# SOME PROBLEMS OF ANISOTROPIC SCATTERING IN PLANETARY ATMOSPHERES*

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**Abstract.** The similarity rules to compare atmospheres with anisotropic and isotropic scattering are reviewed. Omission of a narrow diffraction peak in the scattering pattern is permitted and corresponds to a special application of the similarity rules. It is shown that the extrapolation length for conservative scattering can be found with great precision from a formula involving only $\omega_2$ and $\omega_3$. An asymptotic expression for high-order scattering in a semi-infinite atmosphere is given and it is shown by some examples that this expression can be used to find by interpolation the terms of any order. Finally, the way in which the contrast between a dark and bright area near the center of the disk of Mars is affected by an overlying haze is computed for isotropic and for anisotropic scattering.

Much of the literature on the photometry and spectroscopy of cloudy atmospheres still deals with the cloud particles on the assumption of isotropic or linearly anisotropic light scattering. The reason for this is that calculations with a realistic phase function are complicated and time-consuming. In this paper we report on various approaches which we have recently tried out and which have in common that accurate quantitative answers to certain problems can be given without an excessive amount of computer time.

## 1. Similarity Relations

The question, first clearly posed during a symposium at Tucson 2 years ago, and answered in the Symposium Volume (Van de Hulst and Grossman, 1968) is the following. If a certain multiple scattering problem has been solved on the assumption of isotropic scattering, and we wish to change to an anisotropic law, can we then avoid a completely new computation and transform the total optical depth $b$ and the single scattering albedo $a$ in such a manner that the resulting intensity of reflected radiation is closely similar to the earlier result? The answer to this question is positive and the necessary relations can be summarized by two rules

transform $a$ so that $y = (1 - a)/k = \text{constant}$
transform $b$ so that $ba(1 - g) = \text{constant}$

Here $k$ is the characteristic root, or inverse diffusion length and $g = \overline{\cos\alpha}$ is the anisotropy factor of the single scattering phase function. In the conventional representation of the scattering law by a series of Legendre functions,

$$\Phi(\cos\alpha) = \sum_{n=0}^{N} \omega_n P_n(\cos\alpha)$$

* Condensed version of paper presented at this symposium. Section 3 has been added after the symposium.

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we have $\omega_0 = a$, $\omega_1 = 3ag$. Both rules were derived from the requirement that the diffusion through very thick layers should be made as closely similar as possible. Therefore, these rules should not be expected to give useful results for thin layers, or, generally, in any situation where single scattering, or light scattered only a few times, dominates the result.

Each of these rules degenerates into a simple limit: A semi-infinite layer ($b = \infty$) should remain semi-infinite and a conservative atmosphere ($a = 1$, $k = 0$) should remain conservative.

Examples given in the earlier paper and further tests by Hansen (1969a, b) showed that a difference of 2–5% between the exact results computed for 'similar' situations was not uncommon. This means that the relations are certainly good to 10–20% accuracy, but that they cannot be trusted if a 1% accuracy or better is required.

We now present two additional tests, which do not, however, change this conclusion.

### A. ADDITION OF A FORWARD PEAK

This is a logical test. Strict forward scattering is no scattering at all. Hence, if inside an atmosphere formed by real scatterers and characterized by the values $b'$, $a'$, $g'$, $k'$ we sprinkle a well-mixed medium of fictitious conservative forward scatterers, characterized by the values $b''$, $a'' = 1$, $g'' = 1$, $k'' = 0$, nothing really changes. The combined atmosphere, characterized by the values $b$, $a$, $g$, $k$ therefore gives reflection function, transmission function, and internal radiation field exactly as the atmosphere we started out with.

These identical situations should certainly be called similar. We shall now see if they indeed obey the similarity rules given above. The formal relations for the combined atmospheres are

$$\text{total extinction depth: } b = b' + b''$$

which is the sum of

$$\text{total absorption depth: } b\,(1 - a) = b'(1 - a')$$

and

$$\text{total scattering depth: } ba = b'a' + b''$$

Taking only the forward component of the last relation we have

$$\text{scattering depth times } \overline{\cos\alpha}\text{: } bag = b'a'g' + b''$$

Finally the attenuation suffered by the total flux in the diffusion domain (a description valid if the depth is very large) is

$$\text{diffusion depth: } bk = b'k'$$

Combination of these various equations easily leads to the proportionalities

$$\frac{b}{b'} = \frac{1 - a'}{1 - a} = \frac{a'(1 - g')}{a(1 - g)} = \frac{k'}{k}$$

which agree with the rules stated in Section 1.

The preceding transformation is presented in the form of a logical test, but also has practical applications. First, the actual scattering pattern of particles large compared to the wavelength has a diffraction peak, which, though not infinitely sharp, is very strongly concentrated around the forward direction. The necessity to include this peak can be very bothersome in numerical calculations, because it greatly increases the number of Legendre functions which have to be retained in the expansion of the phase function. A practical consequence of the result just derived is that we can simply omit the diffraction peak, provided it is sharp, and provided we attach to the remaining phase function the appropriate values of $a'$ and $g'$. The same conclusion has been reached by Hansen (1969a).

Secondly, if a result is sought, say for a phase function with a certain $g'$ and $a'$ and we wish to make use of tables (e.g. Van de Hulst, 1968a) available for a somewhat different anisotropy $g$, we can, by arbitrarily adding a fictitious forward peak, quite easily transform to this value $g$. Upon transforming $a'$ to $a$ and $b'$ to $b$ accordingly, we can then obtain the result from the existing tables by interpolation in $a$ and $b$.

## B. CONSERVATIVE, SEMI-INFINITE ATMOSPHERES

Under the particular assumption that $b = \infty$ (semi-infinite atmosphere) and $\omega_0 = a = 1$ (conservative scattering) any phase function is 'similar'. The exact theory (Chandrasekhar, 1950; Busbridge and Orchard, 1968) shows that in this case the value of $\omega_1$ is irrelevant. All quantities must, therefore, be slowly varying function of $\omega_2$, $\omega_3$, etc.

Numerical examples show that this is indeed true. Figure 1 shows by way of illustration the value of the extrapolation length $(1-g)q$. This has the well-known value 0.7104 for isotropic (or linearly anisotropic) scattering. The first two decimals remain the same throughout the figure; only the third and fourth are given. Values for $N=2$ for various $\omega_2$ were taken from Horak and Janousek (1965). Values for $N=3$, representing the combinations $\omega_2=1$, $\omega_3=\frac{1}{2}$ and $\omega_2=1$, $\omega_3=1$ were computed from Busbridge and Orchard (1968). In addition, the figure shows three points for $N=\infty$ corresponding to the Henyey-Greenstein functions for $g=\frac{1}{4}$, $\frac{1}{3}$, and $\frac{1}{2}$ (Van de Hulst, 1968a and unpublished work). The fact that these points fit smoothly into the pattern, although here $\omega_4$ etc. are non-zero, shows that the influence of those higher terms is very small indeed. The linear approximation

$$q(1 - g) = 0.7104 + 0.0020\omega_2 - 0.0010\omega_3$$

gives results good to 0.05% for all points shown in Figure 1 and even beyond that up to the Henyey-Greenstein function with $g=\frac{3}{4}$.

The extrapolation length discussed here forms a particularly favourable case. A less extreme example is the 'escape function', which is the solution of the Milne problem. Its empirical linear approximations in the domain of $(\omega_2, \omega_3)$ corresponding to Figure 1 are in the perpendicular direction

$$K(1) = 1.259 + 0.012\omega_2 - 0.012\omega_3$$

and in grazing directions

$$K(0) = 0.433 - 0.028\omega_2 - 0.021\omega_3$$

which lead to errors of 1 or 2% at most.

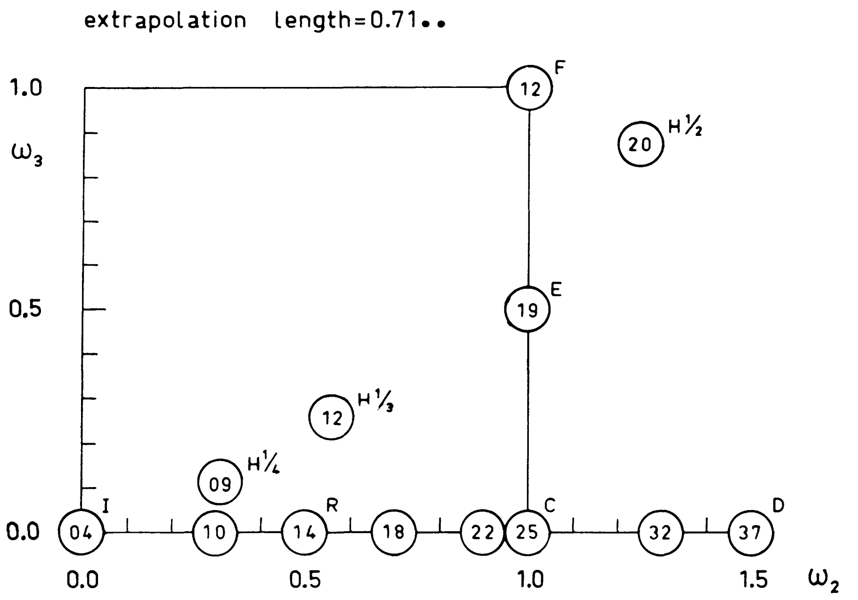


Fig. 1. Values of the extrapolation length for conservative atmospheres plotted against the coefficients $\omega_2$ and $\omega_3$ of the Legendre function. The third and fourth decimals following 0.71 are given.

## 2. High-Order Scattering

In certain problems it would be very attractive to fall back to the very simple description in which the diffusely reflected radiation is made up of the contribution of single scattering, two successive scatterings etc. If the form of the scattering pattern is fixed and only the albedo $a$ is retained as variable we obviously can expand the reflected intensity (in any direction or integrated over a range of directions) in the form

$$f = f_1 a + f_2 a^2 + \cdots + f_n a^n + \cdots$$

in which $f_n$ signifies the contribution given by quanta which have been scattered $n$ times in succession. Such an expansion can be particularly handy if $a$ has to be varied continuously as, e.g., inside an absorption line. Belton (1968) has shown that with known coefficients $f_n$ a theory of the curve of growth follows directly.

It is not too much trouble to find the low-order terms, say $f_1$ to $f_5$, with reasonable accuracy. The principle simply is to take half-steps as follows:

Intensity of $n$-th order
↓ (integration over angles, using single scattering pattern)
source function of $(n + 1)$-th order
↓ (integration over optical depth)
intensity of $(n + 1)$-th order

However, for thick layers the convergence is extremely slow; this is the main reason why this 'successive-order' method has not found a wider use. The asymptotic

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behavior of $f_n$ for $n \to \infty$ is as a geometric series as long as the optical thickness $b$ is finite, but gets a different character for semi-infinite atmospheres. The dominant term in the asymptotic behavior for $b=\infty$ has been derived by Uesugi and Irvine (1969). Further terms can best be found (Van de Hulst, 1970) by starting from the expansion

$$f = G_0 + G_1 t + G_2 t^2 + G_3 t^3 + \cdots$$

which may in turn be derived from a similar expansion in terms of $k$ (e.g. Van de Hulst, 1968b). Here $t=(1-a)^{1/2}$ and $k$ again is the smallest characteristic root, or inverse diffusion length. We find

$$f_n = -(4\pi)^{-1/2} G_1 (n + c)^{-3/2} \{1 + O(n^{-2})\}$$

where

$$c = G_3/G_1 - \tfrac{1}{4}$$

Figure 2 shows by means of a few examples how this knowledge may be used to find the numerical values for all $n$. The abscissa is $(c+4)^{-2}$; the number 4 is chosen arbitrarily to match some average value of $c$. The ordinate is $f_n(n+c)^{3/2}$ with the correct value of $c$. Further specifications for the four examples are given in Table I.

TABLE I

| Example | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Quantity | *URU* | $R(1, 1)$ | *URU* | $R(1, 1)$ |
| anisotropy factor $g$ | 0 | 0 | 0.75 | 0.75 |
| $-G_1$ | 2.31 | 3.66 | 4.62 | 7.33 |
| $c$ | 1.15 | 3.76 | 4.00 | 6.75 |

Here *URU* is the fraction of the energy reflected if the incident radiation has uniform intensity. It also equals the Bond albedo of a planet covered by such an atmosphere. Further $R(1, 1)$ is the reflection function for perpendicular incidence and reflection.

Examples 1 and 2 in Figure 2 vary very little over the entire range; example 1 even starts out at $n=1$ with less than 1% difference from its asymptotic value. Examples 3 and 4 refer to the Henyey-Greenstein function for $g=\frac{3}{4}$, which is highly anisotropic. Here it was fully expected that the radiation emerging after only 2 or 3 scatterings should not yet conform to the asymptotic law. Nevertheless a smooth interpolation between $n=3$ and $n=\infty$ appears possible.

Having found from Figure 2 the values of $f_n$ for any $n$ we may use any summation method, numerical or analytic, to sum the power series and find the function $f$ for an arbitrary value of $a$.

## 3. Contrast of Surface Markings

The blue haze phenomena on Mars have been discussed in this symposium. A number of models have been suggested to explain the visibility of contrast between dark and light surface features if the haze is partially dispersed. The time appears ripe to replace such qualitative suggestions by calculations based on precise models. This is indeed relatively simple.

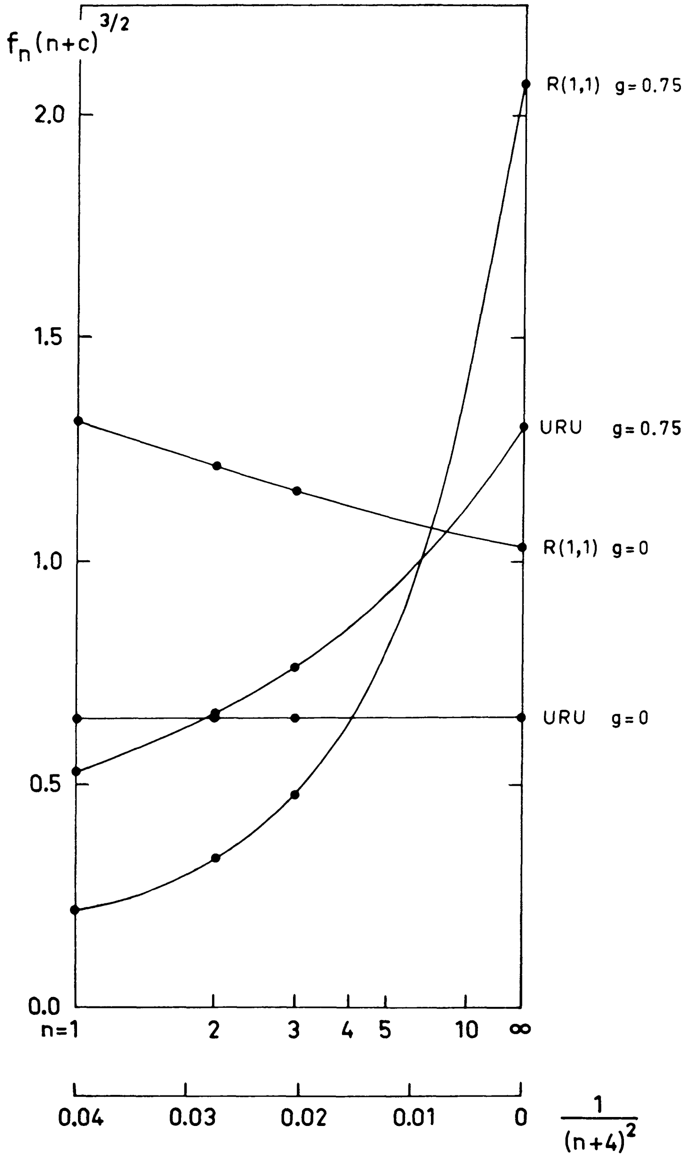



Fig. 2. Graphical interpolation between the contributions of single, double, and triple scattering ($n=1$, 2, 3) and the asymptotic expression for $n \to \infty$, illustrated by four examples.

Let the atmosphere be homogeneous and the ground surface reflect with uniform intensity (i.e., by Lamberts' law) a fraction $p$ of the incident radiation; further let $\mu_0$ and $\mu$ be the cosines of the angles of incidence and reflection, both taken positive, and $\phi - \phi_0$ the azimuth-difference. Define

| | | |
|---|---|---|
| $R'(\mu, \mu_0, \phi - \phi_0; a, b, g)$ | = reflection function | of bare atmosphere |
| $t_1(\mu_0; a, b, g)$ | = transmitted flux | of bare atmosphere |
| $r(a, b, g)$ | = reflected flux with uniform incidence (this was called *URU* in the last section) | of bare atmosphere |
| $R(\mu, \mu_0, \phi - \phi_0; a, b, g; p)$ | = reflection function of atmosphere backed by the ground surface | |

Thus we have the well-known relation, omitting the common variables:

$$R(\mu, \mu_0, \phi - \phi_0) = R'(\mu, \mu_0, \phi - \phi_0) + \frac{p}{1 - pr} t_1(\mu)\, t_1(\mu_0)$$

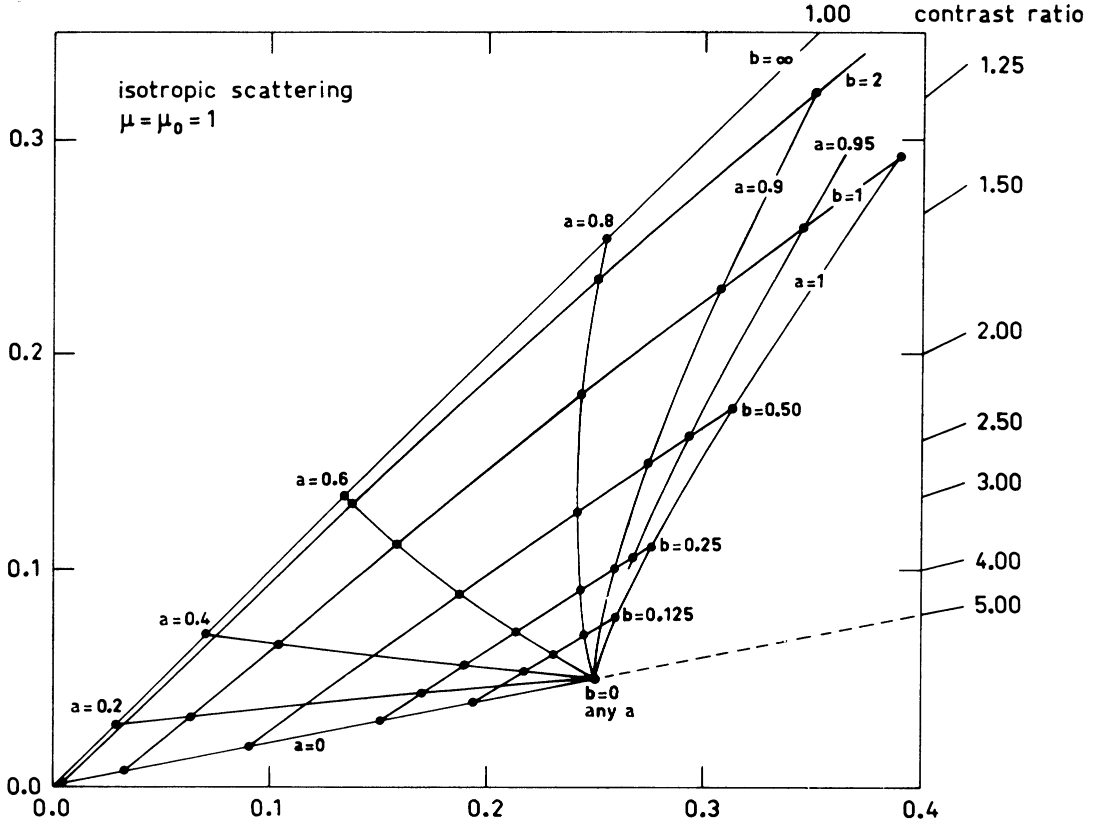
Throughout these definitions reflection from a white Lambert surface has been taken to define the unit reflection function.

The equation just given shows that the brightness $p$ reflected from the bare ground surface may be either increased (by atmospheric scattering) or decreased (by atmospheric extinction) in the presence of an atmosphere. It seems rather difficult, by mere handwaving, to guess what happens to the contrast between a darker and brighter surface area on the planet.

In order to give one definite example, we have selected in Figure 3 two regions seen near the center of the disk at opposition so that $\mu = \mu_0 = 1$. The values of $R(1, 1)$, $t_1(1)$ and $r$ were taken from the tables in a book in preparation. With a view to the actual situation of the blue haze on Mars, we have chosen regions with $p = 0.25$ (bright region; brightness plotted as abscissa) and $p = 0.05$ (dark region; brightness plotted as ordinate). The two diagrams correspond to two scattering laws, isotropic scattering, and Henyey-Greenstein function with $g = 0.5$.

Inside each diagram the single scattering albedo $a$ and the optical depth $b$ of the haze layer are varied. If the haze has completely cleared ($b = 0$), we see the full contrast ratio 5; if it is completely dense ($b = \infty$), the contrast ratio is 1. Somewhat surprisingly, we find that the contrast ratio at intermediate values depends strongly on $b$, but very little on $a$. Separate photometry of the dark and bright markings would in practice be necessary to determine $a$. Although the similarity rules cannot be expected to give a precise answer in this application, where the optical depths are small, the two diagrams are strikingly similar, with the curves of constant $b$ and of constant $a$ shifted approximately as predicted by the similarity rules.

Evidently calculations of this kind for different assumptions of the ground albedo and reflection law, and for different haze scattering patterns, and for different directions of incidence and reflection, are needed before such curves can be used as a firm basis for the interpretation of the observational data.



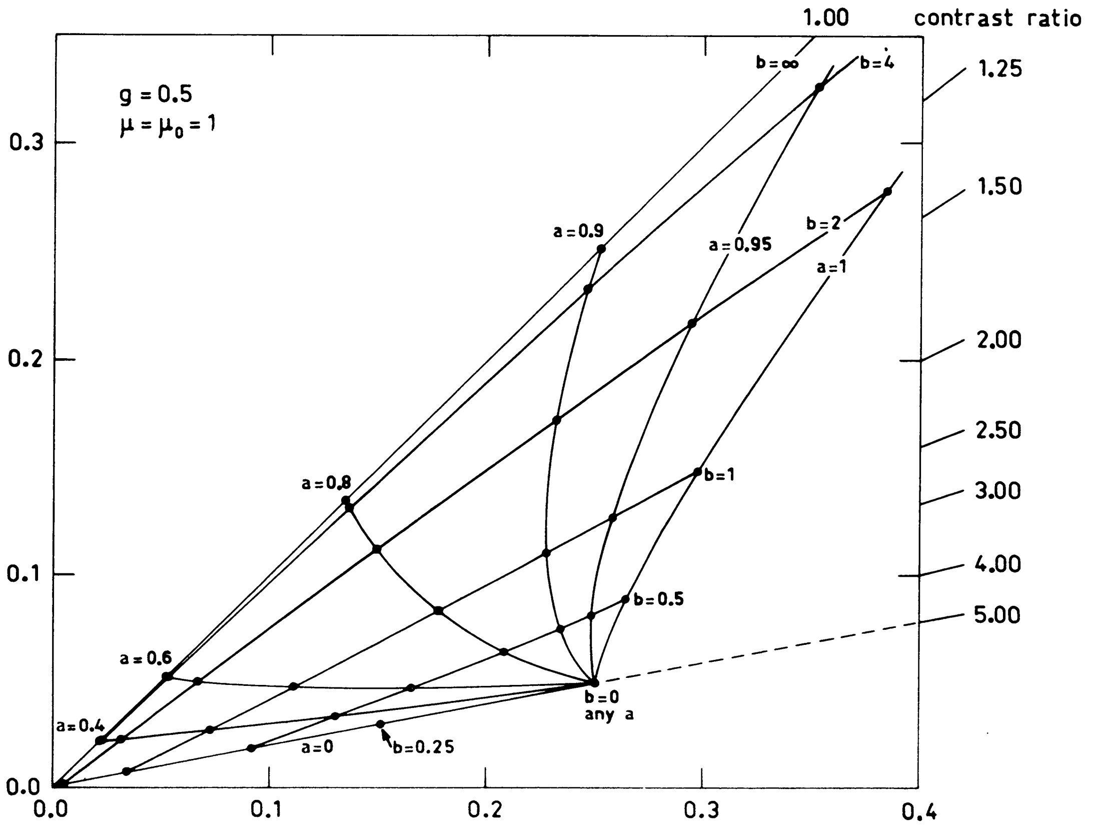




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Fig. 3. Contrast between two areas near the center of the Mars disk, with brightness 0.25 and 0.05, as affected by an overlying haze; $a$ = single scattering albedo, $g$ = anisotropy factor, $b$ = optical thickness of haze layer.