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On doubly transitive permutation groups

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Suppose that G is a doubly transitive permutation group on a finite set Ω, and that for α in Ω the stabilizer G_α of α has a set $\Sigma = \{B_1, \ldots, B_t\}$ of nontrivial blocks of imprimitivity in $\Omega - \{\alpha\}$. If G_α is 3-transitive on Σ it is shown that either G is a collineation group of a desarguesian projective or affine plane or no nonidentity element of G_α fixes B_1 pointwise.

Introduction

Suppose that G is a doubly transitive but not doubly primitive permutation group on a finite set Ω. Let $\Sigma = \{B_1, \ldots, B_t\}$ be a set of nontrivial blocks of imprimitivity in $\Omega - \{\alpha\}$ for the stabilizer G_α of a point $\alpha \in \Omega$.

This paper completes an investigation which began with [11]. In that paper it was shown that if G_α^Σ is the alternating or symmetric group or one of the Mathieu groups in its usual representation then either G is a collineation group of a projective or affine plane or no nonidentity element of G_α fixes B_1 pointwise. In subsequent papers, [10, 12], it was shown that the same conclusions are valid if we assume only that G_α is 3-transitive and not faithful on Σ. The assumption that G_α is not

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faithful on Σ is unattractive but unfortunately it was crucial in the proofs given in those papers. Our aim in this paper is to show that the assumption that G_α is not faithful on Σ is unnecessary. We prove

THEOREM. *Suppose that G is 2-transitive on Ω of degree n, and that for $\alpha \in \Omega$, G_α has a set $\Sigma = \{B_1, \ldots, B_t\}$ of blocks of imprimitivity in $\Omega - \{\alpha\}$, where $t = |\Sigma| \geq 3$, $|B_i| = b > 1$, $bt = n - 1$. Suppose that G_α is 3-transitive on Σ and that G_α contains a nonidentity element fixing B_1 pointwise. Then G is a collineation group of a desarguesian projective or affine plane of order $t - 1$ such that the lines containing α are precisely the sets $B_i \cup \{\alpha\}$, for $i = 1, \ldots, t$.*

We remark that the affine planes arising in the conclusions of [11] and [12] are desarguesian. This was shown in [13]. Most of the notation used here follows the conventions of Wielandt's book [14]. If a group G has a permutation representation on a set Ω then $\text{fix}_\Omega G$ and $\text{supp}_\Omega G$ will denote the subset of Ω fixed by G and the subset of Ω permuted nontrivially by G respectively. By a block design we shall mean a set of v points and a set of b blocks with a relation called incidence between points and blocks, such that any block is incident with k points, where $2 \leq k < v-1$, and any pair of points is incident with λ blocks, where $\lambda > 0$. The number r of blocks incident with a given point is also constant and a counting argument shows $vr = bk$. It is well known that $b \geq v$ and hence that $r \geq k$.

Proof of the theorem

Let G satisfy the hypothesis of the theorem and let K_i, $\overline{K}_i$ denote the setwise and pointwise stabilizers respectively of $B_i \in \Sigma$, for $i = 1, \ldots, t$. By [12] and [13], Proposition D, the theorem is true if either G_α is not faithful on Σ or if $\overline{K}_1$ is 2-transitive on $\Sigma - \{B_1\}$. Thus we may assume that G_α is faithful on Σ and $\overline{K}_1$ is not 2-transitive on $\Sigma - \{B_1\}$. By [13], Theorem A, and since G_α is

3-transitive on Σ, the translates under G of $B_1 \cup \{\alpha\}$ are the blocks of a block design with $\lambda = 1$ preserved by G.

LEMMA 1. *The theorem is true if K_1 has a normal subgroup which acts regularly on $\Sigma - \{B_1\}$.*

Proof. Suppose that K_1 has a normal subgroup N which is regular on $\Sigma - \{B_1\}$. By [5] and since G_α is 3-transitive on Σ, it follows that either G_α^Σ is a normal extension of $\mathrm{PSL}(2, q)$ of degree $q + 1$ or G_α has a normal subgroup M which is regular on Σ. In the latter case, by [14], 11.3, either $t = |M| = 3$ or M is elementary abelian of order 2^a for some $a \geq 2$. If $t = 3$, then $G_\alpha \simeq S_3$ and hence $K_1 \simeq Z_2$. Since K_1 is transitive on B_1, $b = 2$ and $n = 7$. However there is no 2-transitive group G of degree 7 with $G_\alpha \simeq S_3$. If $|M| = 2^a$ then by [4] and since G_α is 3-transitive on Σ, it follows that G has a regular normal subgroup L, say. Then ML is a Frobenius group with Frobenius complement M. However a Frobenius complement contains at most one involution and so we have a contradiction (see [3], 10.3.1). Thus G_α^Σ is a normal extension of $\mathrm{PSL}(2, q)$ with $t = q + 1$.

Now K_1 has a normal subgroup V of order q which is regular on $\Sigma - \{B_1\}$, and since $\overline{K}_1$ is nontrivial, $V \subseteq \overline{K}_1$. Then $K_1 = \left(K_1 \cap K_2\right)\overline{K}_1$, so that $K_1^{B_1} = \left(K_1 \cap K_2\right)^{B_1}$, which is metacyclic. If Z is the largest cyclic normal subgroup of $K_1 \cap K_2$, then Z has order $q - 1$ or $(q-1)/2$. Let $Y = Z \cap \overline{K}_1$ and suppose that Y is nontrivial. Then Y is semi-regular on $\Sigma - \{B_1, B_2\}$, so that $\mathrm{fix}_\Omega Y \subseteq \{\alpha\} \cup B_1 \cup B_2$. Moreover Y, Z are characteristic subgroups of $Z, K_1 \cap K_2$ respectively.

Now there is an element g in G_α which interchanges B_1 and B_2 and hence normalises $K_1 \cap K_2$. It follows that g normalises Y and Z, and hence that $Y = Z \cap \overline{K}_2$. Thus $Y \leq \overline{K}_1 \cap \overline{K}_2$ and so $\mathrm{fix}_\Omega\, Y = \{\alpha\} \cup B_1 \cup B_2$.

If $\Sigma(\beta)$ is the set of blocks of imprimitivity of G_β corresponding to Σ , then $C = \left(B_1 - \{\beta\}\right) \cup \{\alpha\}$ is the block in $\Sigma(\beta)$ containing α and $\overline{K}_1$ is the pointwise stabilizer of $\{\beta\} \cup C$. Now V must be regular on both $\Sigma(\beta) - \{C\}$ and $\Sigma - \{B_1\}$, so that the representations of $\overline{K}_1$ on $\Sigma(\beta) - \{C\}$ and $\Sigma - \{B_1\}$ are equivalent. Thus $\overline{K}_1 \cap K_2$ is the stabilizer in $\overline{K}_1$ of an element C' , say, of $\Sigma(\beta) - \{C\}$. By a similar argument Y fixes C' pointwise, so that $C' = B_2$, which is a contradiction, since $\{\alpha\} \cup B_2$ and $\{\beta\} \cup C'$ are distinct blocks of the block design. Thus $Y = 1$, and Z is faithful and hence semi-regular on B_1 . Therefore b is divisible by $|Z|$.

Now the size $1 + b$ of a block of the block design is at most the number $t = q + 1$ of blocks containing α , and hence b is $q - 1$ or $(q-1)/2$. If b is $q - 1$, the design is an affine plane, which can be shown to be desarguesian as in [13]. So assume that $b = (q-1)/2 = |Z|$. Then q is odd and $G_\alpha \not\trianglerighteq \mathrm{PGL}(2, q)$. In this case V has odd order q , and if $\beta \in B_1$, $G_{\alpha\beta}/V$ is cyclic. Thus a Sylow 2-subgroup of $G_{\alpha\beta}$ is cyclic or trivial. By [1] and [2], respectively, it follows that G has a regular normal subgroup L , say. Now $\mathrm{PSL}(2, q) \leq G_\alpha \leq \mathrm{P\Gamma L}(2, q)$. Let M be a subgroup of $\mathrm{PSL}(2, q)$, which is elementary abelian of order 4 , $M \leq G_\alpha$. An involution g in M fixes either 0 or 2 elements of Σ . In the former case g fixes only the point α of Ω . In the latter case we may assume that $g \in K_1 \cap K_2$ and hence that $g \in Z$. We showed above that Z is semi-regular on B_1 and similarly Z is semi-regular on B_2 . Thus in this case, also, $\mathrm{fix}_\Omega\, g = \{\alpha\}$. Thus M acts fixed point freely on L and so is a Frobenius complement, whereas a Frobenius complement of even order contains only one involution (see [3], 10.3.1). Thus the lemma is proved.

We may assume therefore that K_1 has no normal subgroup acting regularly on $\Sigma - \{B_1\}$; in particular $\overline{K}_1$ is not regular on $\Sigma - \{B_1\}$.

LEMMA 2. *The theorem is true if* $K_1 \cap K_2$ *has a nontrivial abelian normal subgroup.*

Proof. Suppose that $K_1 \cap K_2$ has a nontrivial abelian normal subgroup. By [9], Lemma 1, and [10], the theorem is true unless G_α^Σ has a regular normal elementary abelian 2-subgroup. Thus we may assume that G_α has an elementary abelian normal 2-subgroup M of order $t \geq 4$ which is semiregular on $\Omega - \{\alpha\}$. By [4], G has a regular normal subgroup L , and so M is a Frobenius complement. However, as Frobenius complements contain at most one involution, this is a contradiction (see [3], 10.3.1), and Lemma 2 is proved.

Thus we may assume that $K_1 \cap K_2$ has no nontrivial abelian normal subgroups. By [8], Theorem A, $K_1 \cap K_2$ has a unique minimal normal subgroup S which is a nonabelian simple group and hence has even order. Since $\overline{K}_1$ is not regular on $\Sigma - \{B_1\}$, $\overline{K}_1 \cap K_2$ and similarly $K_1 \cap \overline{K}_2$ are nontrivial normal subgroups of $K_1 \cap K_2$. Thus

$$S \subseteq \left(\overline{K}_1 \cap K_2\right) \cap \left(K_1 \cap \overline{K}_2\right) = \overline{K}_1 \cap \overline{K}_2 = X ,$$

say. Since $K_1 \cap K_2$ is transitive on $\Sigma - \{B_1, B_2\}$, X is $\frac{1}{2}$-transitive on $\Sigma - \{B_1, B_2\}$ with orbits of length, say, x , where $x > 1$, and x divides $t - 2$.

LEMMA 3. $X = \overline{K}_1 \cap \overline{K}_2$ *is not semi-regular on* $\Sigma - \{B_1, B_2\}$.

Proof. If X is semi-regular on $\Sigma - \{B_1, B_2\}$, then, since $|X|$ is even, it follows from [3] and Lemma 1 that $K_1^{\Sigma-\{B_1\}}$ has a unique normal subgroup, say X^1 , which is 2-transitive and simple. Then $X^1 \leq \overline{K}_1^{\Sigma-\{B_1\}}$ which is a contradiction, since $\overline{K}_1$ is not 2-transitive on $\Sigma - \{B_1\}$.

Let $\Sigma(\beta)$ be the set of blocks of imprimitivity for G_β

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corresponding to Σ, where $\beta \in B_1$. Then the element of $\Sigma(\beta)$ containing α is $C = \left(B_1-\{\beta\}\right) \cup \{\alpha\}$. Let $C^1 \in \Sigma(\beta) - \{C\}$ be chosen so that $C^1 \cap B_2$ is non-empty. Then as $\{\beta\} \cup C^1$ and $\{\alpha\} \cup B_2$ are distinct blocks of the block design, $C^1 \cap B_2 = \{\gamma\}$, say. Let Y be the pointwise stabilizer of C^1 in $\overline{K}_1$. Then Y is conjugate to X in G (for $\overline{K}_1$ is normal in $G_{\{\alpha,\beta\}}$, so if $g \in G_{\{\alpha,\beta\}} - G_{\alpha\beta}$, then $Y^g = \overline{K}_1 \cap \overline{K}_i$ where $(C^1)^g = B_i$, and there is an h in $\overline{K}_1$ such that $B_i^h = B_2$ and hence $Y^{gh} = \overline{K}_1 \cap \overline{K}_2 = X$). It follows that $\text{fix}_\Omega\, Y = \{\alpha\} \cup B_1 \cup C^1$, and all Y-orbits in $\text{supp}_\Omega\, Y$ have length a multiple of x. Since Y fixes γ, Y fixes B_2 setwise, and clearly $B_2 - \{\gamma\} \subseteq \text{supp}_\Omega\, Y$. Therefore $b - 1 = |B_2-\{\gamma\}|$ is a multiple of x.

LEMMA 4. *The subset of* Σ *fixed by* Y *is precisely*

$$\text{fix}_\Sigma\, Y = \{B_1\} \cup \{B \in \Sigma;\ B \cap C^1 \neq \emptyset\},$$

and $\text{fix}_\Sigma\, Y$ *is a union of* X*-orbits in* Σ.

Proof. Clearly $\{B_1\} \cup \{B \in \Sigma;\ B \cap C^1 \neq \emptyset\} \subseteq \text{fix}_\Sigma\, Y$, and is the subset of $\text{fix}_\Sigma\, Y$ of those elements of Σ which contain a point of $\text{fix}_\Omega\, Y$. If $\text{fix}_\Sigma\, Y$ contains an additional element B, then $B \subseteq \text{supp}_\Omega\, Y$, and so B is a union of nontrivial Y-orbits; that is, b is divisible by x. However $b - 1$ is divisible by x and $x > 1$, so we conclude that $\text{fix}_\Sigma\, Y = \{B_1\} \cup \{B \in \Sigma;\ B \cap C^1 \neq \emptyset\}$.

Now let $B \in \text{fix}_\Sigma\, Y - \{B_1, B_2\}$ and let Δ be the X-orbit in Σ containing B. It is sufficient to show that $\Delta \subseteq \text{fix}_\Sigma\, Y$. Let $B \cap C^1 = \{\delta\}$ and let Δ' be the X-orbit in Ω containing δ. Then $|\Delta| = x$ and $|\Delta'| = xy$, where $y = |\Delta' \cap B| \geq 1$. Now $Y \subseteq K_1 \cap K_2$, and so Y normalises X. Therefore Y fixes $\Delta' = \{\delta^x;\ x \in X\}$ setwise (for if $y \in Y$, then $\delta^{xy} = \delta^{y^{-1}xy} \in \Delta'$). Thus Y fixes $B \cap \Delta'$ setwise, and so $(B \cap \Delta') - \{\delta\}$ is a union of nontrivial Y-orbits, that is $y - 1 = |(B \cap \Delta')-\{\delta\}| \equiv 0 \pmod{x}$.

Let $a = |\text{supp}_\Sigma Y \cap \Delta|$. Then $0 \le a < x$. If $a > 0$, then $\bigcup\{B' \cap \Delta';\ B' \in \text{supp}_\Sigma Y \cap \Delta\}$ is a union of nontrivial Y-orbits and is a set of ay points. Thus $ay \equiv 0 \pmod{x}$, and so $a \equiv 0 \pmod{x}$, which is a contradiction, since $0 < a < x$. Thus $a = 0$ and $\Delta \subseteq \text{fix}_\Sigma Y$.

LEMMA 5. $X \cap Y = 1$.

Proof. If $X \cap Y$ is nontrivial, then it follows from Lemma 4 that X does not act faithfully on its orbits in $\Sigma - \{B_1, B_2\}$, and so by [8], Proposition 4, $K_1^{\Sigma-\{B_1\}}$ is a normal extension of $L_r(q)$, for some $r \ge 3$ and prime power q , in its natural representation. Then $\overline{K}_1 \supseteq L_r(q)$ and hence is 2-transitive on $\Sigma - \{B_1\}$, which is a contradiction.

Thus $X \cap Y = 1$ and Y normalises X . Similarly since X fixes C and C^1 setwise, X normalises Y . It follows that X and Y centralise each other. Thus, by Lemma 3, the centraliser of X in K_1 is not semiregular on $\Sigma - \{B_1, B_2\}$, and it follows from [6], Corollary B3, and Lemma 2.8 that X is a T.I. set in K_1 (that is distinct conjugates of X by elements of K_1 intersect only in the identity). Thus by [7], Theorem A, and since X is not semiregular on $\Sigma - \{B_1, B_2\}$, $K_1^{\Sigma-\{B_1\}}$ is a normal extension of $L_r(q)$ in its natural representation for some $r \ge 3$ and prime power q . This is impossible, as $\overline{K}_1$ is not 2-transitive on $\Sigma - \{B_1\}$.

Thus the theorem is proved.

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