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BELLWETHERS FOR BOUNDEDNESS OF COMPOSITION OPERATORS ON WEIGHTED BANACH SPACES OF ANALYTIC FUNCTIONS

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Abstract

Let $\mathbb{D}$ be the open unit disc, let $v : \mathbb{D} \to (0, \infty)$ be a typical weight, and let H_v^∞ be the corresponding weighted Banach space consisting of analytic functions f on $\mathbb{D}$ such that $\|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty$. We call H_v^∞ a *typical-growth space*. For φ a holomorphic self-map of $\mathbb{D}$, let C_φ denote the composition operator induced by φ. We say that C_φ is a *bellwether for boundedness* of composition operators on typical-growth spaces if for each typical weight v, C_φ acts boundedly on H_v^∞ only if all composition operators act boundedly on H_v^∞. We show that a sufficient condition for C_φ to be a bellwether for boundedness is that φ have an angular derivative of modulus less than 1 at a point on $\partial\mathbb{D}$. We raise the question of whether this angular-derivative condition is also necessary for C_φ to be a bellwether for boundedness.

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1. Introduction

Let $H(\mathbb{D})$ denote the collection of holomorphic functions on the open unit disc $\mathbb{D}$, let φ denote an element of $H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, and let C_φ be the composition operator induced by φ, so that whenever f is a function defined on $\mathbb{D}$, $C_\varphi f$ is the function defined on $\mathbb{D}$ by $(C_\varphi f)(z) = f(\varphi(z))$. A *typical weight* on the open unit disc $\mathbb{D}$ is a continuous strictly-positive function v on $\mathbb{D}$ that is radial, is nonincreasing with respect to $|z|$, and satisfies $\lim_{|z|\to 1^-} v(z) = 0$. The associated *typical-growth space* H_v^∞ is defined by

$$H_v^\infty = \left\{ f \in H(\mathbb{D}) : \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty \right\}.$$

The norm $\|\cdot\|_v$ gives H_v^∞ a Banach-space structure. Note that convergence in H_v^∞ implies uniform convergence on compact subsets of $\mathbb{D}$. Applying the closed-graph theorem, one sees that C_φ is bounded on H_v^∞ if and only if it maps H_v^∞ into itself.

As a example, consider

$$v_e(z) = \exp(-1/(1-|z|)), \quad \psi(z) = (1+z)/2 \quad \text{and} \quad \varphi(z) = 1/(2-z).$$

Note that $f(z) = \exp(1/(1-z)) \in H_{v_e}^\infty$ but $f(\psi(z)) = \exp(2/(1-z)) \notin H_{v_e}^\infty$, so that C_ψ is not bounded on $H_{v_e}^\infty$. On the other hand, for any $f \in H_{v_e}^\infty$ and $z \in \mathbb{D}$,

$$\begin{aligned} v_e(z)|f(\varphi(z))| &= \frac{v_e(z)}{v_e(\varphi(z))} v_e(\varphi(z))|f(\varphi(z))| \\ &\leq \exp\left(\frac{1}{1-|1/(2-z)|} - \frac{1}{1-|z|}\right)\|f\|_{v_e} \\ &\leq \exp\left(\frac{1}{1-1/(2-|z|)} - \frac{1}{1-|z|}\right)\|f\|_{v_e} \\ &= \exp(1)\|f\|_{v_e}, \end{aligned}$$

and thus C_φ is bounded on $H_{v_e}^\infty$.

In [4], Bonet *et al.* provide characterizations of boundedness for composition operators on H_v^∞, some of which do not require that the weight v be typical. When v is typical, they show that if C_φ is bounded on H_v^∞ for some disc automorphism φ such that $\varphi(0) \neq 0$ then all composition operators are bounded on H_v^∞ [4, proof of Theorem 2.3]. Thus automorphisms that don't fix the origin induce composition operators that are bellwethers for boundedness.

DEFINITION. We say C_φ is a *bellwether for boundedness* of composition operators on typical-growth spaces if for each typical weight v, the boundedness of C_φ on H_v^∞ ensures the boundedness of all composition operators on H_v^∞.

Every disc automorphism not fixing the origin has angular derivatives of modulus less than 1. In Section 3 of this paper, we show that whenever φ has an angular derivative of modulus less than 1, C_φ is a bellwether for boundedness of composition operators on typical-growth spaces. On the other hand, if all angular derivatives of φ exceed 1, then it is easy to see that C_φ is not a bellwether for boundedness (see Corollary 3). Finally, we note that φ having an angular derivative of 1 at a point is not sufficient to ensure that C_φ is a boundedness bellwether: consider the example above where $\varphi(z) = 1/(2-z)$. The mapping φ has angular derivative 1 at 1, and C_φ is bounded on $H_{v_e}^\infty$. However, not all composition operators are bounded on $H_{v_e}^\infty$; in particular, C_ψ, with $\psi(z) = (1+z)/2$, is not. The preceding results raise the following question.

QUESTION. If C_φ is a bellwether for boundedness of composition operators on typical-growth spaces, must φ have angular derivative less than 1 at some point of $\partial\mathbb{D}$?

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2. Preliminaries

2.1. Automorphisms of $\mathbb{D}$ As a corollary to the Schwarz lemma, if $\alpha \in H(\mathbb{D})$ is an automorphism of $\mathbb{D}$ then there must be a $p \in \mathbb{D}$ and a unimodular constant ζ such that $\alpha(z) = \zeta\alpha_p(z)$, where α_p is given by

$$\alpha_p(z) = \frac{z - p}{1 - \bar{p}z}.$$

Observe that the inverse of α_p is α_{-p}. A little algebra yields the following standard and useful identity:

$$1 - |\alpha_p(z)|^2 = \frac{(1 - |z|^2)(1 - |p|^2)}{|1 - \bar{p}z|^2}. \tag{1}$$

LEMMA 1. *Suppose that α is a holomorphic automorphism of $\mathbb{D}$. Then for $0 \le r < 1$,*

$$\max_{|z|=r} |\alpha(z)| = \frac{|\alpha(0)| + r}{1 + |\alpha(0)|r}.$$

PROOF. There is a unimodular constant ζ such that $\alpha(z) = \zeta(z - p)/(1 - \bar{p}z)$. Note that $|\alpha(0)| = |p|$, and that the lemma clearly holds when $\alpha(0) = 0$. Assume that $\alpha(0) \neq 0$, and apply (1), to obtain

$$1 - |\alpha(z)|^2 = \frac{(1 - |z|^2)(1 - |p|^2)}{|1 - \bar{p}z|^2}.$$

The preceding quantity clearly achieves its minimum value on $\{z : |z| = r\}$ when $z = -rp/|p|$, and the lemma follows. □

2.2. Angular derivatives Recall that every bounded function in $H(\mathbb{D})$ has nontangential limits at every point of a subset of $\partial\mathbb{D}$ having full Lebesgue measure. When $f \in H(\mathbb{D})$ has a nontangential limit at ζ, we denote the value of the limit by $f(\zeta)$. The holomorphic self-map φ of $\mathbb{D}$ has angular derivative at $\zeta \in \partial D$ provided that there is a unimodular constant η for which

$$\angle \lim_{z \to \zeta} \frac{\varphi(z) - \eta}{z - \zeta} \tag{2}$$

exists as a complex number, where $\angle \lim_{z \to \zeta}$ denotes the nontangential limit. The limit (2) is written $\varphi'(\zeta)$, and is called the *angular derivative* of φ at ζ. The following classical result provides some alternate ways to view angular derivatives.

JULIA–CARATHÉODORY THEOREM. *Suppose that φ is a holomorphic self-map of $\mathbb{D}$ and $\zeta \in \partial\mathbb{D}$. The following are equivalent:*

(a) *there exists a point η in $\partial\mathbb{D}$ such that*

$$\angle \lim_{z \to \zeta} \frac{\varphi(z) - \eta}{z - \zeta}$$

is finite;

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(b) *both* φ *and* φ' *have finite nontangential limits at* ζ *and* $\varphi(\zeta) = \eta$ *has modulus* 1*;*
(c)
$$\liminf_{z\to\zeta} \frac{1-|\varphi(z)|}{1-|z|} = \delta < \infty.$$

Moreover, when the conditions above hold, $\delta > 0$*; the limit* $\varphi'(\zeta)$ *in (a) is also equal to* $\angle \lim_{z\to\zeta} \varphi'(z)$*, and their common value is* $\eta\bar{\zeta}\delta$ *(thus the limit infimum of (c) is equal to* $|\varphi'(\zeta)|$*); finally,* $\angle \lim_{z\to\zeta}(1-|\varphi(z)|)/(1-|z|) = |\varphi'(\zeta)|$*.*

Note that if, for example, there were some sequence (z_n) in $\mathbb{D}$ with $|z_n|$ approaching 1 for which $|\varphi(z_n)| \geq |z_n|$ for all n, then at any limit point ζ of (z_n), the function φ would have an angular derivative, and $|\varphi'(\zeta)| \leq 1$. Hence we have the following easy corollary of the Julia–Carathéodory theorem.

COROLLARY 2. *Suppose that* φ *has no angular derivatives having modulus* ≤ 1*. Then there is a positive number* $r < 1$ *such that* $|\varphi(z)| < |z|$ *whenever* $r < |z| < 1$*.*

Also useful to us will be the Julia–Carathéodory inequality: suppose that $\liminf_{z\to\zeta}(1-|\varphi(z)|)/(1-|z|) = \delta < \infty$, and η is the nontangential limit of φ at ζ; then for all $z \in \mathbb{D}$,
$$\frac{|\eta-\varphi(z)|^2}{1-|\varphi(z)|^2} \leq \delta \frac{|\zeta-z|^2}{1-|z|^2}. \tag{3}$$
For discussions of the Julia–Carathéodory theorem and inequality, as well as their proofs, see, for example, [9, Ch. 4] or [6, Section 2.3].

2.3. Denjoy–Wolff point For each positive integer n, let $\varphi^{[n]}$ denote the nth iterate of φ, so that, for example, $\varphi^{[2]} = \varphi\circ\varphi$. The Denjoy–Wolff theorem describes the behaviour of iterate sequences for self-maps φ of $\mathbb{D}$. Recall that a disc automorphism is called *elliptic* if it fixes a point in $\mathbb{D}$.

DENJOY–WOLFF THEOREM. *If* φ *is an automorphism of* $\mathbb{D}$ *that is not elliptic, then there is a point* ω *in the closure of* $\mathbb{D}$ *such that*
$$\omega = \lim_{n\to\infty} \varphi^{[n]}(z)$$
for each $z \in \mathbb{D}$*.*

The point ω, called the *Denjoy–Wolff point* of φ, is also characterized as follows: if $|\omega| < 1$, then $\varphi(\omega) = w$ and $|\varphi'(\omega)| < 1$; if $\omega \in \partial\mathbb{D}$, then $\varphi(\omega) = \omega$ and $0 < \varphi'(\omega) \leq 1$.

2.4. Boundedness of composition operators on H_v^∞ We continue to assume that v is a typical weight: that is, a continuous, positive, radial function that is nonincreasing with respect to $|z|$ and satisfies $\lim_{|z|\to 1^-} v(z) = 0$. Let $\tilde{v}$ be the weight associated with v defined on $\mathbb{D}$ by
$$\tilde{v}(z) = (\sup\{|f(z)| : f \in H_v^\infty, \|f\|_v \leq 1\})^{-1},$$

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so that $\tilde{v}(z) = 1/\|\delta_z\|$ where $\delta_z : H_v^\infty \to \mathbb{C}$ is the linear functional of evaluation at $z \in \mathbb{D}$. It is not difficult to show that if v is typical then so is $\tilde{v}$ [4, Proposition 1.1], and that $H_v^\infty = H_{\tilde{v}}^\infty$ with $\|\cdot\|_v = \|\cdot\|_{\tilde{v}}$ [2, p. 144]. The associated weight has the desirable property that for each $z \in \mathbb{D}$ there is a function f_z in the unit ball of H_v^∞ such that $|f_z(z)| = 1/\tilde{v}(z)$.

An easy sufficient condition for boundedness of the composition operator C_φ on H_v^∞ is that

$$\sup_{z\in\mathbb{D}} \frac{v(z)}{v(\varphi(z))} < \infty. \tag{4}$$

In fact, $\|C_\varphi f\|_v \le \sup_{z\in\mathbb{D}} v(z)/v(\varphi(z))\|f\|_v$. Because v is nonincreasing with respect to $|z|$, the preceding observation, together with the Schwarz lemma, shows that C_φ is bounded on H_v^∞, with $\|C_\varphi\| = 1$ whenever $\varphi(0) = 0$. Similarly, C_φ is bounded if φ has no angular derivatives of modulus less than or equal to 1: apply Corollary 2 to see that $v(z)/v(\varphi(z))$ is less than 1 on an annulus of the form $\{z : r < |z| < 1\}$; because v is continuous and positive and $\varphi(\{z : |z| \le r\})$ is bounded away from $\partial\mathbb{D}$, the finiteness of (4) follows. Reference [4, Theorem 2.4] shows that the preceding two situations are the only ones in which C_φ is bounded for all typical weights. Our interest is in the following simple corollary.

COROLLARY 3. *Suppose that all angular derivatives of φ exceed* 1. *Then C_φ is not a bellwether for boundedness of composition operators on typical-growth spaces.*

PROOF. If all angular derivatives of φ exceed 1, then in particular C_φ is bounded on $H_{v_e}^\infty$ for the typical weight $v_e(z) = \exp(-1/(1 - |z|))$. However, as we have seen, not all composition operators are bounded on $H_{v_e}^\infty$. □

The simple sufficient condition (4) becomes necessary when v is replaced by its associated weight. Indeed, [4, Proposition 2.1] provides the following characterization of boundedness: C_φ is bounded on H_v^∞ if and only if

$$\sup_{z\in\mathbb{D}} \frac{\tilde{v}(z)}{\tilde{v}(\varphi(z))} < \infty.$$

This holds even when v is not typical. We remark that information about norms and essential norms of composition operators on H_v^∞ may be found in [3, 5, 8], while some spectral information may be found in, for example, [1].

3. Results

We continue to assume that v is a typical weight. Reference [4, Theorem 2.3] asserts that the following are equivalent:

(i) all composition operators $C_\varphi : H_v^\infty \to H_v^\infty$ are bounded;

(ii) all composition operators $C_\varphi : H_v^0 \to H_v^0$ are bounded;

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(iii)

$$\inf_{n\in\mathbb{N}} \frac{\tilde{v}(1-2^{-n-1})}{\tilde{v}(1-2^{-n})} > 0. \tag{5}$$

Here $\mathbb{N}$ is the set of positive integers, and H_v^0 is the subspace of H_v^∞ defined by $H_v^0 = \{f \in H(\mathbb{D}) : \lim_{|z|\to 1^-} v(z)|f(z)| = 0\}$. We remark that the condition (5) was used by Lusky [7] in his study of the case where H_v^0 is isomorphic to c_0. The proof in [4] of the equivalence of (i) and (iii) is based on the observation that if C_{α_p} is bounded for every $p \in \mathbb{D}$ then all composition operators on H_v^∞ are bounded. The point is that if φ is an arbitrary self-map of $\mathbb{D}$ and $p = \varphi(0)$ then $\psi := \alpha_p \circ \varphi$ induces a bounded composition operator on H_v^∞ because $\psi(0) = 0$. If $C_{\alpha_{-p}}$ is also bounded, then so is $C_\varphi = C_\psi C_{\alpha_{-p}}$. The proof of [4, Theorem 2.3] shows that the boundedness of C_α for a single automorphism α (not fixing the origin) is sufficient for (5) to hold, and that if (5) holds then all automorphisms induce bounded composition operators. We expand upon ideas in the proof of [4, Theorem 2.3] to obtain the following theorem, whose proof provides a direct argument that the boundedness of a single automorphic composition operator C_α, with $\alpha(0) \neq 0$, implies the boundedness of every automorphic composition operator.

For $0 < a < 1$, let

$$\psi_a(z) = az + 1 - a,$$

so that ψ_a is a self-map of $\mathbb{D}$ and $\psi_a^{[n]}(0) = 1 - a^n$ for each $n \in \mathbb{N}$.

THEOREM 4. *Let v be a typical weight. The following are equivalent.*

(i) $C_\varphi : H_v^\infty \to H_v^\infty$ *is bounded for every analytic self-map φ of $\mathbb{D}$;*
(ii) $C_{\psi_a} : H_v^\infty \to H_v^\infty$ *is bounded for every $a \in (0, 1)$;*
(iii) $\inf_{n\in\mathbb{N}}(\tilde{v}(1-a^{n+1})/\tilde{v}(1-a^n)) > 0$ *for every $a \in (0, 1)$;*
(iv) $\inf_{n\in\mathbb{N}}(\tilde{v}(1-a^{n+1})/\tilde{v}(1-a^n)) > 0$ *for some $a \in (0, 1)$;*
(v) $\inf_{t\in(0,1]}(\tilde{v}(1-at)/\tilde{v}(1-t)) > 0$ *for some $a \in (0, 1)$;*
(vi) $C_\alpha : H_v^\infty \to H_v^\infty$ *is bounded for some automorphism α of $\mathbb{D}$ with $\alpha(0) \neq 0$;*
(vii) $C_\alpha : H_v^\infty \to H_v^\infty$ *is bounded for every automorphism α of $\mathbb{D}$.*

PROOF. That (i) implies (ii) is trivial. Suppose that (ii) holds and $a \in (0, 1)$. Then by [4, Proposition 2.1], there is a positive constant M such that

$$\frac{\tilde{v}(z)}{\tilde{v}(\psi_a(z))} < M \quad \forall z \in \mathbb{D}.$$

Letting $z = \psi_a^{[n]}(0)$ in the preceding inequality, where $n \in \mathbb{N}$ is arbitrary, and rearranging,

$$\frac{\tilde{v}(1-a^{n+1})}{\tilde{v}(1-a^n)} > \frac{1}{M}.$$

Because $a \in (0, 1)$ and $n \in \mathbb{N}$ are arbitrary, we deduce (iii).

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That (iii) implies (iv) is trivial. Now suppose that (iv) holds, and that the infimum is $\beta > 0$. Because $\tilde{v}$ is nonincreasing with respect to $|z|$, if $a \le t \le 1$ then

$$\gamma := \frac{\tilde{v}(1-a^2)}{\tilde{v}(0)} \le \frac{\tilde{v}(1-at)}{\tilde{v}(1-t)}.$$

This γ is a positive constant, since $\tilde{v}$ is a positive function on $\mathbb{D}$. Now let k denote a positive integer, and assume that $a^{k+1} \le t < a^k$. Then

$$\frac{\tilde{v}(1-at)}{\tilde{v}(1-t)} \ge \frac{\tilde{v}(1-a^{k+2})}{\tilde{v}(1-a^k)} = \frac{\tilde{v}(1-a^{k+2})}{\tilde{v}(1-a^{k+1})}\frac{\tilde{v}(1-a^{k+1})}{\tilde{v}(1-a^k)} \ge \beta^2.$$

We see that for any $t \in (0, 1]$,

$$\frac{\tilde{v}(1-at)}{\tilde{v}(1-t)} \ge \min\{\gamma, \beta^2\},$$

so that (v) holds.

Suppose (v) holds, with the infimum being $\lambda > 0$. Let $p = (1-a)/(1+a)$, so that p is positive and $(1-p)/(1+p) = a$. We show that the automorphism $\alpha_p(z) = (z-p)/(1-pz)$ induces a bounded composition operator. Note that $\alpha_p(0) \ne 0$ because $p \ne 0$. Using Lemma 1 and the fact that $\tilde{v}$ is nonincreasing and radial, we find that for $z \in \mathbb{D}$,

$$\begin{aligned}
\frac{\tilde{v}(z)}{\tilde{v}(\alpha_p(z))} &\le \frac{\tilde{v}(|z|)}{\tilde{v}((p+|z|)/(1+p|z|))} \\
&= \frac{\tilde{v}(1-(1-|z|))}{\tilde{v}(1-(1-(p+|z|)/(1+p|z|)))} \\
&= \frac{\tilde{v}(1-(1-|z|))}{\tilde{v}(1-(1-p)(1-|z|)/(1+p|z|))} \\
&\le \frac{\tilde{v}(1-(1-|z|))}{\tilde{v}(1-((1-p)/(1+p))\,(1-|z|))} \qquad (\tilde{v} \text{ is nonincreasing}) \\
&= \frac{\tilde{v}(1-(1-|z|))}{\tilde{v}(1-a(1-|z|))} \\
&\le 1/\lambda.
\end{aligned}$$

It follows that C_{α_p} is bounded [4, Proposition 2.1], and we see that (v) implies (vi).

Suppose that (vi) holds: $C_\alpha : H_v^\infty \to H_v^\infty$ is bounded for $\alpha = \zeta\alpha_p$ where $|\zeta| = 1$ and $p \in \mathbb{D} \setminus \{0\}$. Because C_α is bounded,

$$\infty > \sup_{z\in\mathbb{D}} \tilde{v}(z)/\tilde{v}(\alpha(z)) = \sup_{z\in\mathbb{D}} \tilde{v}(z)/\tilde{v}(\alpha_p(z)),$$

where the final equality holds because $\tilde{v}$ is radial. Thus C_{α_p} is also bounded. Note that α_p has Denjoy–Wolff point $-p/|p| \in \partial\mathbb{D}$; hence $|\alpha_p^{[n]}(0)| \to 1$ as $n \to \infty$. Let

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$\tau(z) = \xi(z - q)/(1 - \bar{q}z)$ be an arbitrary disc automorphism. Choose the positive integer n so that $|\alpha_p^{[n]}(0)| > |q|$, and set $s = |\alpha_p^{[n]}(0)|$. We know that $C_{\alpha_p^{[n]}}$ is bounded. Thus there is a constant C such that for every $z \in \mathbb{D}$,

$$\frac{\tilde{v}(z)}{\tilde{v}(\alpha_p^{[n]}(z))} \leq C.$$

Let $z \in \mathbb{D}$ be arbitrary, let $r = |z|$, and choose z_0 with $|z_0| = r$ so that $|\alpha_p^{[n]}(z_0)| = (s + r)/(1 + sr)$ (Lemma 1). Because $x \mapsto (x + r)/(1 + xr)$ is increasing for $-1/r < x < \infty$, $|q| < s$, and $\tilde{v}$ is nonincreasing,

$$\frac{\tilde{v}(z)}{\tilde{v}(\tau(z))} \leq \frac{\tilde{v}(r)}{\tilde{v}((|q| + r)/(1 + |q|r))} \leq \frac{\tilde{v}(r)}{\tilde{v}((s + r)/(1 + sr))} = \frac{\tilde{v}(z_0)}{\tilde{v}(\alpha^{[n]}(z_0))} \leq C,$$

and it follows that C_τ is bounded. Thus, (vi) implies (vii). We have already indicated why (vii) implies (i); see the comments following equation (5) above. Hence the proof is complete. □

Note that the proof of the preceding theorem shows that for each $a \in (0, 1)$ the composition operator C_{ψ_a} induced by $\psi_a(z) = az + 1 - a$ is a bellwether for boundedness: if C_{ψ_a} is bounded on the typical-growth space H_v^∞, then

$$\inf_{n \in \mathbb{N}} \tilde{v}(1 - a^{n+1})/\tilde{v}(1 - a^n) > 0,$$

so that C_φ is bounded on H_v^∞ for all φ. That C_{ψ_a} is a bellwether for boundedness also follows from the next result because the angular derivative of ψ_a at 1 is $a < 1$.

THEOREM 5. *Let v be typical weight. Suppose that $C_\varphi : H_v^\infty \to H_v^\infty$ is bounded and that φ has angular derivative less than 1 at some point $\zeta \in \partial\mathbb{D}$. Then every composition operator on H_v^∞ is bounded.*

PROOF. By the Julia–Carathéodory theorem, there is an $\eta \in \partial\mathbb{D}$ such that φ has nontangential limit η at ζ. Because the composition operators $C_{\bar{\eta}z}$ and $C_{\zeta z}$ are bounded, the composition operator with symbol $\psi(z) = \bar{\eta}\varphi(\zeta z)$ is also bounded. Moreover, $\psi(1) = 1$ and $\psi'(1) = \bar{\eta}\varphi'(\zeta)\zeta = |\varphi'(\zeta)| < 1$. Thus we see that C_ψ is bounded and that ψ has Denjoy–Wolff point 1 with $a := \psi'(1) < 1$.

Applying the Julia–Carathéodory inequality (3) inductively, we see that for every $n \in \mathbb{N}$,

$$\frac{|1 - \psi^{[n]}(z)|^2}{1 - |\psi^{[n]}(z)|^2} \leq a^n \frac{|1 - z|^2}{1 - |z|^2}.$$

Thus, for each $n \in \mathbb{N}$, $1 - |\psi^{[n]}(0)| \leq 2a^n$. Hence, for $n \in \mathbb{N}$,

$$1 - |\psi^{[n]}(0)| = g(n)a^n, \tag{6}$$

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where g is a positive bounded function on $\mathbb{N}$. Because $(\psi^{[n]}(0))$ converges nontangentially to 1 (see, for example, [6, Lemma 2.66, p. 82]), we may apply the Julia–Carathéodory theorem to conclude that

$$\lim_{n\to\infty} \frac{1-|\psi^{[n+1]}(0)|}{1-|\psi^{[n]}(0)|} = a.$$

Equivalently,

$$\lim_{n\to\infty} \frac{g(n+1)}{g(n)} = 1.$$

For each $n \in \mathbb{N}$, set $e_n = n + \log_a(g(n))$. Then

$$e_{n+1} - e_n = 1 + \log_a\left(\frac{g(n+1)}{g(n)}\right).$$

Since $\log_a(g(n+1)/g(n))$ approaches 0 as $n \to \infty$, there is a natural number K such that whenever $n \geq K$, the gap between e_{n+1} and e_n exceeds $1/2$. Because C_ψ is bounded, $(C_\psi)^3 = C_{\psi^{[3]}}$ is also bounded, and there is a constant M such that

$$\frac{\tilde{v}(|z|)}{\tilde{v}(|\psi^{[3]}(z)|)} \leq M \quad \forall z \in \mathbb{D}.$$

In particular, for every $n \in \mathbb{N}$,

$$M \geq \frac{\tilde{v}(1-(1-|\psi^{[n]}(0)|))}{\tilde{v}(1-(1-|\psi^{[n+3]}(0)|))} = \frac{\tilde{v}(1-a^{e_n})}{\tilde{v}(1-a^{e_{n+3}})}.$$

Let $j \in \mathbb{N}$ exceed $K + \log_a(g(K)) = e_K$. Let $n_0 \geq K$ be the greatest positive integer such that $e_{n_0} \leq j$. Note that $e_{n_0+1} > j$. Because the gap between e_{n+1} and e_n exceeds $1/2$ for every $n \geq K$, we have $e_{n_0+3} > e_{n_0+1} + 1 \geq j + 1$. Because $\tilde{v}$ is nonincreasing,

$$\frac{\tilde{v}(1-a^j)}{\tilde{v}(1-a^{j+1})} \leq \frac{\tilde{v}(1-a^{e_{n_0}})}{\tilde{v}(1-a^{e_{n_0+3}})} \leq M,$$

and it follows, since $j \geq e_K$ is arbitrary, that

$$\inf_{n\in\mathbb{N}} \frac{\tilde{v}(1-a^{n+1})}{\tilde{v}(1-a^n)} > 0.$$

Thus (iv) of Theorem 4 holds, and all composition operators on H_v^∞ are bounded. □

Using the Julia–Carathéodory theorem, we can summarize our results (Theorem 5 and Corollary 3) on bellwethers for boundedness as follows. Let

$$\liminf_{|z|\to 1^-} \frac{1-|\varphi(z)|}{1-|z|} = \delta$$

(where we allow $\delta = \infty$). If $\delta < 1$, then C_φ is a bellwether for boundedness of composition operators on typical-growth spaces; if $\delta > 1$, then C_φ is not a boundedness bellwether.

The obvious question is whether $\delta < 1$ is necessary for C_φ to be a bellwether for boundedness.

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References

[1] R. Aron and M. Lindström, 'Spectra of weighted composition operators on weighted Banach spaces of analytic functions', *Israel J. Math.* **141** (2004), 263–276.

[2] K. D. Bierstedt, J. Bonet and J. Taskinen, 'Associated weights and spaces of holomorphic functions', *Studia Math.* **127** (1998), 137–168.

[3] J. Bonet, P. Domański and M. Lindström, 'Essential norm and weak compactness of composition operators on weighted Banach spaces of analytic functions', *Canad. Math. Bull.* **42** (1999), 139–148.

[4] J. Bonet, P. Domański, M. Lindström and J. Taskinen, 'Composition operators between weighted Banach spaces of analytic functions', *J. Austral. Math. Soc. (Series A)* **64** (1998), 101–118.

[5] M. D. Contreras and A. G. Hernandez-Diaz, 'Weighted composition operators in weighted Banach spaces of analytic functions', *J. Austral. Math. Soc. (Series A)* **69** (2000), 41–60.

[6] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions* (CRC Press, Boca Raton, FL, 1995).

[7] W. Lusky, 'On weighted spaces of harmonic and holomorphic functions', *J. London Math. Soc.* **51** (1995), 309–320.

[8] A. Montes-Rodríguez, 'Weighted composition operators on weighted Banach spaces of analytic functions', *J. London Math. Soc.* **61** (2000), 872–884.

[9] J. H. Shapiro, *Composition Operators and Classical Function Theory* (Springer-Verlag, Berlin, 1993).

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