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GENERALISATIONS OF INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE THROUGH CONVEXITY

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Abstract

In this paper, we establish various inequalities for some differentiable mappings that are linked with the illustrious Hermite–Hadamard integral inequality for mappings whose derivatives are s-(α, m)-convex. The generalised integral inequalities contribute better estimates than some already presented. The inequalities are then applied to numerical integration and some special means.

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1. Introduction

Let $f : \emptyset \neq I \subseteq \mathbb{R} \to \mathbb{R}$ be a function defined on the interval I of real numbers. Then f is called convex if

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$. Geometrically, this means that if P, Q and R are three distinct points on the graph of f with Q between P and R, then Q is on or below the chord PR. There are many results associated with convex functions in the area of inequalities, one of these being the classical Hermite–Hadamard inequality:

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

for $a, b \in I$, with $a < b$.

In [5], Hudzik and Maligranda considered, among others, the class of functions which are s-convex in the first and second sense. This class is defined as follows.

Definition 1.1. A function $f : [0, \infty) \to \mathbb{R}$ is said to be s-convex, or f belongs to the class K_s^i, if

$$f(\mu x + \nu y) \leq \mu^s f(x) + \nu^s f(y)$$

holds for all $x, y \in [0, \infty)$, $\mu, \nu \in [0, 1]$ and for some fixed $s \in (0, 1]$.

If $\mu^s + \nu^s = 1$ the above class of convex functions is called s-convex in the first sense and represented by K_s^1; and if $\mu + \nu = 1$ it is called s-convex in the second sense and represented by K_s^2. It may be noted that every 1-convex function is convex.

In the same paper [5], Hudzik and Maligranda discussed a few results connected with s-convex functions in the second sense. Some new results on Hadamard's inequality for s-convex functions are discussed also in [4], as well as many important inequalities connected with 1-convex (convex) functions, one of which is (1.1).

In [10], Mihesan presented the class of (α, m)-convex functions as reproduced below.

Definition 1.2. The function $f : [0, b] \to \mathbb{R}$ is said to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$, if, for every $x, y \in [0, b]$ and $t \in [0, 1]$,

$$f(tx + m(1-t)y) \le t^\alpha f(x) + m(1 - t^\alpha) f(y).$$

Note that for $(\alpha, m) \in \{(0, 0), (\alpha, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$ one obtains the following classes of functions respectively: increasing, α-star-shaped, star-shaped, m-convex, convex and α-convex. For other new proofs, noteworthy extensions, generalisations and numerous applications on inequalities see [6–9, 12].

Denote by $K_m^\alpha(b)$ the set of all (α, m)-convex functions on $[0, b]$ with $f(0) \le 0$. For recent results and generalisations referring to m-convex and (α, m)-convex functions, see [1, 2, 13].

In [4], Dragomir and Pearce discussed inequalities for differentiable and twice differentiable functions connected with the Hermite–Hadamard inequality on the basis of the following lemma.

Lemma 1.3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the interior of I), $a, b \in I$ with $a < b$. If $f' \in L^1[a, b]$, then*

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{(b-a)}{2}\int_0^1 (1-2t) f'(ta + (1-t)b)\,dt. \tag{1.2}$$

In [3], Dragomir and Agarwal established the following result connected with the right-hand side of (1.2) and applied it to some elementary inequalities for real numbers and numerical integration.

Lemma 1.4. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$. If $f' \in L^1[a, b]$, then*

$$\begin{aligned}\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{(b-a)}{2}\int_0^1\int_0^1 (f'(ta + (1-t)b)\\ - f'(ua + (1-u)b))(u - t)\,dt\,du.\end{aligned} \tag{1.3}$$

This paper is organised as follows. In Section 2 we define a generalised convex function and discuss some new integral inequalities of Hermite–Hadamard type for generalised convex functions. In Section 3 we give some new applications for some special means. The inequalities are then applied to numerical integration in Section 4.

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2. Definitions and main results

To establish our principal results, we first give the following definitions.

DEFINITION 2.1. A function $f:[0,\infty)\to[0,\infty)$ is said to be s-(α,m)-convex in the first sense, or f belongs to the class $K^{\alpha,s}_{m,1}$, if for all $x,y\in[0,\infty)$ and $\mu\in[0,1]$, the following inequality holds:

$$f(\mu x+(1-\mu)y)\le(\mu^{\alpha s})f(x)+m(1-\mu^{\alpha s})f\Big(\frac{y}{m}\Big)$$

where $(\alpha,m)\in[0,1]^2$ and for some fixed $s\in(0,1]$.

DEFINITION 2.2. A function $f:[0,\infty)\to[0,\infty)$ is said to be s-(α,m)-convex in the second sense, or f belongs to the class $K^{\alpha,s}_{m,2}$, if for all $x,y\in[0,\infty)$ and $\mu,\nu\in[0,1]$, the following inequality holds:

$$f(\mu x+(1-\mu)y)\le(\mu^{\alpha})^s f(x)+m(1-\mu^{\alpha})^s f\Big(\frac{y}{m}\Big)$$

where $(\alpha,m)\in[0,1]^2$ and for some fixed $s\in(0,1]$.

THEOREM 2.3. *Let $f:I^o\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I^o (the interior of I), $a,b\in I$ with $a<b$, and $f'\in L^1[a,b]$. If the mapping $|f'|$ is s-(α,m)-convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{b-a}{2}\left(v_1|f'(a)|+v_2\left|f'\Big(\frac{b}{m}\Big)\right|\right)\tag{2.1}$$

where $v_1=(1+2^{\alpha s}(\alpha s))/2^{\alpha s}(\alpha s+1)(\alpha s+2)$ and $v_2=m(\frac{1}{2}-v_1)$.

PROOF. From Lemma 1.3,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{b-a}{2}\int_0^1|1-2t||f'(ta+(1-t)b)|\,dt.\tag{2.2}$$

Since $|f'|$ is s-(α,m)-convex on $[a,b]$ for all $t\in[0,1]$, this becomes

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\quad\le\frac{b-a}{2}\int_0^1|1-2t|\left(t^{\alpha s}|f'(a)|+m(1-t^{\alpha s})\left|f'\Big(\frac{b}{m}\Big)\right|\right)dt.\end{aligned}\tag{2.3}$$

Here

$$\int_0^1 t^{\alpha s}|1-2t|\,dt=\int_0^{1/2}(1-2t)t^{\alpha s}\,dt+\int_{1/2}^1(2t-1)t^{\alpha s}\,dt=\frac{1+2^{\alpha s}(\alpha s)}{2^{\alpha s}(\alpha s+1)(\alpha s+2)}\tag{2.4}$$

and

$$\int_0^1(1-t^{\alpha s})|1-2t|\,dt=\frac{1}{2}-\frac{1+2^{\alpha s}(\alpha s)}{2^{\alpha s}(\alpha s+1)(\alpha s+2)}.\tag{2.5}$$

Inequalities (2.3), (2.4) and (2.5) together imply (2.1). □

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REMARK 2.4. For $(\alpha, m) = (1, 1)$ in (2.1), we get [11, Theorem 2].

THEOREM 2.5. *Let the assumptions of Theorem 2.3 be satisfied with $p > 1$, such that $q = p/(p-1)$. If the mapping $|f'|^q$ is s-(α, m)-convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{2(p+1)^{1/p}}\left(\frac{|f'(a)|^q + m\alpha s|f'(\frac{b}{m})|^q}{\alpha s+1}\right)^{1/q}. \tag{2.6}$$

PROOF. By applying Hölder's inequality to the right-hand side of (2.2),

$$\int_0^1 |1-2t||f'(ta+(1-t)b)|\,dt \le \left(\int_0^1 |1-2t|^p\,dt\right)^{1/p}\left(\int_0^1 |f'(ta+(1-t)b)|^q\,dt\right)^{1/q}. \tag{2.7}$$

Here,

$$\int_0^1 |1-2t|^p\,dt = \frac{1}{1+p}. \tag{2.8}$$

Since $|f'|^q$ is s-(α, m)-convex on $[a, b]$ for all $t \in [0, 1]$,

$$|f'(ta+(1-t)b)|^q \le t^{\alpha s}|f'(a)|^q + m(1-t^{\alpha s})|f'(b)|^q,$$

so the second integral on the right-hand side of (2.7) can be simplified by simple integration as:

$$\int_0^1 |f'(ta+(1-t)b)|^q\,dt = \frac{|f'(a)|^q + m\alpha s|f'(\frac{b}{m})|^q}{\alpha s+1}. \tag{2.9}$$

Inequalities (2.7), (2.8) and (2.9) together imply (2.6). □

REMARK 2.6. For $(\alpha, m) = (1, 1)$ in (2.6), we get [11, Theorem 4].

THEOREM 2.7. *Let the assumptions of Theorem 2.3 be satisfied with $q > 1$. If the mapping $|f'|^q$ is s-(α, m)-convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{2^{(p+1)/p}}\left(v_1|f'(a)|^q + v_2\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{1/q} \tag{2.10}$$

where $v_1 = (1 + 2^{\alpha s}(\alpha s))/2^{\alpha s}(\alpha s+1)(\alpha s+2)$ and $v_2 = m(\frac{1}{2} - v_1)$.

PROOF. Inequality (2.2) reduces to the following form:

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\quad\le \frac{b-a}{2}\int_0^1 |1-2t|^{1/p}|1-2t|^{1/q}|f'(ta+(1-t)b)|\,dt\end{aligned} \tag{2.11}$$

where $1/p + 1/q = 1$.

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By applying Hölder's inequality to (2.11), for $q > 1$,

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq \frac{b-a}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1/p}\left(\int_0^1 |1-2t||f'(ta+(1-t)b)|^q\,dt\right)^{1/q}.\end{aligned} \tag{2.12}$$

Applying the s-(α, m)-convexity of $|f'|^q$ on $[a, b]$ for all $t \in [0, 1]$ to the second integral on the right-hand side of (2.12),

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1/p}\left(t^{\alpha s}|1-2t||f'(a)|^q + m(1-t^{\alpha s})|1-2t|\left|f'\left(\frac{b}{m}\right)\right|^q\,dt\right)^{1/q}.\end{aligned} \tag{2.13}$$

Here,

$$\int_0^1 t^{\alpha s}|1-2t|\,dt = \frac{1+2^{\alpha s}(\alpha s)}{2^{\alpha s}(\alpha s+1)(\alpha s+2)} \tag{2.14}$$

and, similarly,

$$\int_0^1 (1-t^{\alpha s})|1-2t|\,dt = \frac{1}{2} - \frac{1+2^{\alpha s}(\alpha s)}{2^{\alpha s}(\alpha s+1)(\alpha s+2)}. \tag{2.15}$$

Inequalities (2.13), (2.14) and (2.15) together imply (2.10). □

Remark 2.8. For $(\alpha, m) = (1, 1)$ in (2.10), we get [11, Theorem 6].

Theorem 2.9. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o (the interior of I), $a, b \in I$ with $a < b$, and $f' \in L^1[a, b]$. If the mapping $|f'|$ is s-(α, m)-convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{2}\left(u_1|f'(a)| + u_2\left|f'\left(\frac{b}{m}\right)\right|\right), \tag{2.16}$$

where $u_1 = ((\alpha s)^2 + 3\alpha s + 4)/2(\alpha s+1)(\alpha s+2)(\alpha s+3)$ and $u_2 = m(\frac{1}{3} - u_1)$.

Proof. Taking absolute values in (1.3),

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq \frac{b-a}{2}\int_0^1\int_0^1 |f'(ta+(1-t)b) - f'(ua+(1-u)b)||u-t|\,dt\,du \\ = (b-a)\int_0^1\int_0^1 |f'(ta+(1-t)b)||u-t|\,dt\,du.\end{aligned} \tag{2.17}$$

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Since $|f'|$ is s-(α, m)-convex on $[a, b]$ for all $t \in [0, 1]$, (2.17) may be written as

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \le (b-a)\int_0^1\int_0^1 \Big(t^{\alpha s}|u-t||f'(a)| + m(1-t^{\alpha s})|u-t|\Big|f'\Big(\frac{b}{m}\Big)\Big|\Big)\,dt\,du.\end{aligned} \tag{2.18}$$

Here,

$$\begin{aligned}\int_0^1\int_0^1 t^{\alpha s}|u-t|\,dt\,du &= \int_0^1\Big(\int_0^u t^{\alpha s}(u-t)\,dt + \int_u^1 t^{\alpha s}(t-u)\,dt\Big)\,du \\ &= \frac{(\alpha s)^2 + 3\alpha s + 4}{2(\alpha s+1)(\alpha s+2)(\alpha s+3)}\end{aligned} \tag{2.19}$$

and, analogously,

$$\int_0^1\int_0^1 (1-t^{\alpha s})|u-t|\,dt\,du = \frac{1}{3} - \frac{(\alpha s)^2 + 3\alpha s + 4}{2(\alpha s+1)(\alpha s+2)(\alpha s+3)}. \tag{2.20}$$

Inequalities (2.18), (2.19) and (2.20) together imply (2.16). □

Remark 2.10. For $(\alpha, m) = (1, 1)$ in (2.16), we get [11, Theorem 8].

Theorem 2.11. *Let the assumptions of Theorem 2.9 be satisfied with $p > 1$, such that $q = p/(p-1)$. If the mapping $|f'|^q$ is s-(α, m)-convex on $[a, b]$, then*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \le (b-a)\Big(\frac{2}{(p+1)(p+2)}\Big)^{1/p}\Big(\frac{|f'(a)|^q + m\alpha s|f'(\frac{b}{m})|^q}{\alpha s+1}\Big)^{1/q}.\end{aligned} \tag{2.21}$$

Proof. By applying Hölder's inequality to the right-hand side of (2.2),

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \le (b-a)\Big(\int_0^1\int_0^1 |f'(ta+(1-t)b)|^q\,dt\,du\Big)^{1/q} \\ \times\Big(\int_0^1\int_0^1 |u-t|^p\,dt\,du\Big)^{1/p}.\end{aligned} \tag{2.22}$$

Here,

$$\int_0^1\int_0^1 |u-t|^p\,dt\,du = \frac{2}{(p+1)(p+2)}. \tag{2.23}$$

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By the s-(α, m)-convexity of $|f'|^q$ on $[a, b]$ for all $t \in [0, 1]$, the first integral on the right-hand side of (2.22) may be solved as

$$\int_0^1 \int_0^1 |f'(ta + (1-t)b)|^q \, dt \, du \leq \frac{|f'(a)| + m\alpha s \left|f'\left(\frac{b}{m}\right)\right|}{\alpha s + 1}. \tag{2.24}$$

Inequalities (2.22), (2.23) and (2.24) together imply (2.21). □

REMARK 2.12. For $(\alpha, m) = (1, 1)$ in (2.21), we get [11, Theorem 10].

THEOREM 2.13. *Let the assumptions of Theorem 2.9 be satisfied with $q > 1$. If the mapping $|f'|^q$ is s-(α, m)-convex on $[a, b]$, then*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\, dx\right| \leq \frac{b-a}{3^{1/p}}\left(u_1|f'(a)|^q + u_2\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{1/q}, \tag{2.25}$$

where $u_1 = ((\alpha s)^2 + 3\alpha s + 4)/2(\alpha s + 1)(\alpha s + 2)(\alpha s + 3)$ and $u_2 = m(\frac{1}{3} - u_1)$.

PROOF. By applying Hölder's inequality to the right-hand side of (2.2),

$$\begin{aligned}\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\, dx\right| &\leq (b-a)\left(\int_0^1\int_0^1 |u-t||f'(ta + (1-t)b)|^q \, dt\, du\right)^{1/q} \\ &\quad \times \left(\int_0^1\int_0^1 |u-t| \, dt\, du\right)^{1/p}.\end{aligned} \tag{2.26}$$

Here,

$$\int_0^1\int_0^1 |u-t|\, dt\, du = \frac{1}{3}. \tag{2.27}$$

By the s-(α, m)-convexity of $|f'|^q$ on $[a, b]$ for all $t \in [0, 1]$, the first integral on the right-hand side of (2.26) may be estimated as

$$\begin{aligned}&\int_0^1\int_0^1 |u-t||f'(ta + (1-t)b)|^q \, dt\, du \\ &\quad \leq \int_0^1\int_0^1 |u-t|\left(t^{\alpha s}|f'(a)|^q + m(1 - t^{\alpha s})\left|f'\left(\frac{b}{m}\right)\right|^q\right) dt\, du,\end{aligned}$$

and

$$\int_0^1\int_0^1 t^{\alpha s}|u-t|\, dt\, du = \frac{\alpha s^2 + 3\alpha s + 4}{2(\alpha s + 1)(\alpha s + 2)(\alpha s + 3)} \tag{2.28}$$

and

$$\int_0^1\int_0^1 (1 - t^{\alpha s}|u-t|\, dt\, du = \frac{1}{3} - \frac{\alpha s^2 + 3\alpha s + 4}{2(\alpha s + 1)(\alpha s + 2)(\alpha s + 3)}, \tag{2.29}$$

which completes the proof. □

REMARK 2.14. For $(\alpha, m) = (1, 1)$ in (2.25), we get [11, Theorem 12].

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3. Application to some special means

Let us recall the following means for any two positive numbers a and b:

(1) the *arithmetic mean*,

$$A \equiv A(a,b) = \frac{a+b}{2};$$

(2) the *geometric mean*,

$$G \equiv G(a,b) = \sqrt{ab};$$

(3) the *harmonic mean*,

$$H \equiv H(a,b) = \frac{2ab}{a+b};$$

(4) the *p-logarithmic mean*,

$$L_p \equiv L_p(a,b) = \begin{cases} a & \text{if } a = b, \\ \left(\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{1/p} & \text{if } a \neq b; \end{cases}$$

(5) the *identric mean*,

$$I \equiv I(a,b) = \begin{cases} a & \text{if } a = b, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)} & \text{if } a \neq b; \end{cases}$$

(6) the *logarithmic mean*,

$$L \equiv L(a,b) = \begin{cases} a & \text{if } a = b, \\ \dfrac{b-a}{\log b - \log a} & \text{if } a \neq b. \end{cases}$$

The following inequalities are well known in the literature (see [14]):

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 3.1. *Let $p > 1$, $0 < a < b$ and $q = p/(p-1)$. Then*

$$|A(a,b) - L(a,b)| \leq \frac{\log b - \log a}{2(p+1)^{1/p}} A^{1/q}(|a|^q, |b|^q). \tag{3.1}$$

Proof. By Theorem 2.5 applied to the mapping $f(x) = e^x$ for $(\alpha, m) = (1,1)$, we have (3.1). □

Proposition 3.2. *Let $p > 1$, $0 < a < b$ and $q = p/(p-1)$. Then*

$$\left|\frac{I(a,b)}{G(a,b)}\right| \leq \exp\left(\frac{b-a}{2} H^{-1/q}(|a|^q, |b|^q)\right).$$

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Proof. The result follows by Theorem 2.7, setting $f(x) = -\log(1-x)$ for $(\alpha, m) = (1, 1)$. □

Another result connected with the p-logarithmic mean $L_p(a, b)$ is the following proposition.

Proposition 3.3. *Let $p > 1$, $0 < a < b$ and $q = p/(p-1)$. Then*

$$|A(a^n, b^n) - L_n^n(a^n, b^n)| \leq |n|^q \frac{b-a}{3} A(|a|^{q(n-1)}, |b|^{q(n-1)}).$$

Proof. The result follows by Theorem 2.13, setting $f(x) = (1-x)^n$, $|n| \geq 2$ and $n \in \mathbb{Z}$ for $(\alpha, m) = (1, 1)$. □

4. Error estimates for the trapezoidal formula

Let D be the partition $\{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ of the interval $[a, b]$ and consider the quadrature formula

$$\int_a^b f(x)\, dx = S(f, D) + R(f, D)$$

where

$$S(f, D) = \sum_{k=0}^{n-1} \frac{f(x_k) + f(x_{k+1})}{2}(x_{k+1} - x_k)$$

for the trapezoidal version and $R(f, D)$ denotes the related approximation error.

Proposition 4.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o such that $f' \in L^1[a, b]$, where $a, b \in I$ with $a < b$ and $|f'|$ is s-convex on $[a, b]$. For every partition D of $[a, b]$ the trapezoidal error approximation satisfies*

$$|R(f, D)| \leq \frac{1}{2^{1/p}} \left(\frac{s \cdot 2^s + 1}{2^s(s+1)(s+2)} \right)^{1/q} \sum_{k=0}^{n-1} \frac{(x_{k+1} - x_k)^2}{2} (|f'(x_k)| + |f'(x_{k+1})|), \tag{4.1}$$

where $p > 1$.

Proof. By applying Theorem 2.7 on the subinterval $[x_k, x_{k+1}]$ of the partition D of $[a, b]$ for $k = 0, 1, 2, \ldots, n-1$, for $(\alpha, m) = (1, 1)$ and using the fact that $\sum_{m=1}^{n-1} (\Phi_m + \Psi_m)^r \leq \sum_{m=1}^{n-1} (\Phi_m)^r + \sum_{m=1}^{n-1} (\Psi_m)^r$ for $0 < r < 1$, where for each m both $\Phi_m, \Psi_m \geq 0$,

$$\begin{aligned} &\left| f\left(\frac{x_{k+1} + x_k}{2} \right) - \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} f(x)\, dx \right| \\ &\quad \leq \frac{x_{k+1} - x_k}{2^{(p+1)/p}} \left(\frac{s \cdot 2^s + 1}{2^s(s+1)(s+2)} \right)^{1/q} (|f'(x_k)| + |f'(x_{k+1})|). \end{aligned} \tag{4.2}$$

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Taking the sum over k from 0 to $n-1$ and taking into account that $|f'|^q$ is s-(α, m)-convex,

$$\left|\int_a^b f(x)\,dx - S(f,K)\right| = \left|\sum_{k=0}^{n-1}\left(\int_{x_k}^{x_{k+1}} f(x)\,dx - (x_{k+1}-x_k)\frac{f(x_{k+1})+f(x_k)}{2}\right)\right|$$
$$\le \sum_{k=0}^{n-1}\left|\left(\int_{x_k}^{x_{k+1}} f(x)\,dx - (x_{k+1}-x_k)\frac{f(x_{k+1})+f(x_k)}{2}\right)\right|.$$

This gives

$$|R(f,D)| \le \sum_{k=0}^{n-1}(x_{k+1}-x_k)\left|\frac{f(x_{k+1})+f(x_k)}{2} - \frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}} f(x)\,dx\right|. \tag{4.3}$$

By combining (4.2) and (4.3), we get (4.1), which completes the proof. □

Proposition 4.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o such that $f' \in L^1[a,b]$, where $a, b \in I$ with $a < b$ and $|f'|^q$ is s-(α, m)-convex on $[a,b]$. Then for every partition D of $[a,b]$ the trapezoidal error approximation satisfies*

$$|R(f,D)| \le \left(\frac{2}{3}\right)^{1/p}\left(\frac{s^2+3s+4}{(s+1)(s+2)(s+3)}\right)^{1/q}\sum_{k=0}^{n-1}\frac{(x_{k+1}-x_k)^2}{2}(|f'(x_k)|^q + |f'(x_{k+1})|^q)^{1/q}$$
$$\le \left(\frac{2}{3}\right)^{1/p}\left(\frac{s^2+3s+4}{(s+1)(s+2)(s+3)}\right)^{1/q}\sum_{k=0}^{n-1}\frac{(x_{k+1}-x_k)^2}{2}(|f'(x_k)| + |f'(x_{k+1})|).$$

Proof. The proof is very similar to that of Proposition 4.1, using Theorem 2.13. □

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