Sigma_\rho}(\mathfrak{o}_v^\times/\ker(\phi_v))$. As in § 4.5 the p-group $\prod_{v\in\Sigma_\rho}(\mathfrak{o}_v^\times/\ker(\phi_v))$ injects in $\mathbb{A}^\times K_0(Q\mathfrak{n}_\varnothing)/\mathbb{A}^\times K_{0,Q}$ and *a fortiori* in $\mathbb{A}^\times K_0(Q\mathfrak{n}_\varnothing)/\mathbb{A}^\times K^Q$. Also the morphisms $Y_Q \to Y_Q^{\mathrm{ad}}$ and $Y_{0,Q} \to Y_{0,Q}^{\mathrm{ad}}$ are étale with group $\mathcal{C}_\rho$. Hence, the étale morphism $Y_Q \to Y_{Q\mathfrak{n}_\varnothing}^{\mathrm{ad}}$ (respectively, $Y_{0,Q} \to Y_{Q\mathfrak{n}_\varnothing}^{\mathrm{ad}}$) factors through an étale morphism $Y_Q \to Y_Q^\Delta$ (respectively, $Y_{0,Q} \to Y_{0,Q}^\Delta$) with group Δ_ρ^ϕ. Then Theorem 2.4(i) applies to each of the five étale morphisms in the following diagram.



(32)

In particular, $\mathrm{H}^d(Y^Q, \mathbb{V}_\mathcal{O})_{\mathfrak{m}_\rho}$ is free over $\mathcal{O}[\Delta_\rho^\phi \times \Delta_Q]$, hence $\mathrm{H}^d(Y^Q, \mathbb{V}_\mathcal{O})_{\mathfrak{m}_\rho}[\phi]$ is free over $\mathcal{O}[\Delta_Q]$ and

$$\mathrm{H}^d(Y^Q, \mathbb{V}_\mathcal{O})_{\mathfrak{m}_\rho}[\phi] \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \cong (\mathrm{H}^d(Y^Q, \mathbb{V}_\mathcal{O})_{\mathfrak{m}_\rho} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O})[\phi] \cong \mathrm{H}^d(Y_{0,Q}, \mathbb{V}_\mathcal{O})_{\mathfrak{m}_\rho}[\phi].$$

Further taking the $[\psi\phi^{-2}, \widetilde{\nu}]$ part, for the action of the prime to p order group $(\mathcal{C}_\rho/\Delta_\rho^\phi) \times \prod_v(\mathfrak{o}_v^\times/\ker(\nu_v))$, and using the argument invoked in the beginning of the proof, yields the desired result. □

So far we have constructed a Taylor–Wiles system $\{\mathcal{R}, \mathcal{M}, (\mathcal{R}_Q, \mathcal{M}^Q)_{Q\in\mathcal{Q}}\}$ for the family $\mathcal{Q}$ of sets Q containing a finite number of primes $\mathfrak{q}$ as in § 5.2. The aim of the next section is to find a subfamily $\{Q_m \mid m \in \mathbb{N}\}$ satisfying the conditions (i) and (ii) of Theorem 5.3.

5.6 Selmer groups

We assume in this section that $P_\rho = \varnothing$. Let $\rho_{f,p}$ be a modular deformation of ρ as in $(\mathbf{Mod}_\rho)$. For $r \geqslant 1$ we put $\rho_r := \rho_{f,p} \mod \varpi^r$, so that $\rho_1 = \rho$.

We use Galois cohomology techniques in order to control the number of generators of $\mathcal{R}_Q$.

DEFINITION 5.10. For $v \mid p$ the subgroup $\mathrm{H}^1_{\mathrm{f}}(F_v, \mathrm{Ad}^0\,\rho_r) \subset \mathrm{H}^1(F_v, \mathrm{Ad}^0\,\rho_r)$ consists of classes corresponding to crystalline extensions of ρ_r by itself.

For $v \nmid p$ the subgroup of unramified classes $\mathrm{H}^1_{\mathrm{f}}(F_v, \mathrm{Ad}^0\,\rho_r) \subset \mathrm{H}^1(F_v, \mathrm{Ad}^0\,\rho_r)$ is defined as $\mathrm{H}^1(\mathcal{G}_{F_v}/I_v, (\mathrm{Ad}^0\,\rho_r)^{I_v})$.

DEFINITION 5.11. The Selmer groups associated with a finite set of primes Σ are defined as

$$\mathrm{H}^1_\Sigma(F, \mathrm{Ad}^0\,\rho_r) = \ker\Big(\mathrm{H}^1(F, \mathrm{Ad}^0\,\rho_r) \to \bigoplus_{v\notin\Sigma} \mathrm{H}^1(F_v, \mathrm{Ad}^0\,\rho_r)/\mathrm{H}^1_{\mathrm{f}}(F_v, \mathrm{Ad}^0\,\rho_r)\Big)$$

and

$$\mathrm{H}^1_\Sigma(F, \mathrm{Ad}^0\,\rho_{f,p} \otimes \mathbb{Q}_p/\mathbb{Z}_p) = \varinjlim \mathrm{H}^1_\Sigma(F, \mathrm{Ad}^0\,\rho_r).$$

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The dual of $\mathrm{Ad}^0\,\rho$ is canonically isomorphic to its Tate twist $\mathrm{Ad}^0\,\rho(1)$. The corresponding dual Selmer group $\mathrm{H}^1_{\Sigma^*}(F, \mathrm{Ad}^0\,\rho(1))$ is defined as the kernel of the map

$$\mathrm{H}^1(F, \mathrm{Ad}^0\,\rho(1)) \to \bigoplus_{v\in\Sigma} \mathrm{H}^1(F_v, \mathrm{Ad}^0\,\rho(1)) \bigoplus_{v\notin\Sigma} \mathrm{H}^1(F_v, \mathrm{Ad}^0\,\rho(1))/\mathrm{H}^1_{\mathrm{f}}(F_v, \mathrm{Ad}^0\,\rho(1)).$$

The Poitou–Tate exact sequence yields the following formula:

$$\frac{\#\mathrm{H}^1_{\Sigma}(F, \mathrm{Ad}^0\,\rho)}{\#\mathrm{H}^1_{\Sigma^*}(F, \mathrm{Ad}^0\,\rho(1))} = \frac{\#\mathrm{H}^0(F, \mathrm{Ad}^0\,\rho)}{\#\mathrm{H}^0(F, \mathrm{Ad}^0\,\rho(1))} \prod_{v\in\Sigma} \frac{\#\mathrm{H}^1(F_v, \mathrm{Ad}^0\,\rho_v)}{\#\mathrm{H}^0(F_v, \mathrm{Ad}^0\,\rho_v)} \prod_{v\mid p\infty} \frac{\#\mathrm{H}^1_{\mathrm{f}}(F_v, \mathrm{Ad}^0\,\rho_v)}{\#\mathrm{H}^0(F_v, \mathrm{Ad}^0\,\rho_v)}. \quad (33)$$

A proof for $F=\mathbb{Q}$ can be found in [Wil95, Proposition 1.6], but as mentioned in [DDT97, Theorem 2.19] the same argument works over an arbitrary number field.

By $(\mathbf{LI}_{\mathrm{Ind}\,\rho})$ we have $\mathrm{H}^0(F, \mathrm{Ad}^0\,\rho) = \mathrm{H}^0(F, \mathrm{Ad}^0\,\rho(1)) = 0$. Since ρ is totally odd, for all $v \mid \infty$ we have $\dim \mathrm{H}^0(F_v, \mathrm{Ad}^0\,\rho_v) = 1$. Since ρ is crystalline at all places v dividing p we have

$$\dim \mathrm{H}^1_{\mathrm{f}}(F_v, \mathrm{Ad}^0\,\rho_v) - \dim \mathrm{H}^0(F_v, \mathrm{Ad}^0\,\rho_v) \leqslant [F_v : \mathbb{Q}_p] \quad (34)$$

(cf. [Fuj06a, Theorem 3.20] and also [DFG04, Corollary 2.3]). Finally, for all $\mathfrak{q}\in Q$, $\dim \mathrm{H}^0(F_{\mathfrak{q}}, \mathrm{Ad}^0\,\rho_{\mathfrak{q}}(1)) = 1$. Putting all of this together we obtain the following result.

Lemma 5.12. *We have* $\dim \mathrm{H}^1_Q(F, \mathrm{Ad}^0\,\rho) \leqslant \mathrm{H}^1_{Q^*}(F, \mathrm{Ad}^0\,\rho(1)) + \#Q$.

Finally, by the same arguments as in [Wil95, § 3] we obtain the following lemma.

Lemma 5.13. *Let $m \geqslant 1$ be an integer. Then for each non-zero element $x \in \mathrm{H}^0_{\varnothing^*}(F, \mathrm{Ad}^0\,\rho(1))$ there exists a prime $\mathfrak{q}$ such that:*

- $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) \equiv 1 \pmod{p^m}$;
- *ρ is unramified at $\mathfrak{q}$ and $\rho(\mathrm{Frob}_{\mathfrak{q}})$ has two distinct eigenvalues in κ; and*
- *the image by the restriction map of x in $\mathrm{H}^1_{\mathrm{f}}(F_{\mathfrak{q}}, \mathrm{Ad}^0\,\rho(1))$ is non-trivial.*

Put $r := \dim \mathrm{H}^0_{\varnothing^*}(F, \mathrm{Ad}^0\,\rho(1))$. For each $m \geqslant 1$, let Q_m be the set of primes $\mathfrak{q}$ corresponding by the above lemma to the elements of a basis of $\mathrm{H}^0_{\varnothing^*}(F, \mathrm{Ad}^0\,\rho(1))$. Then $\mathrm{H}^0_{Q_m^*}(F, \mathrm{Ad}^0\,\rho(1)) = 0$ and by Lemma 5.12 we obtain $\dim \mathrm{H}^0_{Q_m}(F, \mathrm{Ad}^0\,\rho) \leqslant \#Q_m$. Therefore, $\mathcal{R}_m$ is generated by at most $\#Q_m = r$ elements. This completes the proof of Theorem 5.1.

6. Raising the level

6.1 Numerical invariants

Definition 6.1. For a local complete Noetherian $\mathcal{O}$-algebra A endowed with a surjective homomorphism $\theta_A : A \to \mathcal{O}$, we define the following two invariants:

- the congruence ideal $\eta_A := \theta_A(\mathrm{Ann}_A(\ker \theta_A)) \subset \mathcal{O}$; and
- the module of relative differentials $\Phi_A := \Omega^1_{A/\mathcal{O}} = \ker \theta_A/(\ker \theta_A)^2$.

Here we state Wiles' numerical criterion.

Theorem 6.2 [DDT97, Theorem 3.40]. *Let $\pi : \mathcal{R} \twoheadrightarrow \mathcal{T}$ be a surjective homomorphism such that $\theta_{\mathcal{R}} = \pi \circ \theta_{\mathcal{T}}$. Assume that $\mathcal{T}$ is finite and flat over $\mathcal{O}$ and $\eta_{\mathcal{T}} \neq (0)$. Then the following three*

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conditions are equivalent:

(i) $\#\Phi_{\mathcal{R}} \leqslant \#(\mathcal{O}/\eta_{\mathcal{T}})$;

(ii) $\#\Phi_{\mathcal{R}} = \#(\mathcal{O}/\eta_{\mathcal{T}})$; *and*

(iii) $\mathcal{R}$ *and* $\mathcal{T}$ *are complete intersections over* $\mathcal{O}$ *and* π *is an isomorphism.*

We consider couples $(\mathcal{T}, \mathcal{M})$ consisting of a finite and flat $\mathcal{O}$-algebra $\mathcal{T}$ and a $\mathcal{T}$-module $\mathcal{M}$ which is a finitely generated free $\mathcal{O}$-module endowed with a perfect $\mathcal{T}$-linear pairing $\langle\cdot,\cdot\rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{O}$ and such that $\mathcal{M} \otimes E$ is free over $\mathcal{T} \otimes E$ of a given rank (in our application this rank will be 2^d). The pairing induces an isomorphism of $\mathcal{T}$-modules $\mathcal{M} \xrightarrow{\sim} \mathrm{Hom}(\mathcal{M}, \mathcal{O})$.

From [DDT97, Lemma 4.17] and [Dia97b, Theorem 2.4] we deduce the following.

Proposition 6.3. *Let $(\mathcal{T}, \mathcal{M})$ and $(\mathcal{T}', \mathcal{M}')$ be two couples as above. Assume that we have a surjective homomorphism $\mathcal{T}' \twoheadrightarrow \mathcal{T}$ and a $\mathcal{T}'$-linear injective homomorphism $\xi : \mathcal{M} \hookrightarrow \mathcal{M}'$ inducing via $\langle\cdot,\cdot\rangle$ a surjective homomorphism $\widehat{\xi} : \mathcal{M}' \twoheadrightarrow \mathcal{M}$.*

If $\mathcal{M}$ is free over $\mathcal{T}$ and if $\widehat{\xi} \circ \xi(\mathcal{M}) = T \cdot \mathcal{M}$ for some $T \in \mathcal{T}$ then

$$\#(\mathcal{O}/\eta_{\mathcal{T}})\#(\mathcal{O}/\theta_{\mathcal{T}}(T)) \leqslant \#(\mathcal{O}/\eta_{\mathcal{T}'}).$$

Moreover, equality holds if, and only if, $\mathcal{M}'$ is free over $\mathcal{T}'$.

6.2 Proof of Theorem A

Let Σ be a finite set of primes containing P_ρ. We start by redefining $\mathcal{T}_\Sigma$ geometrically.

Let Y_Σ be the Hilbert modular variety of level K_Σ defined in § 4.4.

Let S be a finite set of primes containing $\Sigma_\rho \cup \Sigma \cup \{v \mid p\} \cup \{\mathfrak{u}\} \cup \{v \mid \phi\widetilde{\nu}_v \text{ ramified}\} = \Sigma_{K_\Sigma} \cup \{v \mid p\}$.

Let $\mathbb{T}'_\Sigma$ be the image of $\mathbb{T}^S$ in the ring of $\mathcal{O}$-linear endomorphisms of

$$\mathcal{M}_\Sigma := \mathrm{H}^d(Y_\Sigma, \mathbb{V}_{\mathcal{O}})[\psi, \widetilde{\nu}\phi]_{(\mathfrak{m}_\rho, U_{\mathfrak{u}} - \alpha_{\mathfrak{u}}, U''_{\mathfrak{q}}; \mathfrak{q} \in \Sigma)}. \tag{35}$$

By Theorem 2.3(i) $\mathcal{M}_\Sigma$ is free of finite rank over $\mathcal{O}$.

For every Hilbert modular newform f occurring in $\mathcal{T}_\Sigma$ we denote by $\theta_f^\Sigma : \mathcal{T}_\Sigma \to \mathcal{O}$ the projection on the f-component and by η_f^Σ the corresponding congruence ideal.

Lemma 6.4. *We have the following:*

(i) *there is a unique isomorphism of $\mathbb{T}^S$-algebras $\mathbb{T}'_\Sigma \cong \mathcal{T}_\Sigma$;*

(ii) *$\mathcal{M}_\Sigma \otimes \mathbb{C}$ is free of rank 2^d over $\mathcal{T}_\Sigma \otimes \mathbb{C}$ and $U''_{\mathfrak{q}}$ acts as zero on it for all $\mathfrak{q} \in \Sigma$.*

Proof. We follow closely the proofs of Lemmas 5.4 and 5.7. The main point here is to show that, if f is a Hilbert modular newform occurring in $\mathcal{T}_\Sigma \otimes \mathbb{C}$ and π denotes the corresponding automorphic representation, then for all $\mathfrak{q} \in \Sigma$,

$$(\pi_{\mathfrak{q}})^{K''_{\mathfrak{q}}}[\phi_{\mathfrak{q}}\widetilde{\nu}_{\mathfrak{q}}] = (\pi_{\mathfrak{q}} \otimes \phi_{\mathfrak{q}}^{-1}\widetilde{\nu}_{\mathfrak{q}}^{-1})^{K_1(\mathfrak{q}^{c_{\mathfrak{q}}}) \cap K_0(\mathfrak{q}^{c_{\mathfrak{q}}+d_{\mathfrak{q}}})}$$

contains a unique eigenline for $U''_{\mathfrak{q}}$ with eigenvalue congruent to zero modulo ϖ (and this eigenvalue is actually zero). We distinguish three cases.

- If $(\widetilde{\nu}_{\mathfrak{q}}\phi_{\mathfrak{q}})^{-1} \otimes \rho_{f,p}$ is unramified at $\mathfrak{q}$, then necessarily $d_{\mathfrak{q}} = 2$, $c_{\mathfrak{q}} = 0$ and

$$\dim\big((\pi_{\mathfrak{q}} \otimes \phi_{\mathfrak{q}}^{-1}\widetilde{\nu}_{\mathfrak{q}}^{-1})^{K_0(\mathfrak{q}^2)}\big) = 3.$$

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The characteristic polynomial of $U''_{\mathfrak{q}} = [K_0(\mathfrak{q}^2) \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{q}} \end{smallmatrix}\right) K_0(\mathfrak{q}^2)]$ acting on it is given by

$$X(X^2 - c(f,\mathfrak{q})X + \psi(\mathfrak{q})\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q})^{w_0+1}) = \theta_f^{\Sigma}(X(X^2 - T'_{\mathfrak{q}}X + S'_{\mathfrak{q}}\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}))),$$

and $X=0$ is a simple root modulo ϖ of this polynomial.

- If $\dim((\widetilde{\nu}_{\mathfrak{q}}\phi_{\mathfrak{q}})^{-1} \otimes \rho_{f,p})^{I_{\mathfrak{q}}} = 1$, then $d_{\mathfrak{q}} \geqslant 1$ and

$$\dim\big((\pi_{\mathfrak{q}} \otimes \phi_{\mathfrak{q}}^{-1}\widetilde{\nu}_{\mathfrak{q}}^{-1})^{K_1(\mathfrak{q}^{c_{\mathfrak{q}}}) \cap K_0(\mathfrak{q}^{c_{\mathfrak{q}}+d_{\mathfrak{q}}})}\big) = 2.$$

The characteristic polynomial of $U''_{\mathfrak{q}} = [K_0(\mathfrak{q}^{c_{\mathfrak{q}}+d_{\mathfrak{q}}}) \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{q}} \end{smallmatrix}\right) K_0(\mathfrak{q}^{c_{\mathfrak{q}}+d_{\mathfrak{q}}})]$ acting on it is given by

$$X(X - c(f,\mathfrak{q})) = \theta_f^{\Sigma}(X(X - U'_{\mathfrak{q}})),$$

where $U'_{\mathfrak{q}} = [K_0(\mathfrak{q}^{c_{\mathfrak{q}}+d_{\mathfrak{q}}-1}) \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{q}} \end{smallmatrix}\right) K_0(\mathfrak{q}^{c_{\mathfrak{q}}+d_{\mathfrak{q}}-1})]$ and $X=0$ is simple root modulo ϖ of this polynomial.

- Finally, if $((\widetilde{\nu}_{\mathfrak{q}}\phi_{\mathfrak{q}})^{-1} \otimes \rho_{f,p})^{I_{\mathfrak{q}}} = \{0\}$, then

$$\dim\big((\pi_{\mathfrak{q}} \otimes \phi_{\mathfrak{q}}^{-1}\widetilde{\nu}_{\mathfrak{q}}^{-1})^{K_1(\mathfrak{q}^{c_{\mathfrak{q}}}) \cap K_0(\mathfrak{q}^{c_{\mathfrak{q}}+d_{\mathfrak{q}}})}\big) = 1,$$

and $U''_{\mathfrak{q}} = 0$ on it.

This completes the proof. □

By § 1.2 we have a surjection $\pi_{\Sigma} : \mathcal{R}_{\Sigma} \to \mathcal{T}_{\Sigma}$. Therefore, we may endow $\mathcal{R}_{\Sigma}$ with a surjective homomorphism $\theta_f^{\Sigma} \circ \pi_{\Sigma} : \mathcal{R}_{\Sigma} \to \mathcal{O}$ and we denote by Φ_f^{Σ} the corresponding numerical invariant.

Proposition 6.5 (Wiles [Wil95, Proposition 1.2]). *We have*

$$\mathrm{Hom}_{\mathcal{O}}(\Phi_f^{\Sigma}, E/\mathcal{O}) \cong \mathrm{H}^1_{\Sigma}(F, \mathrm{Ad}^0(\rho_{f,p}) \otimes \mathbb{Q}_p/\mathbb{Z}_p).$$

By (16) there exists a perfect $\mathcal{T}_{\Sigma}$-linear pairing:

$$\langle \cdot, \cdot \rangle : \mathcal{M}_{\Sigma} \times \mathcal{M}_{\Sigma} \to \mathcal{O}, \tag{36}$$

analogous to that defined in [DFG04, 1.5.3, 1.8.1] in the case $F = \mathbb{Q}$ (note that since $\Sigma \supset P_{\rho}$ we do not need the rather technical [DFG04, Lemma 1.5]).

Theorem A is implied by the first part of the following.

Theorem 6.6. *Let $\rho : \mathcal{G}_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation satisfying $(\mathbf{LI}_{\mathrm{Ind}\rho})$ and $(\mathbf{Mod}_{\rho})$. Let Σ be a finite set of primes containing P_{ρ}. Then $\pi_{\Sigma} : \mathcal{R}_{\Sigma} \to \mathcal{T}_{\Sigma}$ is an isomorphism of complete intersections over $\mathcal{O}$ and $\mathcal{M}_{\Sigma}$ is free of finite rank over $\mathcal{T}_{\Sigma}$. In particular, all Σ-ramified deformations of ρ are modular.*

Moreover, for all Hilbert modular newforms f such that $\rho_{f,p}$ is a Σ-ramified deformations of ρ:

$$\#\mathrm{H}^1_{\Sigma}(F, \mathrm{Ad}^0(\rho_{f,p}) \otimes \mathbb{Q}_p/\mathbb{Z}_p) = \#(\mathcal{O}/\eta_f^{\Sigma}) < \infty. \tag{37}$$

Proof. We proceed by induction on $\#\Sigma$. Assume first that $\Sigma = P_{\rho}$. We already know that $\pi_{P_{\rho}} : \mathcal{R}_{P_{\rho}} \to \mathcal{T}_{P_{\rho}}$ is an isomorphism of complete intersections over $\mathcal{O}$ and $\mathcal{M}_P := \mathcal{M}$ is free of rank 2^d over $\mathcal{T}_{P_{\rho}}$ (cf. Theorem 5.1 if $P_{\rho} = \varnothing$ and Proposition 5.5 together with Fujiwara [Fuj06a, Theorem 9.1] in general).

Assume now that the theorem holds for some $\Sigma \supset P_{\rho}$, that is to say $\pi_{\Sigma} : \mathcal{R}_{\Sigma} \to \mathcal{T}_{\Sigma}$ is an isomorphism of complete intersections over $\mathcal{O}$ and that $\mathcal{M}_{\Sigma}$ is free over $\mathcal{T}_{\Sigma}$. In particular, we have $\#\Phi_f^{\Sigma} = \#(\mathcal{O}/\eta_f^{\Sigma})$, where f is a newform contributing to $\mathcal{M}$.

Let $\mathfrak{q}$ be a prime outside Σ not dividing p. Put $\Sigma' = \Sigma \cup \{\mathfrak{q}\}$.

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It follows directly from Proposition 6.5 and Definition 5.11 that

$$\#\Phi_f^{\Sigma'} \leqslant \#\Phi_f^{\Sigma} \cdot \#\mathrm{H}^0(F_{\mathfrak{q}}, (\mathrm{Ad}^0(\rho_{f,p}) \otimes \mathbb{Q}_p/\mathbb{Z}_p)(1)).$$

By Theorem 6.2 and Proposition 6.3, the theorem will hold for Σ' if we construct a surjective homomorphism $\mathcal{T}_{\Sigma'} \twoheadrightarrow \mathcal{T}_{\Sigma}$ compatible with the surjections θ_f^{Σ} and $\theta_f^{\Sigma'}$ and a $\mathcal{T}_{\Sigma'}$-linear injective homomorphism $\xi : \mathcal{M}_{\Sigma} \hookrightarrow \mathcal{M}_{\Sigma'}$ inducing a surjection $\widehat{\xi} : \mathcal{M}_{\Sigma'} \twoheadrightarrow \mathcal{M}_{\Sigma}$ such that $\widehat{\xi} \circ \xi(\mathcal{M}_{\Sigma}) = T \cdot \mathcal{M}_{\Sigma}$ for some $T \in \mathcal{T}_{\Sigma}$ satisfying

$$\#(\mathcal{O}/\theta_f^{\Sigma}(T)) = \#\mathrm{H}^0(F_{\mathfrak{q}}, (\mathrm{Ad}^0(\rho_{f,p}) \otimes \mathbb{Q}_p/\mathbb{Z}_p)(1)). \tag{38}$$

This is done on a case-by-case basis, depending on the local behavior of ρ at $\mathfrak{q}$ (cf. Definition 4.2).

The case $\mathfrak{q} \in \Sigma_\rho \backslash S_\rho$ is relatively straightforward, since adding such a prime does not change $\mathcal{M}_{\Sigma}$. We distinguish two more cases.

(1) Assume that $\mathfrak{q} \notin \Sigma_\rho$. In this case $\rho_{\mathfrak{q}} \otimes \nu_{\mathfrak{q}}^{-1}$ is unramified.

By Theorem 3.1 and Proposition 3.3, the homomorphism

$$\mathrm{pr}_3^* \mathrm{pr}_1^* + \mathrm{pr}_3^* \mathrm{pr}_2^* + \mathrm{pr}_4^* \mathrm{pr}_1^* : \mathcal{M}_{\Sigma}^{\oplus 3} = \mathrm{H}^d(Y_{\Sigma}, \mathbb{V}_{\mathcal{O}})[\psi, \widetilde{\nu}\phi]_{\mathfrak{m}_\rho}^{\oplus 3} \to \mathrm{H}^d(Y_{\Sigma'}, \mathbb{V}_{\mathcal{O}})[\psi, \widetilde{\nu}\phi]_{\mathfrak{m}_\rho}$$

is injective with flat cokernel.

The characteristic polynomial of $U''_{\mathfrak{q}}$ acting on $\mathcal{M}_{\Sigma}^{\oplus 3}$ is $X(X^2 - T_{\mathfrak{q}}X + S_{\mathfrak{q}}\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}))$ and $X = 0$ is simple root modulo ϖ of this polynomial. Hence, the localization of the above injection at $(U''_{\mathfrak{q}}, U_{\mathfrak{u}} - \alpha_{\mathfrak{u}})$ yields another injection with flat cokernel:

$$\xi : \mathcal{M}_{\Sigma} \xrightarrow{\sim} (\mathcal{M}_{\Sigma}^{\oplus 3})_{U''_{\mathfrak{q}}} \hookrightarrow \mathcal{M}_{\Sigma'}.$$

This gives a surjective homomorphism $\mathcal{T}_{\Sigma'} \twoheadrightarrow (\mathcal{T}_{\Sigma}^3)_{U''_{\mathfrak{q}}} \cong \mathcal{T}_{\Sigma}$. Computations performed by Wiles [Wil95, § 2] and Fujiwara [Fuj06a, § 10] show that $\widehat{\xi} \circ \xi(\mathcal{M}_{\Sigma}) = T \cdot \mathcal{M}_{\Sigma}$ with

$$T = (\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) - 1)(T_{\mathfrak{q}}^2 - S_{\mathfrak{q}}(\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) + 1)^2).$$

Then (38) follows by a straightforward computation.

(2) Assume that $\mathfrak{q} \in S_\rho$. In this case $\dim(\rho_{\mathfrak{q}} \otimes \nu_{\mathfrak{q}}^{-1})^{I_v} = 1$.

By Proposition 3.3 there is an exact sequence whose last arrow has a flat cokernel:

$$0 \to \mathrm{H}^d(Y_{K_\Sigma \cdot K'''_{\mathfrak{q}}}, \mathbb{V}_{\mathcal{O}})[\psi, \widetilde{\nu}\phi]_{\mathfrak{m}_\rho} \xrightarrow{(\mathrm{pr}_1^*, -\mathrm{pr}_2^*)} \mathrm{H}^d(Y_{\Sigma}, \mathbb{V}_{\mathcal{O}})[\psi, \widetilde{\nu}\phi]_{\mathfrak{m}_\rho}^{\oplus 2} \xrightarrow{\mathrm{pr}_3^* + \mathrm{pr}_4^*} \mathrm{H}^d(Y_{\Sigma'}, \mathbb{V}_{\mathcal{O}})[\psi, \widetilde{\nu}\phi]_{\mathfrak{m}_\rho},$$

where $K'''_{\mathfrak{q}} = \ker(K_1(\mathfrak{q}^{c_{\mathfrak{q}}-1}) \xrightarrow{\det} \mathfrak{o}_{\mathfrak{q}}^{\times} \xrightarrow{\widetilde{\nu}_{\mathfrak{q}}\phi} \mathcal{O}^{\times})$.

The characteristic polynomial of $U''_{\mathfrak{q}}$ acting on $(\mathrm{pr}_3^* + \mathrm{pr}_4^*)(\mathcal{M}_{\Sigma}^{\oplus 2})$ is $X(X - U'_{\mathfrak{q}})$ and $X = 0$ is simple root modulo ϖ of this polynomial. Hence, the localization of the map $\mathrm{pr}_3^* + \mathrm{pr}_4^*$ at $\mathfrak{m}_{\Sigma'} = (\mathfrak{m}_{\Sigma}, U''_{\mathfrak{q}})$ yields an injection with flat cokernel:

$$\xi : \mathcal{M}_{\Sigma} \xrightarrow{\sim} (\mathcal{M}_{\Sigma}^{\oplus 2})_{U''_{\mathfrak{q}}} \hookrightarrow \mathcal{M}_{\Sigma'}.$$

This gives a surjective homomorphism $\mathcal{T}_{\Sigma'} \twoheadrightarrow (\mathcal{T}_{\Sigma}^2)_{U''_{\mathfrak{q}}} \cong \mathcal{T}_{\Sigma}$. Computations performed by Wiles [Wil95, § 2] and Fujiwara [Fuj06a, § 10] show that $\widehat{\xi} \circ \xi(\mathcal{M}_{\Sigma}) = T \cdot \mathcal{M}_{\Sigma}$ with

$$T = \begin{cases} \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) - 1 & \text{if } \rho_{\mathfrak{q}} \text{ is decomposable,} \\ \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q})^2 - 1 & \text{if } \rho_{\mathfrak{q}} \text{ is indecomposable.} \end{cases}$$

As above, (38) is obtained by a straightforward computation. □

6.3 Towards the modularity of a quintic threefold

We now give an example coming from the geometry where Theorem A applies. Consani and Scholten [CS01] consider the middle degree cohomology of a quintic threefold $\widetilde{X}$ (a proper and smooth $\mathbb{Z}[1/30]$-scheme with Hodge numbers $h^{3,0}=h^{2,1}=1$, $h^{2,0}=h^{1,0}=0$ and $h^{1,1}=141$). They show that the $\mathcal{G}_{\mathbb{Q}}$-representation $\mathrm{H}^3(\widetilde{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)$ is induced from a two-dimensional representation $\widetilde{\rho}$ of $\mathcal{G}_{\mathbb{Q}(\sqrt{5})}$ and conjecture the modularity of $\widetilde{\rho}$. As explained in [DD06], Theorem 6.6 implies the following proposition.

Proposition 6.7 [DD06]. *Assume that $p \geqslant 7$ and that $\widetilde{\rho}$ is congruent modulo p to the p-adic Galois representation attached to a Hilbert modular form on $\mathbb{Q}(\sqrt{5})$ of weight $(2,4)$ and some prime to p level. Then $\widetilde{\rho}$ is modular and, in particular, the L-function associated with $\mathrm{H}^3(\widetilde{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)$ has an analytic continuation to the whole complex plane and satisfies a functional equation.*

7. Cardinality of the adjoint Selmer group

In this section we give a proof of Theorem B. It is enough to establish part (ii), since then the finiteness of $\mathrm{H}^1_{\mathrm{f}}(F, \mathrm{Ad}^0(\rho_{f,p}) \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ implies part (i) by the same argument as in [DFG04, § 2.2].

Choose a finite set Σ of primes not dividing p, containing the auxiliary prime $\mathfrak{u}$ and all primes (not dividing p) at which $\mathrm{Ad}^0(\rho_{f,p})$ is ramified. Denote by f_Σ the automorphic form contributing to $\mathcal{M}_\Sigma$ corresponding to the newform f of Theorem B.

7.1 Periods of automorphic forms

For $J \subset J_F$ denote by ϵ_J the corresponding character of the Weyl group $\left(\begin{smallmatrix}1 & 0\\ 0 & \pm 1\end{smallmatrix}\right)^{J_F} \subset \mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$. Put $F_\infty = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)^{J_F} \in \mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$.

We fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$ extending the embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Since $\mathcal{M}_\Sigma$ is free over the principal domain $\mathcal{O}$, Lemma 6.4(ii) implies that $\mathcal{M}_\Sigma[f, \epsilon_J]$ is a free $\mathcal{O}$-module of rank one, where $[\epsilon_J]$ denotes the eigenspace for this character and $[f]$ denotes $\bigcap_{v \notin S} \ker(T_v - c(f,v))$.

Let $S_k(K_\Sigma; \psi, \widetilde{\nu}\phi)$ be the $\mathbb{C}$-vector space of automorphic forms on $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})$ which are holomorphic of weight $(k; w_0)$ at infinity and right $K_0(\mathfrak{n}_\Sigma)$-equivariant for the character $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \mapsto \psi(a)\widetilde{\nu}\phi((ad-bc)/a^2)$. In particular, such a form is right K_Σ-invariant. Let

$$\begin{aligned} \delta_J &: S_k(K_\Sigma; \psi, \widetilde{\nu}\phi) \to \mathrm{H}^d_{\mathrm{cusp}}(Y_\Sigma, \mathbb{V}_{\mathbb{C}})[\psi, \widetilde{\nu}\phi, \epsilon_J], \quad \text{and} \\ \delta &: \bigoplus_{J \subset J_F} S_{k,J}(K_\Sigma; \psi, \widetilde{\nu}\phi) \to \mathrm{H}^d_{\mathrm{cusp}}(Y_\Sigma, \mathbb{V}_{\mathbb{C}})[\psi, \widetilde{\nu}\phi] \end{aligned} \tag{39}$$

denote the Matsushima–Shimura–Harder isomorphisms (cf. [Hid94, Proposition 3.1, Equation (4.2)]).

Definition 7.1. For every $J \subset J_F$, we fix a basis $b_{f,J}$ of $\mathcal{M}_\Sigma[f, \epsilon_J]$ and define the period $\Omega_f^J = \delta_J(f_\Sigma)/b_{f,J} \in \mathbb{C}^\times/\mathcal{O}^\times$.

Remark 7.2. Classically, the Matsushima–Shimura–Harder periods of a newform f of level $\mathfrak{n}$ are defined using a basis of the free rank-one $\mathcal{O}$-module $\mathrm{H}^d(Y_1(\mathfrak{n}), \mathbb{V}_{\mathcal{O}})[f, \epsilon_J]$ (cf. [Dim05, § 4.2]). As shown in [Dim05, Theorem 6.6, §§ 4.4 and 4.5] the value at one of the imprimitive adjoint L-function divided by those periods measures the congruences modulo p between f and other Hilbert modular eigenforms of the same weight, level and central character. However, in general the corresponding local Hecke algebra does not have a Galois theoretic interpretation whereas, as proved in Theorem 6.6, $\mathcal{T}_\Sigma$ does, hence our choice to define the periods using $\mathcal{M}_\Sigma[f, \epsilon_J]$.

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Next we explain the relation between the Petersson inner product and the modified Poincaré pairing defined in § 2.5 under the Matsushima–Shimura–Harder isomorphism.

The Atkin–Lehner involution $\iota = \begin{pmatrix} 0 & -1 \\ \mathfrak{n}_\Sigma & 0 \end{pmatrix}$ induces an isomorphism

$$S_k(K_\Sigma; \psi, \widetilde{\nu}\phi) \xrightarrow{\sim} S_k(K_\Sigma; \psi^{-1}, (\widetilde{\nu}\phi)^{-1}), \quad f \mapsto f(\cdot\iota) \otimes \psi^{-1}. \tag{40}$$

The Hecke operator $[K_\Sigma x K_\Sigma]$ acts on the left $S_k(K_\Sigma; \psi, \widetilde{\nu}\phi)$ by sending f on $\sum_i f(\cdot x_i)$, where $K_\Sigma x K_\Sigma = \coprod_i x_i K_\Sigma$. One can easily check that for diagonal x one can choose the x_i so that we simultaneously have $K_\Sigma x K_\Sigma = \coprod_i x_i K_\Sigma$ and $K_\Sigma^\iota x K_\Sigma^\iota = \coprod_i x_i K_\Sigma^\iota$, where $K_\Sigma^\iota = \iota K_\Sigma \iota^{-1}$. In the following commutative diagram the horizontal arrows are isomorphisms.

$$\begin{array}{ccccc} S_k(K_\Sigma; \psi, \widetilde{\nu}\phi) & \xrightarrow{\quad\iota\quad} & S_k(K_\Sigma^\iota; \psi, \psi\phi^{-1}\widetilde{\nu}^{-1}) & \xrightarrow{\otimes\psi^{-1}} & S_k(K_\Sigma; \psi^{-1}, (\widetilde{\nu}\phi)^{-1}) \\ \downarrow {\scriptstyle [K_\Sigma \iota^{-1} x \iota K_\Sigma]} & & \downarrow {\scriptstyle [K_\Sigma^\iota x K_\Sigma^\iota]} & & \downarrow {\scriptstyle \psi(\det(x))[K_\Sigma x K_\Sigma]} \\ S_k(K_\Sigma; \psi, \widetilde{\nu}\phi) & \xrightarrow{\quad\iota\quad} & S_k(K_\Sigma^\iota; \psi, \psi\phi^{-1}\widetilde{\nu}^{-1}) & \xrightarrow{\otimes\psi^{-1}} & S_k(K_\Sigma; \psi^{-1}, (\widetilde{\nu}\phi)^{-1}) \end{array} \tag{41}$$

Finally, for $f_1, f_2 \in S_k(K_\Sigma; \psi, \widetilde{\nu}\phi)$ we define the normalized Petersson inner product by

$$(f_1, f_2) = [K(1) : K_0(\mathfrak{n}_\Sigma)]^{-1} \int_{Y_\Sigma^{\mathrm{ad}}} f_1(g)\overline{f_2(g)}|\det(g)|^{w_0}\, dg. \tag{42}$$

We have $([K_\Sigma x K_\Sigma] \cdot f_1, f_2) = |\det(x)|^{-w_0}(f_1, [K_\Sigma x^{-1} K_\Sigma] \cdot f_2) = \psi(\det(x))(f_1, [K_\Sigma \iota x \iota^{-1} K_\Sigma] \cdot f_2)$.

It follows that the Hecke eigenvalues of f_Σ are complex conjugates of those of

$$f_\Sigma(\cdot\iota) \otimes \psi^{-1} = f_\Sigma(\cdot\iota)\psi(\det(\cdot)^{-1}) = f_\Sigma(\det(\cdot)^{-1} \cdot \iota)|\det(\cdot)|^{-w_0} = f_\Sigma((\cdot)^*\iota)|\det(\cdot)|^{-w_0},$$

where we use the notation from § 2.5. Using Lemma 6.4(ii) we deduce by strong multiplicity one that these two forms differ by a constant, which turns out to be in $\mathcal{O}^\times$ (the arguments of [DFG04, Lemma 2.13] involving local epsilon factors can be adapted to our setting). Hence, in the computation that follows, this constant can be ignored, as well as $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n}_\Sigma)$ and powers of two:

$$(f_\Sigma, f_\Sigma)\mathcal{O} = [\delta(f_\Sigma), \delta(f_\Sigma((\cdot)^*\iota F_\infty))]\mathcal{O} = \langle \delta(f_\Sigma), \delta(f_\Sigma(\cdot F_\infty))\rangle\mathcal{O} = \langle \delta_J(f_\Sigma), \delta_{J_F \setminus J}(f_\Sigma)\rangle\mathcal{O}.$$

From here and Definition 7.1 we obtain the relation we have been looking for:

$$\langle b_{f,J}, b_{f,J_F\setminus J}\rangle\mathcal{O} = \frac{(f_\Sigma, f_\Sigma)}{\Omega_f^J \Omega_f^{J_F \setminus J}}\mathcal{O}. \tag{43}$$

7.2 The Rankin–Selberg method

The Rankin–Selberg method relating the Petersson inner product to the value at one of the adjoint L-function has been carried out by Shimura for Hilbert modular newforms f of level $K_1(\mathfrak{n})$. Since the level structures K_Σ that we consider are more general, the resulting formula in our case slightly differs from Shimura's. While Shimura's formula relates the Petersson inner product of f with the imprimitive adjoint L-function, in our setting the Petersson inner product of f_Σ will be related to adjoint L-function outside Σ. We follow Jacquet's adelic version of the Rankin–Selberg method for GL_2 and our main reference is Bump [Bum97].

All integrals that we consider are with respect to Haar measures on the corresponding algebraic groups. The normalized Petersson inner product (42) can be rewritten as

$$(f_\Sigma, f_\Sigma) = \int_{\mathrm{GL}_2(F)\mathbb{A}^\times\backslash \mathrm{GL}_2(\mathbb{A})} |f_\Sigma(g)|^2 |\det(g)|^{w_0}\, dg. \tag{44}$$

The automorphic form f_Σ admits an adelic Fourier expansion:

$$f(g) = \sum_{y\in F^\times} W\left(\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix} g\right), \tag{45}$$

where $W(g) = \int_{\mathbb{A}/F} \overline{\lambda}(x) f\left(\left(\begin{smallmatrix} 1 & x\\ 0 & 1\end{smallmatrix}\right) g\right) dx$ is the adelic Whittaker function with respect to an additive unitary character λ. Explicitly, one can take $\lambda : \mathbb{A}/F \to \mathbb{A}_\mathbb{Q}/\mathbb{Q} \to \mathbb{C}^\times$ where the first map is the trace, whereas the second is the usual non-trivial additive character $\lambda_\mathbb{Q}$ such that $\lambda_\mathbb{Q}|_\mathbb{R} = \exp(2i\pi\cdot)$ and for every prime number ℓ, $\ker(\lambda_\mathbb{Q}|_{\mathbb{Q}_\ell}) = \mathbb{Z}_\ell$. Hence, for every finite place v we have $\ker(\lambda_v) = v^{-\delta_v}\mathfrak{o}_v$, where δ_v denotes the valuation at v of the different $\mathfrak{d}$ of F.

The following decomposition can be found in [Bum97, Theorem 3.5.4], but one should be careful to replace the usual $k_\tau/2$ by $(-w_0 - k_\tau)/2$ since we are using the arithmetic (non-unitary) normalization (cf. [DT04, pp. 566–567]):

$$W\left(\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix}\right) = \prod_{\tau\in J_F} y_\tau^{-(w_0+k_\tau)/2} \exp(-2\pi y_\tau) \prod_v W_v\left(\begin{pmatrix} y_v & 0\\ 0 & 1\end{pmatrix}\right). \tag{46}$$

Let φ be the Schwartz function on $\mathbb{A}\times\mathbb{A}$ defined as product of the following local functions:

$$\varphi_\tau(x, y) = \exp(-\pi(x^2+y^2)) \quad\text{and}\quad \varphi_v = \begin{cases} \mathrm{char}(\mathfrak{o}_v)\otimes \mathrm{char}(\mathfrak{o}_v) & \text{for } v\notin\Sigma;\\ \mathrm{char}(v^{c_v+d_v})\otimes \mathrm{char}(\mathfrak{o}_v^\times) & \text{for } v\in\Sigma.\end{cases} \tag{47}$$

For $g\in \mathrm{GL}_2(\mathbb{A})$ put $\varepsilon(g) = \zeta_{F,\Sigma}(2s)^{-1}\pi^{sd}\Gamma(s)^{-d}|\det(g)|^s \int_{\mathbb{A}_F^\times} |t|^{2s}\varphi(t(0,1)g)\, dt$.

Then ε is a right $K_0(\mathfrak{n}_\Sigma)\,\mathrm{SO}_2(F\otimes_\mathbb{Q}\mathbb{R})$-invariant function on $\mathrm{GL}_2(\mathbb{A})$ such that $\varepsilon(1) = 1$ and $\varepsilon\left(\left(\begin{smallmatrix} y & x\\ 0 & y'\end{smallmatrix}\right) g\right) = |y/y'|^s\varepsilon(g)$. Consider as in [Bum97, § 3.7] the Eisenstein series:

$$E(g, s) = \sum_{B(F)\backslash \mathrm{GL}_2(F)} \varepsilon(\gamma g). \tag{48}$$

The Rankin–Selberg unfolding yields (cf. [Bum97, pp. 372–373]):

$$\begin{aligned}&\int_{\mathrm{GL}_2(F)\mathbb{A}^\times\backslash \mathrm{GL}_2(\mathbb{A})} E(g,s)|f_\Sigma(g)|^2|\det(g)|^{w_0}\, dg\\ &\quad = \int_{B(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})}\int_{\mathbb{A}_{F+}^\times} \left|W\left(\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix} g\right)\right|^2 \varepsilon\left(\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix} g\right) |y|^{w_0-1}|\det(g)|^{w_0}\, dy\, dg.\end{aligned}$$

Here $\mathbb{A}_{F+}^\times = (F\otimes\widehat{\mathbb{Z}})^\times (F\otimes_\mathbb{Q}\mathbb{R})_+^\times$ denote the subgroup of ideles with totally positive infinite part. In [Bum97] the integration is over $\mathbb{A}_F^\times$ but this makes no difference, since $\mathbb{A}_F^\times = \mathbb{A}_{F+}^\times F^\times$ and the adelic Fourier expansion of $f(g)$ is supported only by totally positive elements. Using Iwasawa decomposition

$$\mathrm{GL}_2(\mathbb{A}) = B(\mathbb{A})\,\mathrm{GL}_2(\mathfrak{o}\otimes\widehat{\mathbb{Z}})\,\mathrm{SO}_2(F\otimes_\mathbb{Q}\mathbb{R}),$$

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and the right $\mathrm{SO}_2(F\otimes_{\mathbb{Q}}\mathbb{R})$-invariance of the integrand, we further rewrite this integral as $\prod_\tau Z_\tau \prod_v Z_v$, where

$$Z_v=\int_{\mathrm{GL}_2(\mathfrak{o}_v)}\int_{F_v^\times}\left|W_v\left(\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix} g\right)\right|^2 \varepsilon_v\left(\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix} g\right)|y|^{w_0-1}\,dy\,dg,$$

and

$$\begin{aligned}Z_\tau&=\int_0^\infty\left|W_\tau\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix}\right|^2\varepsilon_\tau\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix}|y|^{w_0-1}d^\times y\\&=\int_0^\infty \exp(-4\pi y)y^{s+k_\tau-1}d^\times y=(4\pi)^{-s-k_\tau+1}\Gamma(s+k_\tau-1).\end{aligned}$$

Furthermore, for $v\notin\Sigma$ (respectively $v\in\Sigma$) the function $|W_v|^2=|W_v\cdot(\phi_v\widetilde{\nu}_v)^{-1}\circ\det|^2$ is right $\mathrm{GL}_2(\mathfrak{o}_v)$-invariant (respectively, $K_0(v^{c_v+d_v})$-invariant). Moreover $\varepsilon_v\left(\left(\begin{smallmatrix} y & 0\\ 0 & 1\end{smallmatrix}\right)g\right)|y|^{-s}$ is by definition the characteristic function of $\mathrm{GL}_2(\mathfrak{o}_v)$ (respectively, $K_0(v^{c_v+d_v})$). Hence, for all v:

$$Z_v=\int_{F_v^\times}\left|W_v\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix}\right|^2|y|^{s+w_0-1}\,dy.$$

For $v\notin\Sigma$ we have $Z_v=\mathrm{N}_{F/\mathbb{Q}}(v^{\delta_v})^s(1+\mathrm{N}_{F/\mathbb{Q}}(v)^{-s})L_v(\mathrm{Ad}^0(\rho_{f,p}),s)$ (cf. [Bum97, Proposition 3.8.1]).

For $v\in\Sigma$, W_v is annihilated by U_v, hence $Z_v=\mathrm{N}_{F/\mathbb{Q}}(v^{\delta_v})^s$. Therefore,

$$\int_{\mathrm{GL}_2(F)\mathbb{A}^\times\backslash\mathrm{GL}_2(\mathbb{A})}E(g,s)|f_\Sigma(g)|^2|\det(g)|^{w_0}\,dg=\frac{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{d})^s\mathrm{L}_\Sigma(\mathrm{Ad}^0(\rho_{f,p}),s)}{\zeta_{F,\Sigma}(2s)\zeta_{F,\Sigma}(s)^{-1}}\prod_{\tau\in J_F}\frac{\Gamma(s+k_\tau-1)}{(4\pi)^{s+k_\tau-1}}.$$

By [Bum97, Proposition 3.7.5], $E(g,s)$ has a pole at $s=1$ with residue independent of g and equal to the residue at $s=1$ of the function $\zeta_{F,\Sigma}(2)^{-1}\pi^d\int_{\mathbb{A}}\int_{\mathbb{A}^\times}|t|^{2s}\varphi(t,tx)\,dt\,dx$. One readily computes

$$\int_{\mathbb{R}}\int_{\mathbb{R}^\times}|t|^2\varphi(t,tx)\,dt\,dx=\frac{1}{\pi}\int_{\mathbb{R}}\frac{dx}{(1+x^2)}=1,$$

and

$$\begin{aligned}&\int_{F_v}\int_{F_v^\times}|t|^{2s}\varphi(t,tx)\,dt\,dx\\&\quad=\begin{cases}(1-\mathrm{N}_{F/\mathbb{Q}}(v)^{1-2s})^{-1} & \text{for } v\notin\Sigma;\\(1-\mathrm{N}_{F/\mathbb{Q}}(v)^{-1})(1-\mathrm{N}_{F/\mathbb{Q}}(v)^{1-2s})^{-1}\mathrm{N}_{F/\mathbb{Q}}(v)^{(1-2s)(c_v+d_v)} & \text{for } v\in\Sigma.\end{cases}\end{aligned}$$

$$(f_\Sigma,f_\Sigma)=\frac{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n}_\Sigma\mathfrak{d})}{2^{|k|}}\Gamma(\mathrm{Ad}^0(\rho_{f,p}),1)\mathrm{L}_\Sigma(\mathrm{Ad}^0(\rho_{f,p}),1)=\frac{\mathrm{L}_\Sigma(\mathrm{Ad}^0(\rho_{f,p}),1)}{\pi^{|k|+d}}\frac{\prod_\tau(k_\tau-1)!}{4^{|k|}\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n}_\Sigma\mathfrak{d})^{-1}}.\qquad(49)$$

Since, by our assumptions, $\prod_\tau(k_\tau-1)!/4^{|k|}\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n}_\Sigma\mathfrak{d})^{-1}\in\overline{\mathbb{Z}}_{(p)}^\times$ it follows that $(\mathrm{L}_\Sigma(\mathrm{Ad}^0(\rho_{f,p}),1))/(\pi^{|k|+d}(f_\Sigma,f_\Sigma))\in\overline{\mathbb{Z}}_{(p)}^\times$. Since, by definition,

$$\Gamma(\mathrm{Ad}^0(\rho_{f,p}),s)=\prod_{\tau\in J_F}\pi^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)(2\pi)^{-(s+k_\tau-1)}\Gamma(s+k_\tau-1)$$

we obtain

$$\frac{\Gamma(\mathrm{Ad}^0(\rho_{f,p}),1)\mathrm{L}_\Sigma(\mathrm{Ad}^0(\rho_{f,p}),1)}{(f_\Sigma,f_\Sigma)} \in \overline{\mathbb{Z}}_{(p)}^\times. \tag{50}$$

7.3 End of the proof of Theorem B(ii)

Recall that $\mathcal{M}_\Sigma$ is endowed with a perfect $\mathcal{T}_\Sigma$-linear pairing: $\langle\cdot,\cdot\rangle_\Sigma : \mathcal{M}_\Sigma \times \mathcal{M}_\Sigma \to \mathcal{O}$.

Since for all $J \subset J_F$, $\mathcal{M}_\Sigma[f,\epsilon_J]$ is free of rank one over $\mathcal{T}_\Sigma$ it follows that

$$(\eta_f^\Sigma)^2 = \mathrm{disc}(\mathcal{M}_\Sigma[f,\epsilon_J] \oplus \mathcal{M}_\Sigma[f,\epsilon_{J_F\setminus J}]) = \langle b_{f,J}, b_{f,J_F\setminus J}\rangle^2\mathcal{O}.$$

Using (43) we obtain

$$\eta_f^\Sigma = \langle b_{f,J}, b_{f,J_F\setminus J}\rangle\mathcal{O} = \frac{(f_\Sigma,f_\Sigma)}{\Omega_f^J\Omega_f^{J_F\setminus J}}\mathcal{O}. \tag{51}$$

Keeping the notation of § 5.6, since $\rho|_{\mathcal{G}_{F(\zeta_p)}}$ is irreducible by $(\mathbf{LI}_{\mathrm{Ind}\rho})$, Schur's lemma imply

$$\mathrm{H}^0(F, \mathrm{Ad}^0(\rho_{f,p})\otimes\mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{H}^0(F, (\mathrm{Ad}^0(\rho_{f,p})\otimes\mathbb{Q}_p/\mathbb{Z}_p)(1)) = 0.$$

Then [DFG04, Lemma 2.1], which remains valid over F, yields

$$\begin{aligned}&\mathrm{Fitt}_\mathcal{O}(\mathrm{H}^1_\Sigma(F, \mathrm{Ad}^0(\rho_{f,p})\otimes\mathbb{Q}_p/\mathbb{Z}_p))\\ &\quad= \mathrm{Fitt}_\mathcal{O}(\mathrm{H}^1_\mathrm{f}(F, \mathrm{Ad}^0(\rho_{f,p})\otimes\mathbb{Q}_p/\mathbb{Z}_p))\prod_{v\in\Sigma}\mathrm{Fitt}_\mathcal{O}(\mathrm{H}^1_\mathrm{f}(F_v, \mathrm{Ad}^0(\rho_{f,p})(1))^*),\end{aligned}$$

where $(\ \ )^*$ denotes the Pontryagin dual. By [FP94, Proposition I.4.2.2(i)] and [DFG04, p. 708, Lemma 2.16]

$$\mathrm{Tam}(\mathrm{Ad}^0(\rho_{f,p})) = \prod_\tau \mathrm{Tam}_\tau(\mathrm{Ad}^0(\rho_{f,p}))\prod_v \mathrm{Tam}_v(\mathrm{Ad}^0(\rho_{f,p})) = \prod_{v\in\Sigma}\mathrm{Tam}_v(\mathrm{Ad}^0(\rho_{f,p})).$$

Furthermore, by [DFG04, (57)] and by [FP94, Proposition I.4.2.2(ii)] and its proof, for $v \in \Sigma$ we have

$$\begin{aligned}\mathrm{Tam}_v(\mathrm{Ad}^0(\rho_{f,p})) &= \mathrm{Tam}_v(\mathrm{Ad}^0(\rho_{f,p})(1)) = \mathrm{Fitt}_\mathcal{O}(\mathrm{H}^1\left(I_v, \mathrm{Ad}^0(\rho_{f,p})(1)\right)^{\mathcal{G}_{F_v}}_{\mathrm{tor}})\\ &= \mathrm{Fitt}_\mathcal{O}((\mathrm{H}^1(I_v, \mathrm{Ad}^0(\rho_{f,p})(1))^{\mathcal{G}_{F_v}}_{\mathrm{tor}})^*)\\ &= \frac{\mathrm{Fitt}_\mathcal{O}(\mathrm{H}^1_\mathrm{f}(F_v, \mathrm{Ad}^0(\rho_{f,p})(1))^*)}{\mathrm{Fitt}_\mathcal{O}(\mathrm{H}^1(\mathcal{G}_{F_v}/I_v, (\mathrm{Ad}^0(\rho_{f,p})(1))^{I_v})^*)}\\ &= \frac{\mathrm{Fitt}_\mathcal{O}(\mathrm{H}^1_\mathrm{f}(F_v, \mathrm{Ad}^0(\rho_{f,p})(1))^*)}{\mathrm{Fitt}_\mathcal{O}(\mathrm{H}^0(F_v, (\mathrm{Ad}^0(\rho_{f,p})\otimes\mathbb{Q}_p/\mathbb{Z}_p)(1)))}\\ &= \mathrm{L}_v(\mathrm{Ad}^0(\rho_{f,p}),1)\,\mathrm{Fitt}_\mathcal{O}(\mathrm{H}^1_\mathrm{f}(F_v, \mathrm{Ad}^0(\rho_{f,p})(1))^*).\end{aligned}$$

From the three previous equations we deduce that

$$\begin{aligned}&\mathrm{Tam}(\mathrm{Ad}^0(\rho_{f,p}))\,\mathrm{Fitt}_\mathcal{O}(\mathrm{H}^1_\mathrm{f}(F, \mathrm{Ad}^0(\rho_{f,p})\otimes\mathbb{Q}_p/\mathbb{Z}_p))\\ &\quad= \prod_{v\in\Sigma}\mathrm{L}_v(\mathrm{Ad}^0(\rho_{f,p}),1)\,\mathrm{Fitt}_\mathcal{O}(\mathrm{H}^1_\Sigma(F, \mathrm{Ad}^0(\rho_{f,p})\otimes\mathbb{Q}_p/\mathbb{Z}_p)).\end{aligned} \tag{52}$$

Finally, since $\rho_{f,p}$ is a Σ-ramified deformation of $\rho = \overline{\rho}_{f,p}$ (cf. Definition 4.6) and $\Sigma \supset P_\rho$ (cf. Definition 4.2), Theorem 6.6 yields

$$\mathrm{Fitt}_\mathcal{O}(\mathrm{H}^1_\Sigma(F, \mathrm{Ad}^0(\rho_{f,p})\otimes\mathbb{Q}_p/\mathbb{Z}_p) = \eta_f^\Sigma. \tag{53}$$

The theorem results by putting together the equations (50), (51), (52) and (53).

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