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ON THE RESOLUTION DIAGRAMS OF THE BRIESKORN SINGULARITIES (2,q,r) OF TYPE II

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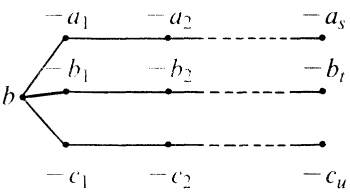
1. Statement of results. Let p, q, r be pairwise coprime integers with $2 \leqq p < q < r$. The equation $z_1^p + z_2^q + z_3^r = 0$ defines a complex hypersurface $V(p, q, r) \subset \mathbf{C}^3$ which has an isolated singular point at the origin. We call the singularity the *Brieskorn singularity* (p, q, r). An algorithm of resolving this singularity is known [**1**]. According to the algorithm, the resolution diagram which describes the configuration of the pre-image of the singular point in the resolved surface is a star-shaped tree $\Gamma_{p,q,r}$ with three branches:

(1.1) $\Gamma_{p,q,r} = -b$  $(a_i, b_j, c_k \geqq 2)$.

The positive integers (weights) a_i, b_j, c_k, b are given as follows: Let x, y, z, b be integers satisfying

$$(1.2)\quad \begin{aligned} &xqr \equiv -1 (\mathrm{mod}\ p),\ ypr \equiv -1 (\mathrm{mod}\ q),\ zpq \equiv -1 (\mathrm{mod}\ r),\\ &0 < x < p,\ 0 < y < q,\ 0 < z < r,\\ &bpqr = xqr + ypr + zpq + 1. \end{aligned}$$

Then

$$p/x = [a_1, a_2, \ldots, a_s],$$

$$q/y = [b_1, b_2, \ldots, b_t],$$

$$r/z = [c_1, c_2, \ldots, c_u],$$

where $[n_1, n_2, \ldots, n_v]$ denotes the continued fraction

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$$n_1 - \cfrac{1}{n_2 - \cfrac{1}{\ddots - \cfrac{1}{n_\nu}}} \quad (n_i \geqq 2).$$

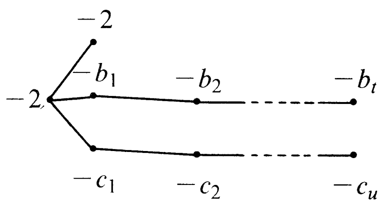
Remark. The number b is equal to either 1 or 2.

Definition. A weighted graph (such as $\Gamma_{p,q,r}$) is of *type* II if all the weights are even integers.

In what follows we confine ourselves to the case $p = 2$, and our aim in this paper is to characterize those weighted graphs of type II which appear as the (minimal) resolution diagrams of the Brieskorn singularities of type $(2, q, r)$.

In our case $p = 2$, the resolution diagram of type II has the following form:

(1.3)



$(b_j, c_k \geqq 2)$.

The above weighted (planar) graphs are in 1 to 1 correspondence to the arrays of positive integers

$$\begin{bmatrix} b_1, b_2, \ldots, b_t \\ c_1, c_2, \ldots, c_u \end{bmatrix}.$$

Thus our task will be to characterize these arrays.

Before stating our results, we introduce a semi-group S of arrays of integers:

$$S := \left\{ \begin{bmatrix} m_1, m_2, \ldots, m_\mu \\ n_1, n_2, \ldots, n_\nu \end{bmatrix} \middle| m_i, n_j \in \mathbf{Z}, \mu, \nu \geqq 0 \right\}.$$

Product in S is defined by juxtaposition:

$$\begin{bmatrix} m_1, \ldots, m_\mu \\ n_1, \ldots, n_\nu \end{bmatrix} \begin{bmatrix} m'_1, \ldots, m'_\xi \\ n'_1, \ldots, n'_\eta \end{bmatrix} = \begin{bmatrix} m_1, \ldots, m_\mu, m_1', \ldots, m'_\xi \\ n_1, \ldots, n_\nu, n'_1, \ldots, n'_\eta \end{bmatrix}.$$

The identity element in S is $\begin{bmatrix} \emptyset \\ \emptyset \end{bmatrix}$, where $\emptyset$ is the empty sequence.

We define three special types of elements in S called 'joints', 'molecules' and 'head and tail', respectively.

A) Joints. There are four elements $\bar{Z}$, $\underline{Z}$, $\bar{T}$, $\underline{T} \in S$ called *joints*.

$$\bar{Z} = \begin{bmatrix} 2, 2, 6 \\ 6, 2, 2 \end{bmatrix}, \underline{Z} = \iota(\bar{Z}), \bar{T} = \begin{bmatrix} 2, 2, 4, 2, 2 \\ 8 \end{bmatrix}, \underline{T} = \iota(\bar{T}),$$

where $\iota : S \rightarrow S$ is the involution defined by

$$\iota(X) = \begin{bmatrix} n_1, \ldots, n_\nu \\ m_1, \ldots, m_\mu \end{bmatrix} \text{ for } X = \begin{bmatrix} m_1, \ldots, m_\mu \\ n_1, \ldots, n_\nu \end{bmatrix}.$$

We call $\iota(X)$ the *inverse* (i.e., upside-down) of $X \in S$.

B) Molecules. First we define auxiliary elements (called *particles*) $\bar{e}$, $\underline{e}$, $\bar{p}_n$, $\underline{p}_n \in S$ as follows:

$$\bar{e} = \begin{bmatrix} 2 \\ \emptyset \end{bmatrix}, \quad \underline{e} = \iota(\bar{e}), \bar{p}_n = \begin{bmatrix} (8n - 1)*2 \\ 2n + 2 \end{bmatrix}, \quad \underline{p}_n = \iota(\bar{p}_n),$$

where $n \geqq 1$ and the notation $m*2$ stands for a sequence $2, 2, \ldots, 2$ consisting of m $2's$ ($m \in \mathbf{Z}$, $m \geqq 1$).

A *molecule* M is a product in S of these particles of the following form:

$$M = \begin{cases} \bar{e}\,\bar{p}_{n(1)}\underline{p}_{n(2)} \cdots \underline{p}_{n(\mu)}\underline{e} \text{ or its inverse } (\mu\text{: even} \geqq 0), \\ \bar{e}\,\bar{p}_{n(1)}\underline{p}_{n(2)} \cdots \bar{p}_{n(\mu)}\bar{e} \text{ or its inverse } (\mu\text{: odd} \geqq 1). \end{cases}$$

The explicit construction rule of a molecule is as follows:

(i) A molecule begins with either $\bar{e}$ or $\underline{e}$.
(ii) If it begins with $\bar{e}$ (resp. $\underline{e}$), the first $p_{n(1)}$ has an upper bar (resp. a lower bar).
(iii) p_n with upper bar and p_n with lower bar appear alternately.
(iv) If $\mu \geqq 1$, a molecule ends with $\bar{e}$ or $\underline{e}$ according as the last $p_{n(\mu)}$ has an upper bar or a lower bar. The molecule with $\mu = 0$ is $\bar{e}\underline{e}$ or $\underline{e}\bar{e}$.

Examples. (1) $\bar{e}\underline{e} = \underline{e}\bar{e} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. This is the simplest molecule.

(2) $$\bar{e}\,\bar{p}_n\bar{e} = \begin{bmatrix} 2 \\ \emptyset \end{bmatrix}\begin{bmatrix} (8n-1)*2 \\ 2n + 2 \end{bmatrix}\begin{bmatrix} 2 \\ \emptyset \end{bmatrix} = \begin{bmatrix} (8n+1)*2 \\ 2n + 2 \end{bmatrix}.$$

(3) $$\bar{e}\,\bar{p}_n\underline{p}_m\underline{e} = \begin{bmatrix} 2 \\ \emptyset \end{bmatrix}\begin{bmatrix} (8n-1)*2 \\ 2n+2 \end{bmatrix}\begin{bmatrix} 2m+2 \\ (8m-1)*2 \end{bmatrix}\begin{bmatrix} \emptyset \\ 2 \end{bmatrix} = \begin{bmatrix} 8n*2, 2m+2 \\ 2n+2, 8m*2 \end{bmatrix}.$$

C) Head and tail. The *head* $H \in S$ is defined by $H = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, the *tail* L by $L = \begin{bmatrix} \emptyset \\ 2, 2 \end{bmatrix}$. H has the same form as the simplest molecule.

Now we state our main result. Let $D(X)$ denote the weighted graph (1.3), where

$$X = \begin{bmatrix} b_1, \dots, b_t \\ c_1, \dots, c_u \end{bmatrix} \in S.$$

THEOREM 1.1. *$D(X)$ is a weighted graph of type* II *which appears as the resolution diagram of a Brieskorn singularity* $(2, q, r)$ $(q < r)$ *if and only if X is written as the following product in S*:

$$(1.4) \qquad X = HM_1J_1M_2J_2 \cdots M_{\nu-1}J_{\nu-1}M_\nu L \quad (\nu \geqq 1),$$

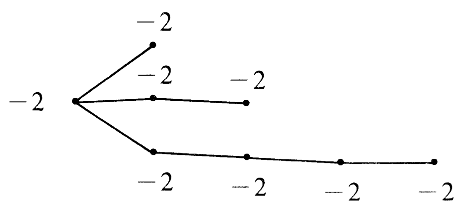
where H is the head, L is the tail, each M_i is a molecule and each J_i is a joint.

Remark. The above decomposition of X is unique.

Examples. (1) The simplest example is

$$X = H(\bar{e}\underline{e})L = \begin{bmatrix} 2, 2 & \\ 2, 2, 2, 2 \end{bmatrix}.$$

The corresponding diagram $D(X)$ is the Dynkin diagram E_8:



This is the resolution diagram of the Brieskorn singularity (2, 3, 5).

$$(2) \qquad H(\underline{e}\underline{p}_1\underline{e})L = \begin{bmatrix} 2 \\ 2 \end{bmatrix}\begin{bmatrix} 4 \\ 9*2 \end{bmatrix}\begin{bmatrix} \emptyset \\ 2, 2 \end{bmatrix} = \begin{bmatrix} 2, 4 \\ 12*2 \end{bmatrix}.$$

The diagram $D(H(\underline{e}\underline{p}_1\underline{e})L)$ appears as the resolution diagram of the singularity (2, 7, 13). This diagram also represents the indecomposable inner product space Γ_{16} of type II in dimension 16 [**2**].

$$(3) \qquad H(\underline{e}\underline{p}_n\underline{e})L = \begin{bmatrix} 2 \\ 2 \end{bmatrix}\begin{bmatrix} 2n+2 \\ (8n+1)*2 \end{bmatrix}\begin{bmatrix} \emptyset \\ 2, 2 \end{bmatrix} = \begin{bmatrix} 2, 2n+2 \\ (8n+4)*2 \end{bmatrix}.$$

$D(H(\underline{e}\underline{p}_n\underline{e})L)$ is the resolution diagram of the singularity $(2, 4n + 3, 8n + 5)$.

(4) $$H(\underline{e}\,\underline{p}_l\bar{p}_m\underline{p}_n\underline{e})L = \begin{bmatrix}2\\2\end{bmatrix}\begin{bmatrix}2l+2, (8m-1)*2, 2n+2\\ 8l*2, 2m+2, 8n*2\end{bmatrix}\begin{bmatrix}\emptyset\\2, 2\end{bmatrix}.$$

$D(H(\underline{e}\,\underline{p}_l\bar{p}_m\underline{p}_n\underline{e})L)$ is the resolution diagram of the singularity $(2, q, r)$, where

$$q = 64\,lmn + 32\,lm + 16mn + 4\,l + 8m + 4n + 3,$$
$$r = 128\,lmn + 48\,lm + 32mn + 8\,l + 12m + 8n + 5.$$

(5) An example with two joints is

$$H(\bar{e}\,\bar{p}_2\bar{e})\underline{Z}(\bar{e}\,\bar{p}_2\underline{p}_1\underline{e})\bar{T}(\underline{e}\,\underline{p}_3\bar{p}_2\bar{e})L.$$

The corresponding singularity is of the type (2, 8998141, 13759411).

THEOREM 1.2. *Suppose that* $D(HM_1J_1\cdots J_{\nu-1}M_\nu L)$ *and* $D(H\iota(M_1J_1\cdots J_{\nu-1}M_\nu)L)$ *are resolution diagrams of the singularities* $(2, q, r)$ *and* $(2, q', r')$, *respectively. Then* (q, r) *and* (q', r') *are related by*

$$q' = -4q + 3r$$
$$r' = -5q + 4r.$$

COROLLARY 1.2.1. *If the resolution diagram of the singularity* $(2, q, r)(q < r)$ *is of type* II, *then so is the resolution diagram of the singularity* $(2, -4q + 3r, -5q + 4r)$.

By Theorem 6.3 in [**3**], if the resolution diagram of the singularity $(2, q, r)$ is of type II, then $r < 2q$.

This, together with Corollary 1.2.1, yields

$$-5q + 4r < 2(-4q + 3r),$$

that is $3q < 2r$. Thus we have

COROLLARY 1.2.2. *If the resolution diagram of the singularity* $(2, q, r)(q < r)$ *is of type* II, *then* $(3/2)q < r < 2q$.

Remark. The lower bound 3/2 and the upper bound 2 given for r/q by Corollary 1.2.2 are best possible because there exist sequences $\{X_n\}$ and $\{X'_n\}$ in S such that $D(X_n)$ and $D(X'_n)$ are weighted graphs of type II appearing as the resolution diagrams of Brieskorn singularities $(2, q_n, r_n)$ $(q_n < r_n)$ and $(2, q'_n, r'_n)$ $(q'_n < r'_n)$ and the corresponding sequences $\{r_n/q_n\}$ and $\{r'_n/q'_n\}$ of rational numbers approach 3/2 and 2, respectively, as n tends to infinity:

$X_n := H\bar{e}\,\bar{p}_n\bar{e}L$ which corresponds to $(2, 8n+3, 12n+5)$,

$X'_n := H\underline{e}\,\underline{p}_n\underline{e}L$ which corresponds to $(2, 4n+3, 8n+5)$.

(Cf. Example (3).)

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2. Fundamental lemma. Define four polynomials of variables η, η', ζ and ζ':

$$F_{\mathrm{I}}(\eta, \eta', \zeta, \zeta') := 1 + \eta\zeta - 4(\eta - \eta')(\zeta - \zeta')$$

$$F_{\mathrm{II}}(\eta, \eta', \zeta, \zeta') := 1 + \eta\zeta' - 4(\eta - \eta')(\zeta - \zeta')$$

$$F_{\mathrm{III}}(\eta, \eta', \zeta, \zeta') := 1 + \eta'\zeta - 4(\eta - \eta')(\zeta - \zeta')$$

$$F_{\mathrm{IV}}(\eta, \eta', \zeta, \zeta') := 1 + \eta'\zeta' - 4(\eta - \eta')(\zeta - \zeta').$$

Let

$$Y = \begin{bmatrix} y_0, y_1, \ldots, y_{t+1} \\ z_0, z_1, \ldots, z_{u+1} \end{bmatrix} \in S.$$

For each pair (k, l) of integers with $0 \leqq k \leqq t$ and $0 \leqq l \leqq u$, we define four integers $(Y|k, l)_{\mathrm{I}}$, $(Y|k, l)_{\mathrm{II}}$, $(Y|k, l)_{\mathrm{III}}$ and $(Y|k, l)_{\mathrm{IV}}$:

(2.1) $\quad (Y|k, l)_* := F_*(y_k, y_{k+1}, z_l, z_{l+1}), \; * = \mathrm{I, II, III, IV}.$

Let

$$X = \begin{bmatrix} b_1, b_2, \ldots, b_t \\ c_1, c_2, \ldots, c_u \end{bmatrix} \in S$$

be written as (1.4). A *subarray*

$$X' = \begin{bmatrix} b_1, \ldots, b_k \\ c_1, \ldots, c_l \end{bmatrix} \text{ of } X$$

with $1 \leqq k \leqq t$ and $1 \leqq l \leqq u$ is said to *end with M* if there exists an integer λ such that

$$1 \leqq \lambda \leqq \nu \quad \text{and} \quad X' = HM_1J_1 \ldots M_{\lambda-1}J_{\lambda-1}M_\lambda.$$

Similarly, a subarray X' of X is said to *end with J* if there exists an integer λ such that

$$0 \leqq \lambda \leqq \nu - 1 \quad \text{and} \quad X' = J_0M_1J_1 \ldots M_\lambda J_\lambda$$

where J_0 stands for H. Further, we classify manners of ending of subarrays at the 'particle level' as follows: Suppose that a molecule $M(\in S)$ is decomposed as a product of μ particles $q_1, q_2, \ldots, q_\mu$ whose arrangement is subject to the construction rule stated in Section 1. Then, the product in S of its first i particles $q_1, \ldots, q_i$ $(1 \leqq i < \mu)$ is called the *i-th section* of the decomposition of M. For example, let

$$M = \bar{e}\,\bar{p}_{n(1)}\underline{p}_{n(2)}\bar{p}_{n(3)}\underline{p}_{n(4)}\underline{e}.$$

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Then, $\bar{e}\bar{p}_{n(1)}\underline{p}_{n(2)}$ is the 3rd section of M. For convenience, we divide particles into two groups:

(1) $\bar{e}$ and $\underline{p}_n (n \geqq 1)$, called particles of type $\underline{p}$,

(2) $\underline{e}$ and $\bar{p}_n (n \geqq 1)$, called particles of type $\bar{p}$.

We say that a subarray X' of X *ends with* $\underline{p}$ if X' is expressed as

$$X' = HM_1J_1 \dots M_{\lambda-1}J_{\lambda-1}q_1q_2 \dots q_i \quad (1 \leqq \lambda \leqq \nu, i \geqq 1) \tag{2.2}$$

where $q_1q_2 \dots q_i$ is the i-th section of $M_\lambda (q_1q_2 \dots q_i \neq M_\lambda)$ and q_i is of type $\underline{p}$. Similarly, X' is said to *end with* $\bar{p}$ if X' is written as (2.2) and q_i is of type $\bar{p}$.

LEMMA. *Let*

$$X = \begin{bmatrix} b_1, \dots, b_t \\ c_1, \dots, c_u \end{bmatrix} \in S$$

be written as (1.4).

Define an array

$$Y = \begin{bmatrix} y_0, \dots, y_{t+1} \\ z_0, \dots, z_{u+1} \end{bmatrix} \in S$$

by the following formulas:

$$\begin{cases} y_{t+1} = 0, y_t = 1, y_{k-1} = b_k y_k - y_{k+1} & (1 \leqq k \leqq t), \\ z_{u+1} = 0, z_u = 1, z_{l-1} = c_l z_l - z_{l+1} & (1 \leqq l \leqq u). \end{cases} \tag{2.3}$$

Then, the following four propositions hold:

I. *if a subarray* $\begin{bmatrix} b_1, \dots, b_k \\ c_1, \dots, c_l \end{bmatrix}$ *of X ends with M, then* $(Y|k, l)_{\mathrm{I}} = 0$;

II. *if a subarray* $\begin{bmatrix} b_1, \dots, b_k \\ c_1, \dots, c_l \end{bmatrix}$ *of X ends with $\underline{p}$, then* $(Y|k, l)_{\mathrm{II}} = 0$;

III. *if a subarray* $\begin{bmatrix} b_1, \dots, b_k \\ c_1, \dots, c_l \end{bmatrix}$ *of X ends with $\bar{p}$, then* $(Y|k, l)_{\mathrm{III}} = 0$;

IV. *if a subarray* $\begin{bmatrix} b_1, \dots, b_k \\ c_1, \dots, c_l \end{bmatrix}$ *of X ends with J, then* $(Y|k, l)_{\mathrm{IV}} = 0$.

This lemma is proved according to a "network induction" scheme illustrated by Fig. 1. Observe that, starting at the arrow marked with L (tail) in Fig. 1, one advances along arrows and finally goes out along the arrow marked with H (head) to obtain an array of the form (1.4), in 'tail-to-head' direction. One may pass through the same arrow any number of times.

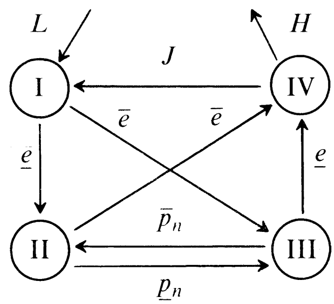

Fig. 1

Thus, it suffices to show the following eight assertions:

1° $(Y|t, u - 2)_{\mathrm{I}} = 0$;

2° if $(Y \mid k, l)_{\mathrm{I}} = 0$ and

$$\begin{bmatrix} b_1, \dots, b_k \\ c_1, \dots, c_l \end{bmatrix} = \begin{bmatrix} b_1, \dots, b_k \\ c_1, \dots, c_{l-1} \end{bmatrix} \underline{e}$$

(or equivalently, $c_l = 2$), then $(Y|k, l - 1)_{\mathrm{II}} = 0$;

3° if $(Y|k, l)_{\mathrm{I}} = 0$ and

$$\begin{bmatrix} b_1, \dots, b_k \\ c_1, \dots, c_l \end{bmatrix} = \begin{bmatrix} b_1, \dots, b_{k-1} \\ c_1, \dots, c_l \end{bmatrix} \bar{e}$$

(or equivalently, $b_k = 2$), then $(Y|k - 1, l)_{\mathrm{III}} = 0$;

4° if $(Y|k, l)_{\mathrm{II}} = 0$ and

$$\begin{bmatrix} b_1, \dots, b_k \\ c_1, \dots, c_l \end{bmatrix} = \begin{bmatrix} b_1, \dots, b_{k-1} \\ c_1, \dots, c_l \end{bmatrix} \bar{e}$$

(or equivalently, $b_k = 2$), then $(Y|k - 1, l)_{\mathrm{IV}} = 0$;

5° if $(Y|k, l)_{\mathrm{II}} = 0$ and

$$\begin{bmatrix} b_1, \dots, b_k \\ c_1, \dots, c_l \end{bmatrix} = \begin{bmatrix} b_1, \dots, b_{k-1} \\ c_1, \dots, c_{l-8n+1} \end{bmatrix} \underline{p}_n$$

(or equivalently, $b_k = 2n+2$ and $c_{l-8n+2} = c_{l-8n+3} = \dots = c_l = 2$) for some $n \geqq 1$, then $(Y|k - 1, l - 8n + 1)_{\mathrm{III}} = 0$;

6° if $(Y|k, l)_{\mathrm{III}} = 0$ and

$$\begin{bmatrix} b_1, \dots, b_k \\ c_1, \dots, c_l \end{bmatrix} = \begin{bmatrix} b_1, \dots, b_{k-8n+1} \\ c_1, \dots, c_{l-1} \end{bmatrix} \bar{p}_n$$

(or equivalently, $b_{k-8n+2} = \dots = b_k = 2$ and $c_l = 2n + 2$) for some $n \geqq 1$, then $(Y|k - 8n + 1, l - 1)_{\text{II}} = 0$;

7° if $(Y|k, l)_{\text{III}} = 0$ and

$$\begin{bmatrix} b_1, \dots, b_k \\ c_1, \dots, c_l \end{bmatrix} = \begin{bmatrix} b_1, \dots, b_k \\ c_1, \dots, c_{l-1} \end{bmatrix} \underline{e}$$

(or equivalently, $c_l = 2$), then $(Y|k, l - 1)_{\text{IV}} = 0$;

8° If $(Y|k, l)_{\text{IV}} = 0$ and

$$\begin{bmatrix} b_1, \dots, b_k \\ c_1, \dots, c_l \end{bmatrix} = \begin{bmatrix} b_1, \dots, b_{k'} \\ c_1, \dots, c_{l'} \end{bmatrix} J$$

where J is a joint, then $(Y|k', l')_{\text{I}} = 0$.

Each assertion is shown by straightforward calculations.

Proof of the "if" part of Theorem 1.1. Let

$$X = \begin{bmatrix} b_1, \dots, b_t \\ c_1, \dots, c_u \end{bmatrix} \in S$$

be written as (1.4). Define

$$Y = \begin{bmatrix} y_0, \dots, y_{t+1} \\ z_0, \dots, z_{u+1} \end{bmatrix} \in S$$

by (2.3). Since the subarray $\begin{bmatrix} b_1 \\ c_1 \end{bmatrix}$ $(= H)$ of X ends with J, we have

$$(Y|1, 1)_{\text{IV}} = 1 + y_2 z_2 - 4(y_1 - y_2)(z_1 - z_2) = 0$$

by the lemma. Substituting $y_2 = 2y_1 - y_0$ and $z_2 = 2z_1 - z_0$, we obtain

$$1 + y_0 z_0 + 2y_1 z_0 + 2y_0 z_1 = 4y_0 z_0.$$

Since $b_k \geqq 2$ for all k and $c_l \geqq 2$ for all l, it is clear that

$$y_{k+1} < y_k \quad (0 \leqq k \leqq t) \quad \text{and} \quad z_{l+1} < z_l \quad (0 \leqq l \leqq u).$$

In particular, $0 < y_1 < y_0$ and $0 < z_1 < z_0$. Clearly

$$y_0/y_1 = [b_1, \dots, b_t], \quad z_0/z_1 = [c_1, \dots, c_u],$$

by the definition of Y. Therefore, 2, y_0 and z_0 are pairwise coprime integers such that $D(X)$ is the weighted graph of type II which appears as the resolution diagram of the Brieskorn singularity $(2, y_0, z_0)$. (Cf. Algorithm in Section 1.) Using Theorem 1.2, one can prove $y_0 < z_0$.

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The "only if" part of Theorem 1.1 is proved by the network induction again, but in reverse direction. The argument is elementary, except that it involves careful estimation of the relevant quantities appearing in expansions into continued fractions. The proof of Theorem 1.2 proceeds similarly according to the network induction. (See also [**5**].)

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