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AN ARGUMENT OF A FUNCTION IN $H^{1/2}$

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Abstract Let $H^{1/2}$ be the Hardy space on the open unit disc. For two non-zero functions f and g in $H^{1/2}$, we study the relation between f and g when $f/g \geqslant 0$ a.e. on ∂D. Then we generalize a theorem of Neuwirth and Newman and Helson and Sarason with a simple proof.

Keywords: Hardy space; argument; boundary value

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For $0 < p \leqslant \infty$, H^p denotes the usual Hardy space on the open unit disc D.

When f and g are in $H^{1/2}$, and $f/g \geqslant 0$ a.e. on ∂D, we want to know the relation between f and g. Neuwirth and Newman [**5**] showed that if $g = 1$, then $f = \gamma g$ for some positive constant γ. That is, they proved that there exists no non-constant positive function in $H^{1/2}$. Independently, Helson and Sarason [**2**] showed that if $g = z^n$ and $n \geqslant 0$, then f is a polynomial with degree in the range $[n, 2n]$. In fact, they proved that f/g is a rational function with degree less than or equal to $2n$. In order to generalize the result of Helson and Sarason, suppose $g = z^n h$, where h is in $H^{1/2}$ and h^{-1} is in H^∞. Then $f/g = h^{-1}f/z^n$ and $h^{-1}f$ is in $H^{1/2}$. Hence, if $f/g \geqslant 0$ a.e. on ∂D, then by the result of Helson and Sarason, $h^{-1}f$ is a polynomial p with degree in the range $[n, 2n]$ and so $f = ph$.

For $0 < p \leqslant \infty$, a non-zero function h in H^p is called strongly outer (or p-strongly outer) if h satisfies the following: if f is a non-zero function in H^p such that $f/g \geqslant 0$ a.e. on ∂D, then $f = \gamma g$ for some positive constant γ. It is known [**3**] that there is no strongly outer function in H^p when $0 < p < \frac{1}{2}$. When $\frac{1}{2} \leqslant p \leqslant \infty$, if h is in H^p and h^{-1} is in H^∞, then h is a p-strongly outer function by the Neuwirth–Newman Helson–Sarason Theorem (see Lemma 4). Examples of 1-strongly outer functions are known, for instance, when h^{-1} is in H^1 or when $\operatorname{Re} h \geqslant 0$. We have two characterizations of 1-strongly outer functions [**1**, **4**]. But these characterizations are not easy to check. A 1-strongly outer function is also called a rigid function, and if it has a unit norm, then it is an exposed point in the unit ball of H^1.

Unfortunately, we do not know of any examples except the above for $H^{1/2}$, that is, when h^{-1} belongs to H^∞. Moreover, we do not have any characterization for $\frac{1}{2}$-strongly outer functions. However, it is natural to ask the following question.

Question 1. *Let f be a non-constant function in $H^{1/2}$, let $n \geqslant 0$ and let h be a strongly outer function in $H^{1/2}$. If $f/z^n h \geqslant 0$ a.e. on ∂D, then does $f = ph$ hold for some polynomial p with degree in the range $[n, 2n]$?*

In this paper we answer the above question positively.

Theorem 2. *Suppose n is a non-negative integer and h is a strongly outer function in $H^{1/2}$. If f is a non-zero function in $H^{1/2}$ such that $f/z^n h \geqslant 0$ a.e. on ∂D, then $f = ph$ and p is a polynomial with degree in the range $[n, 2n]$. In particular,*

$$p = \gamma \prod_{j=1}^{n} (z - a_j)(1 - \bar{a}_j z),$$

where γ is some positive constant and a_j, $1 \leqslant j \leqslant n$, are some complex constants.

Lemma 3. *Suppose h_0^2 is strongly outer in $H^{1/2}$ and $0 \leqslant j < \infty$. If $\bar{z}^j \bar{h}_0/h_0 = \bar{Q}\bar{k}/k$, where Q is inner and k is outer in H^1, then Q is a Blaschke product with degree less than or equal to j.*

Proof. If $Q = q_1 \cdots q_{j+1}$ and q_ℓ is a non-constant inner function for $1 \leqslant \ell \leqslant j+1$, then

$$\bar{q}_\ell = \frac{1 - \overline{q_\ell(0)} q_\ell}{q_\ell - q_\ell(0)} \frac{1 - q_\ell(0)\bar{q}_\ell}{1 - \overline{q_\ell(0)} q_\ell} = \bar{z}\bar{\tilde{q}}_\ell \frac{1 - q_\ell(0)\bar{q}_\ell}{1 - \overline{q_\ell(0)} q_\ell}$$

and so

$$\bar{z}^j \frac{\bar{h}_0}{h_0} = \bar{z}^{j+1} \prod_{\ell=1}^{j+1} \bar{\tilde{q}}_\ell \frac{\prod_{\ell=1}^{j+1}(1 - q_\ell(0)\bar{q}_\ell)}{\prod_{\ell=1}^{j+1}(1 - \overline{q_\ell(0)} q_\ell)} \frac{\bar{k}}{k},$$

where $\tilde{q}_\ell$ is inner for $1 \leqslant \ell \leqslant j+1$. Hence, setting

$$g = \prod_{\ell=1}^{j+1} (1 - \overline{q_\ell(0)} q_\ell) k \quad \text{and} \quad \tilde{Q} = \prod_{\ell=1}^{j+1} \tilde{q}_\ell,$$

we then obtain that g is still outer and

$$\frac{\bar{h}_0}{h_0} = \bar{z}\bar{\tilde{Q}} \frac{\bar{g}}{g} = \frac{\overline{(1+z)}\overline{(1+\tilde{Q})}\bar{g}}{(1+z)(1+\tilde{Q})g}.$$

Hence, $h_0^2 = \gamma(1+z)^2(1+\tilde{Q})^2 g^2$ for some constant $\gamma > 0$ because h_0^2 is strongly outer in $H^{1/2}$. Therefore, $z(1+\tilde{Q})^2 g^2/h_0^2 \geqslant 0$ and so $h_0^2 = \gamma z(1+\tilde{Q})^2 g^2$ for some constant $\gamma > 0$. This contradicts the statement that h_0 is outer and so Q is a finite Blaschke product of degree $\ell \leqslant j$. $\square$

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Proof of Theorem 2. Let $h = h_0^2$ for an outer function h_0 in H^1. Let $f = qk^2$ for an outer function k in H^1 and an inner function q. Let $\phi = |f|/f$. Then

$$\phi = \bar{z}^n \frac{\bar{h}_0}{h_0} = \bar{q}\frac{\bar{k}}{k}.$$

In particular, by Lemma 3, q is a Blaschke product of degree less than or equal to n. Hence, $H^1 \cap \bar{\phi}\bar{H}^1$ contains $\{z^j h_0\}_{j=0}^n$ and qk. Since $h_0(0) \neq 0$, there exists a polynomial p_n in $\mathcal{P}_n$ such that $qk - p_n h_0 = z^{n+1}s$ and $s \in H^1$ where $\mathcal{P}_n$ is the set of all analytic polynomials of degree less than or equal to n. If $qk \notin \mathcal{P}_n \times h_0$, then $s \not\equiv 0$. Hence, if g is the outer part of s, then $0 \neq z^{n+1}g \in H^1 \cap \bar{\phi}\overline{H^1}$. Therefore, there exists a function $\psi \in H^1$ such that $z^{n+1}g = \bar{\phi}\bar{\psi}$. Since $|\phi| = 1$, $\psi = Qg$ for some inner function Q. Thus, $\bar{z}^n\bar{h}_0/h_0 = \bar{z}^{n+1}\bar{Q}\bar{g}/g$. This contradicts Lemma 3 because g is outer and $z^{n+1}Q$ is inner. Thus, $qk = p_n h_0$ for some p_n in $\mathcal{P}_n$ with degree less than or equal to n. Now it is enough to prove the theorem only when the degree of p_n is just n. Hence,

$$qk = \gamma_1 \prod_{j=1}^{n} (z - \alpha_j) h_0,$$

where $\gamma_1 \in \mathbb{C}$, $\alpha_j \in \mathbb{C}$, $|\alpha_j| < 1$, $1 \leqslant j \leqslant \ell$, and $|\alpha_j| \geqslant 1$, $j \geqslant \ell + 1$, and so

$$q = \prod_{j=1}^{\ell} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z}.$$

Hence,

$$k = \gamma_1 \prod_{j=1}^{\ell} (1 - \bar{\alpha}_j z) \prod_{j=\ell+1}^{n} (z - \alpha_j) h_0.$$

Therefore,

$$f = qk^2 = \gamma_1^2 \prod_{j=1}^{\ell} (z - \alpha_j)(1 - \bar{\alpha}_j z) \prod_{j=\ell+1}^{n} (z - \alpha_j)^2 h_0^2.$$

Since

$$\left(\prod_{j=1}^{\ell} (z - \alpha_j)(1 - \bar{\alpha}_j z) \right) \frac{1}{z^\ell} \geqslant 0,$$

we have

$$\left(\gamma_1^2 \prod_{j=\ell+1}^{n} (z - \alpha_j)^2 \right) \frac{1}{z^{n-\ell}} \geqslant 0,$$

and necessarily $|\alpha_j| = 1$, $\ell + 1 \leqslant j \leqslant n$, because if $|\alpha_j| > 1$ and $(z - \alpha_j)^2/z \geqslant 0$, then

$$\frac{z - \alpha_j}{1 - \bar{\alpha}_j z} |1 - \bar{\alpha}_j z|^2 = \frac{(z - \alpha_j)^2}{z} \geqslant 0.$$

This contradiction shows $|\alpha_j| = 1$, $\ell + 1 \leqslant j \leqslant n$. Now the theorem follows. □

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Lemma 4. *If g is a function in $H^{1/2}$ such that g^{-1} belongs to H^∞, then g is a strongly outer function in $H^{1/2}$.*

Proof. Suppose f is in $H^{1/2}$ and $f/g \geqslant 0$ a.e. on ∂D. Then $f = qh^2$, where q is inner and h is outer in H^1. Since g is outer, $g = g_0^2$, where $g_0 \in H^1$ and g_0^{-1} belongs to H^∞. Since $qh^2/g_0^2 \geqslant 0$ a.e. on ∂D, $qhg_0^{-1} = \bar{h}\bar{g}_0^{-1}$. Hence, qhg_0^{-1} is a constant c because $H^1 \cap \overline{H^1} = \mathbb{C}$. Therefore, hg_0^{-1} and q are constants. Thus, qh^2/g_0^2 is a positive constant. This implies the lemma. □

Corollary 5. *Suppose F is a non-zero non-negative function such that qF belongs to $H^{1/2}$ for some inner function q. If q is a constant, then F is a non-negative constant. If*

$$q = \prod_{j=1}^{n} \frac{z - b_j}{1 - \bar{b}_j z}$$

and $|b_j| < 1$, $1 \leqslant j \leqslant n$, then there are complex numbers a_j, $1 \leqslant j \leqslant n$, such that

$$F = \gamma \prod_{j=1}^{n} \frac{(z - a_j)(1 - \bar{a}_j z)}{(z - b_j)(1 - \bar{b}_j z)},$$

where γ is some positive constant.

Proof. If q is a constant, then F is a non-negative constant because 1 is strongly outer in $H^{1/2}$. If $f = qF$, then f belongs to $H^{1/2}$. Since

$$q = z^n \prod_{j=1}^{n} \frac{|1 - \bar{b}_j z|^2}{(1 - \bar{b}_j z)^2} \quad \text{and} \quad \frac{f}{q} \geqslant 0 \text{ a.e. on } \partial D,$$

we have

$$\frac{f}{z^n} \prod_{j=1}^{n} (1 - \bar{b}_j z)^{-2} \geqslant 0 \quad \text{a.e. on } \partial D.$$

By Theorem 2 and Lemma 4, there exist a positive constant γ and complex numbers a_j, $1 \leqslant j \leqslant n$, such that

$$f = \gamma \prod_{j=1}^{n} (z - a_j)(1 - \bar{a}_j z) \times \prod_{j=1}^{n} (1 - \bar{b}_j z)^{-2}$$

and so

$$F = \gamma \prod_{j=1}^{n} \frac{(z - a_j)(1 - \bar{a}_j z)}{(z - b_j)(1 - \bar{b}_j z)}.$$

□

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