OPEN MAPPINGS ON SPHERES

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1. Introduction

In this paper we study open mappings of the sphere, S^n, onto itself. In particular, sufficient conditions are given that such a mapping be a homeomorphism. For the cases $n \leqq 2$ many of the results could be obtained from the work of G. T. Whyburn [7], [8], and [10]. For the cases $n \geqq 3$ the useful results of A. V. Cernavskii, [1], [4], proved to be sufficient. An application is made concerning a finite to one open mapping of one n cell onto itself. It is interesting to note that for $n \leqq 2$ that we could use similar proofs to show that certain quasi-monotone mappings of S^n onto S^n are necessarily monotone mappings.

2. Notation

A mapping f will always mean a continuous function from the space X to the space Y, where X and Y are spheres or cells of the same dimension. The mapping f is open provided that the image of every open set is also an open set and light if the inverse image of each point is totally disconnected. The mapping f is said to be monotone if all point inverses are compact and connected. Following Hemmingsen and Church [2] the set of all points at which the mapping fails to be a local homeomorphism is denoted by B_f. The multiplicity function of f, M, assigns to each point $x \in X$ the number of points in $f^{-1}f(x)$ if this set is finite and is $+\infty$ otherwise. A mapping f is said to have bounded multiplicity if M has a finite upper bound. By a region U is meant an open connected subset and the frontier of U, denoted by FrU, is $\overline{U}-U$.

3. Main results

A mapping f is said to be quasi-monotone provided that for any continuum K in the range space with non-vacuous interior $f^{-1}(K)$ has a finite

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number of components and each of these map onto K under f. This definition is due to A. D. Wallace [5] as well as the following characterization of quasi-monotone mappings. A mapping of one locally connected continuum A onto another such continuum B is quasi-monotone if and only if each component of the inverse of any region R in B maps onto R under f. On locally connected continua quasi-monotone mappings are equivalent to the quasi-open mappings of G. T. Whyburn [6]. The first theorem is just a very special case of a theorem in [7].

THEOREM 3.1. *Let $f(X) = Y$ be an open mapping where X and Y are one-dimensional spheres. If there is a point $x_1 \in X$ for which $f^{-1}f(x_1) = \{x_1\}$, then f is a homeomorphism.*

PROOF. Let $z \in Y$, $z \neq f(x_1)$, and let C be a component of $X - f^{-1}(z)$. Since f is an open mapping $f(C) = Y - \{z\}$. Thus $f^{-1}f(x_1) \subset C$ so that C is the only component of $X - f^{-1}(z)$. Consequently $f^{-1}(z)$ is connected and by openness of f, $f^{-1}(z)$ is degenerate. Therefore f is $1-1$ and necessarily a homeomorphism.

THEOREM 3.2. *Let $f(X) = Y$ be a quasi-monotone mapping, where X and Y are one-dimensional spheres. If there is a point $x_1 \in X$ for which $f^{-1}f(x_1)$ is connected, then f is monotone.*

PROOF. The proof is similar to that of 3.1 using the characterization of quasi-monotone mappings mentioned earlier.

Before proving the corresponding theorems in the case of 2-spheres the following lemma is needed. A more general form of this lemma was proved by G. T. Whyburn in [10].

LEMMA 3.3. *Let R_1, R_2, and R_3 be pairwise disjoint regions in the plane or on a 2-sphere. There do not exist three distinct points which are accessible from each region.*

THEOREM 3.4. *Let $f(X) = Y$ be a light-open mapping, where X and Y are 2-spheres. If there exist three distinct points x_1, x_2, and x_3 such that $f^{-1}f(x_i) = x_i$, $i = 1, 2, 3$, then f is a homeomorphism.*

PROOF. Let $z \in Y$, $z \neq f(x_i)$, $i = 1, 2, 3$, and let S^1 be a simple closed curve in Y containing z, $f(x_1)$, $f(x_2)$, and $f(x_3)$. Let R_1 and R_2 be the two regions of $Y - S^1$. By openness of f each component C of $f^{-1}(R_i)$, $i = 1, 2$, maps onto R_i and $\{x_1, x_2, x_3\} \subset FrC$. Since f is a light open mapping each such C has property S so that each of x_1, x_2, and x_3 is accessible from C. By 3.3 and the proof up to this point $f^{-1}(R_1)$ and $f^{-1}(R_2)$ are connected. Each component of $f^{-1}(S^1)$ must map onto S^1 so that $f^{-1}(S^1)$ is connected and $Frf^{-1}(R_1) = f^{-1}(S^1) = Frf^{-1}(R_2)$ since f is open.

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We now show $f^{-1}(S^1)$ is a simple closed curve by showing no point separates it while every pair of distinct points do separate it. Suppose there is a point $x^1 \in f^{-1}(S^1)$ such that $f^{-1}(S^1)-f^{-1}f(x^1) = A \cup B$ is a separation. The mapping $f|(f^{-1}(S^1)-f^{-1}f(x^1))$ is an open and closed mapping of $f^{-1}(S^1)-f^{-1}f(x^1)$ onto $S^1-f(x)$. Thus $f(A)$ is open and closed in $S^1-f(x^1)$, so that $f(A) = S^1-f(x^1)$. Similarly $f(B) = S^1-f(x^1)$. The point $f(x^1)$ is at most one of the three points $f(x_1)$, $f(x_2)$, of $f(x_3)$, thus each of A and B contains at least two of x_1, x_2, x_3 and this is contradictory. Suppose now that x', x'' are distinct points of $f^{-1}(S^1)$. There is a simple arc (x', x'') spanning $f^{-1}(R_1)$ and a simple arc (x', x'') spanning $f^{-1}(R_2)$. The union of these two simple arcs is a simple closed curve J that meets $f^{-1}(S^1)$ precisely in the points x' and x''. Let M_1 and M_2 be the two regions of $X-J$. Each of M_1 and M_2 contain points of $f^{-1}(R_1)$ and $f^{-1}(R_2)$ and hence each contains points of $f^{-1}(S^1)$. Therefore each pair of points of $f^{-1}(S^1)$ separates $f^{-1}(S^1)$.

The mapping $f|f^{-1}(S^1)$ is an open mapping satisfying the conditions of 3.1 so that it is necessarily a homeomorphism. Now $f^{-1}(z)$ is a single point which in turn implies f is $1-1$.

There is an alternate proof for 3.4 which can be obtained by using the characteristic equation of f, see reference [7], p. 98 or [8], p. 202.

Assuming the hypotheses of theorem 3.4 and applying theorem 1.1, p. 98, [7], one obtains the characteristic equation $k\chi(Y)-\chi(X) = kr-n$, where k is the degree of f, r is the number of singular points of f on Y and n is the number of inverse points in X of these r singular points. Since X and Y are 2-spheres the Euler Characteristic $\chi(X) = \chi(Y) = 2$. Let $q_1, q_2, \cdots, q_r$ be the singular points of f in Y and n_i the number of points in $f^{-1}(q_i)$. It can be assumed that $q_1 = f(x_1)$, $q_2 = f(x_2)$, and $q_3 = f(x_3)$ so that the characteristic equation becomes

$$2k-2 = 3k-3+\sum_{4}^{r}(k-n_i).$$

This reduces to

$$1 = k+\sum_{4}^{r}(k-n_i),$$

and since the last term is non-negative, it then follows that $k = 1$ and hence f is a homeomorphism. Furthermore in case there are two points with unique inverses a similar type of argument proves that no other point of Y can be a singular point. This method of proof was communicated to me by G. T. Whyburn.

THEOREM 3.5. *Let $f(X) = Y$ be a quasi-monotone, where X and Y are 2-spheres. If there exist three distinct points x_1, x_2, and x_3, such that $f^{-1}f(x_i)$, $i = 1, 2, 3$, is a continuum, then f is monotone.*

Proof. For all $y \in Y$, $f^{-1}(y)$ does not separate X, for if C_1 is a component of $X-f^{-1}(y)$, then $f(C_1)$ is open and $f(C_1)$ is closed in $Y-\{y\}$, thus $f(C_1) = Y-\{y\}$. Each such C_1 must contain at least two of the three continua $f^{-1}f(x_i)$, $i = 1, 2, 3$, which in turn implies $Y-f^{-1}(y)$ is connected. It now follows that no component of $f^{-1}(y)$ can separate X.

Let $f = Lm$ be the unique monotone light factorization of f and let M be the middle space. It is well known [9], [3] that L is light-open and f generates an upper semi-continuous decomposition of X into continua which do not separate X and consequently M is a topological 2-sphere. We can now apply **3.4** to the light-open mapping $L(M) = Y$ since there are at least three distinct points z for which $L^{-1}L(z) = z$ and obtain the result that L is a homeomorphism. This now implies that f is necessarily a monotone mapping.

In order to treat the higher dimensional case, we need the following result of A. V. Cernavskii [1] which was also proved by J. Väisälä [4] using more elementary methods.

Theorem 3.6. *Let M and N be connected n-manifolds without boundary and $f(M) = N$ a finite to one open and closed mapping. Then f has bounded multiplicity, the multiplicity function takes its maximum value on $M-f^{-1}f(B_f)$, where it is constant, and dim $f^{-1}f(B_f) \leqq n-2$.*

Theorem 3.7. *Let $f(X) = Y$ be finite to one open mapping, where X and Y are n-spheres. If C is an $(n-1)$ dimensional set in X for which $x \in C$ implies $f^{-1}f(x)$ is a single point, then f is a homeomorphism.*

Proof. By (3.6) dim $f^{-1}f(B_f) \leqq n-2$ so that $(X-f^{-1}f(B_f)) \cap C \neq \emptyset$. Since the multiplicity function is constant on $X-f^{-1}f(B_f)$, it must take on the value one. Consequently f is $1-1$ on $X-f^{-1}f(B_f)$ and therefore by the openness of f $1-1$ on the closure of $X-f^{-1}f(B_f)$ which is X itself.

It is possible to construct simple examples of finite to one open mappings of the n-sphere, $n > 1$, onto itself for which the dimension of $f^{-1}f(B_f)$ is $n-2$. It is not known whether a light-open mapping of an n-sphere, $n > 2$, onto itself is necessarily finite to one. In fact the latter is not even known for the case when f in addition to being light and open also preserves the dimension of all closed subsets.

We use **3.7** and a simple construction to solve a conjecture of G. T. Whyburn [11], which can be stated as follows. 'If f is a finite to one open mapping on a **3**-cell X with boundary B such that f is $1-1$ on B and $f(B)$ is disjoint from $f(X-B)$, then f is a homeomorphism'. The following technique reduces the problem for n-cells to an application of **3.7** and gives an affirmative answer. Consider the boundary B of the n-cell X to be such that X is the set of all points on the unit sphere S^n in E^{n+1} with last coordinate non-positive and B to be all the points of X with last coordinate zero. Extend the mapping f to a light open mapping F of S^n onto S^n in the following manner. Suppose f has coordinate functions $f_0, f_1, \cdots, f_n$ and $z \in S^n$, $z = (x_0, \cdots, x_n)$, $x_n > 0$.

Define $F(x)$ as follows.

$$F(x) = (f_0(x_0, \cdots, x_{n-1}, -x_n), f_1(x_0, \cdots, x_{n-1}, -x_n), \cdots,\\ f_{n-1}(x_0, \cdots, x_{n-1}, -x_n), -f_n(x_0, \cdots, x_{n-1}, -x_n))$$

The mapping F of S^n onto S^n thus defined is a finite to one open mapping and it is at least $1-1$ on the $n-1$ dimensional set B, so by theorem 3.7 F is a homeomorphism and consequently f is a homeomorphism.

THEOREM 3.8. *Let $f(X) = Y$ be a finite to one open mapping of an n-cell onto an n-cell. If B is the boundary of X and $f(B) = B$, $f^{-1}f(B) = B$, and f/B is $1-1$, then f is a homeomorphism.*

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