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Graded Jacobi operators on the algebra of differential forms

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Abstract. One-to-one correspondences are established between the set of all nondegenerate graded Jacobi operators of degree -1 defined on the graded algebra $\Omega(M)$ of differential forms on a smooth, oriented, Riemannian manifold M, the space of bundle isomorphisms $L: TM \to TM$, and the space of nondegenerate derivations of degree 1 having null square. Derivations with this property, and Jacobi structures of odd $\mathbb{Z}_2$-degree are also studied through the action of the automorphism group of $\Omega(M)$.

Key words: differential forms, graded Poisson brackets, graded symplectic forms, Jacobi structures

Introduction

Let M be a smooth manifold, and let $\Omega(M)$ be its graded algebra of differential forms. In this note we center our attention on the graded differential operators D of order $\leqslant 2$ acting on $\Omega(M)$, that define a graded Poisson structure on this algebra through the bracket they generate in the sense of [Kz]. These are called graded Jacobi operators. A Jacobi operator is nondegenerate if its corresponding Poisson structure is nondegenerate. We determine here all the nondegenerate Jacobi operators of degree -1 by establishing a one-to-one correspondence between them, and bundle isomorphisms $L: TM \to TM$ (Thm. 2.3). This is done through an interesting duality defined, amongst the differential operators on $\Omega(M)$, by the Hodge operator associated to a Riemannian metric (Thm. 1.4). As a by-product, a relationship is established between graded Jacobi operators and differentials on $\Omega(M)$; that is, derivations whose square is zero (Thms. 1.8, and 2.2). We also describe all the nondegenerate differentials, and all the nondegenerate Jacobi operators of odd $\mathbb{Z}_2$-degree in terms of the action of the automorphism group of the algebra $\Omega(M)$ (Thm. 3.4): It is shown that the former are obtained as $\varphi \circ \mathrm{d} \circ \varphi^{-1}$ (Prop. 2.4), whereas the latter as $\varphi \circ \delta \circ \varphi^{-1}$ (Prop. 3.1), where φ is an automorphism of $\Omega(M)$ restricting to the identity on $C^\infty(M) = \Omega^0(M)$, and δ is the codifferential associated to a given Riemannian metric. We exhibit two main families of non-

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degenerate graded Jacobi operators parametrized respectively by nondegenerate bivectors, and by Riemannian metrics on M (Prop. 4.3). We finally compute the Hamiltonian vector fields associated to the generators of the algebra $\Omega(M)$ for these families of examples (Prop. 4.4).

1. Graded Jacobi operators on $\Omega(M)$

Let M be a smooth manifold of dimension n, and let $\Omega(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$ be its $\mathbb{Z}$-graded algebra of differential forms. An $\mathbb{R}$-linear operator $D\colon \Omega(M) \to \Omega(M)$ is said to be of degree $|D|$ if $D(\alpha) \in \Omega^{k+|D|}(M)$, for all $\alpha \in \Omega^k(M)$. It is not difficult to see that $\operatorname{End}\Omega(M) = \bigoplus_{k=-n}^{n} \operatorname{End}^k \Omega(M)$, where $\operatorname{End}^k \Omega(M)$ is the subspace of linear operators of degree k. Let F and G be two linear operators having degrees $|F|$, and $|G|$, respectively. Their graded commutator is the linear operator defined by $[F, G] = F \circ G - (-1)^{|F|\,|G|} G \circ F$. A linear operator $D \in \operatorname{End}\Omega(M)$ is a differential operator of order 0 if for any $\alpha \in \Omega(M)$, $[D, \mu_\alpha] = 0$, where $\mu_\alpha \in \operatorname{End}\Omega(M)$ denotes the linear operator of left multiplication by α. Note that $\alpha \in \Omega^k(M)$ defines via μ_α a differential operator of degree k and order 0, and all such operators are precisely of this form. Now $D \in \operatorname{End}\Omega(M)$ is a differential operator of order $\leqslant r$, iff for each $\alpha \in \Omega(M)$, $[D, \mu_\alpha]$ is a differential operator of order $\leqslant r - 1$. We shall write $[D, \alpha]$ instead of $[D, \mu_\alpha]$ when no confusion may arise. We shall denote by $\mathcal{D}_r^k(M)$ the set of all differential operators of order $\leqslant r$ which have degree k. We shall also refer ourselves to the $\mathbb{Z}_2$-graded structures of both, the algebra $\Omega(M)$, and the $\Omega(M)$-submodules $\Sigma_k \mathcal{D}_r^k(M)$. These are naturally inherited from their corresponding $\mathbb{Z}$-gradings. We fix the convention that a reference to an element of 'odd degree' is to be understood with respect to its inherited $\mathbb{Z}_2$-graded structure: Therefore, it will be a sum of elements of every possible odd $\mathbb{Z}$-degree, unless stated otherwise.

DEFINITION 1. ([Kz]). Let D be a differential operator of order $\leqslant 2$, of odd degree, and such that $D(1) = 0$. The bracket on $\Omega(M)$ generated by D is:

$$[\![\alpha, \beta]\!]_D = (-1)^{|\alpha|}(D(\alpha \wedge \beta) - D(\alpha) \wedge \beta - (-1)^{|\alpha|} \alpha \wedge D(\beta)).$$

Remark. Let D and D' be two differential operators of order $\leqslant 2$, and odd degree, with $D(1) = D'(1) = 0$. It is easy to see that $[\![\,,]\!]_D = [\![\,,]\!]_{D'}$ if and only if $D - D' \in \mathcal{D}_1^*(M)$; that is, if and only if they differ by some derivation. This defines an equivalence class of operators. In this work we shall only be interested in the equivalence classes obtained in this manner. A reference to the 'equivalence classes of operators', shall always mean these equivalence classes.

From the work in [Gr] on abstract Jacobi structures we adopt the following terminology (we refer the reader to [BM2] for the basics on graded Poisson structures):

DEFINITION 2. Let D be a differential operator of order $\leqslant 2$ and odd degree, such that $D(1) = 0$. D is a graded Jacobi operator if its associated bracket, $[\![\,,]\!]_D$, is a graded Poisson bracket on $\Omega(M)$.

The first step in understanding the structure of graded Jacobi operators is the following characterization given by Koszul

PROPOSITION 3. ([Kz]). *Let D be a differential operator of order $\leqslant 2$, of odd degree, and such that $D(1) = 0$. Then D is a graded Jacobi operator if and only if $D^2 = D \circ D$ is in fact a differential operator of order $\leqslant 2$.*

Remark. Observe that $\mathcal{D}^*_*(M) = \Sigma_k\big(\cup_r \mathcal{D}^k_r(M)\big)$ has the structure of an associative $\mathbb{R}$-algebra, and also the structure of a (left) $\Omega(M)$-module. Both structures are filtered by the order and graded by the degree. The graded commutator satisfies,

$$[\mathcal{D}^k_r(M), \mathcal{D}^\ell_s(M)] \subset \mathcal{D}^{k+\ell}_{r+s-1}(M).$$

Note that a differential operator D like in the proposition above, satisfies $D^2 = \frac{1}{2}[D, D]$, and it is, in general, a differential operator of order $\leqslant 3$. This explains the strength of Proposition 3, and the methods of this paper provide a criterion for deciding whether or not $D^2 \in \mathcal{D}^{-2}_2(M)$ when $D \in \mathcal{D}^{-1}_2(M)$. This is possible by letting Riemannian metrics come into the description of differential operators; a fact that was already used in [BM1] to obtain a decomposition theorem for operators in $\mathcal{D}^{-1}_2(M)$. Here we exploit the duality relationship between differential operators of different orders and degrees established by the Hodge operator. This however, requires the manifold M to be orientable. The precise statement is the following

THEOREM 4. *Let M be an oriented Riemannian manifold of dimension n, and let $* \in \operatorname{End}\Omega(M)$ be the Hodge operator defined by the Riemannian metric g on M. The map $D \mapsto \bar{D}$, where $\bar{D} = (-1)^{(2p-k-1)k/2} *^{-1} \circ D \circ *|_{\Omega^p(M)}$, restricts to a one-to-one correspondence between $\mathcal{D}^k_r(M)$, and $\mathcal{D}^{-k}_{k+r}(M)$ for all $k + r \geqslant 0$.*

Proof. Since $*$ maps $\Omega^\ell(M)$ into $\Omega^{n-\ell}(M)$, it is easy to check directly that the degree of the operator $\bar{D}$ is $-k$, whenever the degree of D is k. The nontrivial assertion is that the order of $\bar{D}$ is $\leqslant k + r$ when D is of order $\leqslant r$. This, however, can be proved by induction on $r + k$. The actual source of the induction process is found in the following Lemma. In the course of its proof, use is made of the fact that any differential operator of order r and degree $-r$ on $\Omega(M)$ is uniquely of the form i_Q for some $Q \in \Gamma(\Lambda^r TM)$; that is, it is given by total contraction against Q. Indeed: Let $D \in \mathcal{D}^{-r}_r(M)$ ($r \geqslant 1$). Then, $[D, f] = 0$ for any $f \in C^\infty(M)$, and $D(\beta) = 0$ for all $\beta \in \Omega^s(M)$, with $0 \leqslant s \leqslant r - 1$. In particular, D is tensorial and it is completely determined by its value on r-forms. Let $\alpha \in \Omega^r(M)$. Since the elements of $\mathcal{D}^*_*(M)$ are local operators, the map $\alpha \mapsto D(\alpha) \in \Omega^0(M)$ defines a unique $Q \in \Gamma(\Lambda^r TM)$, such that $D(\alpha) = i_Q\alpha$. (The notation i_Q is explained as follows: Let $\Gamma(TM)$ be the $C^\infty(M)$-module of vector fields on M. Each $X \in \Gamma(TM)$ defines a differential operator of order 1 and degree -1 on $\Omega(M)$; namely, the contraction i_X against X. The fact that $i_X \circ i_X = 0$ yields a unique $C^\infty(M)$-linear extension of i to $\Gamma(\Lambda TM)$, and $i_{X_1\wedge\cdots\wedge X_r} = i_{X_1} \circ \cdots \circ i_{X_r}$, on generators).

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LEMMA 5. *Let* $* \in \operatorname{End}\Omega(M)$ *be the Hodge operator defined by a given Riemannian metric* g *on the oriented* n*-dimensional manifold* M*. Let* $\mu_\alpha \in \mathcal{D}_0^1(M)$*, and* $i_X \in \mathcal{D}_1^{-1}(M)$ *be the differential operators defined by multiplication by* $\alpha \in \Omega^1(M)$*, and contraction against* $X \in \Gamma(TM)$*, respectively. Then,*

$$(-1)^{k-1} *^{-1} \circ \mu_\alpha \circ *|_{\Omega^k(M)} = i_{g^\sharp(\alpha)}, \quad (-1)^k *^{-1} \circ i_X \circ *|_{\Omega^k(M)} = \mu_{g^\flat(X)},$$

where $g^\sharp\colon \Gamma(T^*M) \to \Gamma(TM)$*, and* $g^\flat\colon \Gamma(TM) \to \Gamma(T^*M)$ *are the natural isomorphisms associated to* g*.*

Proof of the Lemma. Let $\nu_g \in \Omega^n(M)$ be the volume form associated to the Riemannian metric g. Let β be a k-form. By definition, the Hodge operator on $\Omega(M)$ is given by $*\beta = (-1)^{k(k-1)/2} i_\beta\, \nu_g$, where i_β is defined through the metric g^{-1} on $\Omega(M)$, via $g^{-1}(i_\beta\, \eta, \xi) = (-1)^{k(k-1)/2} g^{-1}(\eta, \mu_\beta\, \xi)$, and $*$ fixed by the property, $*\nu_g = 1$. Since, $g^{-1}(\alpha, g^\flat(X)) = \alpha(X) = i_X\, \alpha = g(g^\sharp(\alpha), X)$ for any 1-form α, and vector field X, and since g^{-1} is defined on $\Omega(M)$ through a determinant, one verifies in a straightforward fashion that the map $\eta \mapsto i_\alpha\, \eta$ is a derivation of degree -1, whenever $\alpha \in \Omega^1(M)$, and that it is actually equal to $i_{g^\sharp(\alpha)}$. Since $g^\sharp$ and $g^\flat$ are extended to $\Gamma(\Lambda T^*M) \to \Gamma(\Lambda TM)$, and $\Gamma(\Lambda TM) \to \Gamma(\Lambda T^*M)$ as algebra isomorphisms, it also follows that $i_\beta = i_{g^\sharp(\beta)} \in \mathcal{D}_k^{-k}(M)$ for any k-form β, and that $i_{g^\sharp(\beta)}\, \gamma = (-1)^{k(k-1)/2} g(g^\sharp(\beta), g^\sharp(\gamma))$ for any β, and $\gamma \in \Omega^k(M)$.

We now compute the commutator $[\alpha, i_\beta]$. This is a differential operator of degree $-(k-1)$ and order $\leqslant k-1$. Therefore, it is of the form i_Q, where Q is a multivector of degree $k-1$. We claim that $Q = g^\sharp(i_\alpha \beta)$ $(= g^\sharp(i_{g^\sharp(\alpha)}\, \beta))$. In fact, Q is completely determined by the value of $[\alpha, i_\beta]$ on a $k-1$ form γ, but,

$$\begin{aligned}
[\alpha, i_\beta]\gamma &= -(-1)^k i_\beta(\alpha \wedge \gamma) = -(-1)^k i_{g^\sharp(\beta)}(\alpha \wedge \gamma) \\
&= -(-1)^{k+(k(k-1)/2)} g(g^\sharp \beta, g^\sharp(\alpha \wedge \gamma)) \\
&= -(-1)^{k(k+1)/2} g^{-1}(i_{g^\sharp(\alpha)}(\beta), \gamma) \\
&= -(-1)^{k(k+1)/2} g\,(g^\sharp(i_{g^\sharp(\alpha)}(\beta)),\ g^\sharp(\gamma)) \\
&= -(-1)^{(k(k+1)+(k-1)(k-2))/2} i_{g^\sharp(i_{g^\sharp(\alpha)}(\beta))}(\gamma) \\
&= -(-1)^{k^2-k+1} i_{g^\sharp(i_{g^\sharp(\alpha)}(\beta))}(\gamma) \\
&= i_{g^\sharp(i_{g^\sharp(\alpha)}(\beta))}(\gamma),
\end{aligned}$$

which proves our claim. Finally,

$$\begin{aligned}
*^{-1} \circ \alpha \circ *(\beta) &= (-1)^{k(k-1)/2} *^{-1} \circ \alpha \circ\ i_\beta\,(\nu_g) \\
&= (-1)^{k(k-1)/2} *^{-1} \circ [\alpha, i_\beta]\,(\nu_g)
\end{aligned}$$

$$= (-1)^{k(k-1)/2} *^{-1} \circ i_{g^\sharp(i_{g^\sharp(\alpha)}(\beta))}(\nu_g)$$

$$= (-1)^{k-1} i_{g^\sharp(\alpha)}(\beta).$$

Thus, $(-1)^{k-1} *^{-1} \circ\alpha \circ *|_{\Omega^k(M)} = i_{g^\sharp(\alpha)}$. The second assertion is now a consequence of this one, and the well known formulae for $* \circ *$ (*cf*, [Lh]). □

We may now conclude the proof of Theorem 4: The first step in the induction process corresponds to $k + r = 0$, since the degree k of a differential operator is always greater than or equal to $-r$; r being the order. Now, use the fact that any differential operator of order $\leqslant r$ and degree $-r$ is given by total contraction against some r-vector, $Q \in \Gamma(\Lambda^r TM)$. Then, the lemma implies that conjugation of the operator i_Q by the Hodge operator yields, up to a sign, the operator of multiplication by the r-form $g^\flat(Q)$. Therefore, $D \in \mathcal{D}_r^{-r}(M)$, does imply, $\bar{D} \in \mathcal{D}_0^r(M)$, where $\bar{D} = (-1)^{(2p+r-1)r/2} *^{-1} \circ D \circ *|_{\Omega^p(M)}$. (Similarly, the lemma implies that for any $\alpha \in \Omega^\ell(M)$, we have $(-1)^{(2(n-p)+\ell-1)\ell/2} * \circ\alpha \circ *^{-1}|_{\Omega^p(M)} = i_{g^\sharp\alpha}$).

Now, let s be a natural number greater than 0. We suppose by induction that the following statement holds true: If $D \in \mathcal{D}_r^k(M)$, with $k + r \leqslant s$ then, $\bar{D} \in \mathcal{D}_{k+r}^{-k}(M)$. Let $D \in \mathcal{D}_r^k(M)$ with $k + r = s + 1$. We claim that $[\bar{D}, \alpha] \in \mathcal{D}_{k+r-1}^{\ell-k}(M)$, whenever $\alpha \in \Omega^\ell(M)$. Indeed, this commutator is equal (up to a sign) to $*^{-1} \circ [D, i_{g^\sharp(\alpha)}] \circ *$, since, acting on $\Omega^p(M)$,

$$\begin{aligned}
&[\bar{D}, \alpha] \\
&\quad = (-1)^{(2p-k-1+2\ell)k/2}(*^{-1} \circ D \circ * \circ \alpha - \alpha \circ *^{-1} \circ D \circ *) \\
&\quad = (-1)^{(2p-k+\ell-1)(k-\ell)/2}(*^{-1} \circ D \circ i_{g^\sharp(\alpha)} \circ * - (-1)^{\ell k} *^{-1} \circ i_{g^\sharp(\alpha)} \circ D \circ *) \\
&\quad = (-1)^{(2p-k+\ell-1)(k-\ell)/2} *^{-1} \circ [D, i_{g^\sharp(\alpha)}] \circ *.
\end{aligned}$$

Now, $i_{g^\sharp(\alpha)}$ belongs to $\mathcal{D}_\ell^{-\ell}(M)$. This implies that, $[D, i_{g^\sharp(\alpha)}] \in \mathcal{D}_{r+\ell-1}^{k-\ell}(M)$. Applying now the induction hypothesis (as, $(k-\ell) + (r+\ell-1) = k+r-1 = s$), we conclude that $\overline{[D, i_{g^\sharp(\alpha)}]}$ is a differential operator of order $\leqslant k+r-1$ and degree $\ell-k$. Therefore, $[\bar{D}, \alpha] \in \mathcal{D}_{k+r-1}^{\ell-k}(M)$ as claimed. Whence, $\bar{D} \in \mathcal{D}_{k+r}^{-k}(M)$ which follows from the definition of the order of a differential operator. □

Remark. The duality established in the Theorem 4 between differential operators makes $\mathcal{D}_r^{-(r-1)}(M)$ to correspond in a one-to-one fashion with $\mathcal{D}_1^{r-1}(M)$. In particular, the well-known decomposition theorem for derivations of degree k (Thm. 6, below) gives rise – via conjugation by $*$ – to a decomposition theorem for differential operators of order $\leqslant 1 + k$ and degree $-k$. We recall that $\mathcal{D}_1^*(M) = \Sigma_{k\geq -1}\mathcal{D}_1^k(M)$ is a graded Lie submodule of $\mathcal{D}_*^*(M)$ (since, $[\mathcal{D}_1^k(M), \mathcal{D}_1^\ell(M)] \subset \mathcal{D}_1^{k+\ell}(M)$), and that $\mathcal{D}_1^*(M) \simeq \operatorname{Der}\Omega(M) \oplus \Omega(M)$ (indeed: given $D \in \mathcal{D}_1^k(M)$, for any homogeneous α, and β in $\Omega(M)$, one has $[[D, \alpha], \beta] = 0$. Using this, it

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easily follows that $D - D(1) \in \mathrm{Der}\,\Omega(M)$. The summand $\Omega(M)$ in $\mathcal{D}_1^*(M)$ is precisely $\mathcal{D}_0^*(M)$).

THEOREM 6. ([FN]). *Let* $\mathrm{Der}^k\,\Omega(M)$ *be the subspace of* $\mathrm{End}^k\,\Omega(M)$ *consisting of derivations of degree* k *(*$k \geqslant -1$*). For each* $D \in \mathrm{Der}^k\,\Omega(M)$ *there exist unique* TM*-valued forms* $K \in \Omega^{k+1}(M;TM)$*, and* $L \in \Omega^k(M;TM)$*, such that,* $D = i_K + [i_L, \mathrm{d}]$.

Remark. One observation to be made from this decomposition is that the operator $[i_L, \mathrm{d}]$ is not tensorial, and it is completely characterized by the fact that it commutes with d; i.e., $[[i_L, \mathrm{d}], \mathrm{d}] = 0$. We adhere ourselves to the now standard notation $\mathcal{L}_L$ for the derivation $[i_L, \mathrm{d}]$ (the generalized Lie derivative with respect to $L \in \Omega(M;TM)$), and call it *the nontensorial part of* D.

The next result follows now as an easy consequence of Theorems 4 and 6.

PROPOSITION 7. *There exists a one-to-one correspondence,*

$$\left\{ \begin{array}{c} \textit{Equiv. Classes} \\ \textit{of Operators in} \\ \mathcal{D}_2^{-1}(M) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Data:} \\ K \in \Omega^2(M;TM) \\ L \in \Omega^1(M;TM) \end{array} \right\}.$$

In fact, given $D \in \mathcal{D}_2^{-1}(M)$*, there exist unique* TM*-valued forms* $K \in \Omega^2(M;TM)$*, and* $L \in \Omega^1(M;TM) \simeq \Gamma(\mathrm{End}\,TM)$*, such that,*

$$D = (-1)^{n-p}(* \circ i_K \circ *^{-1} + * \circ \mathcal{L}_L \circ *^{-1})|_{\Omega^p(M)} \quad (\mathrm{mod}\,\mathrm{Der}^{-1}\Omega(M)).$$

Proof. From Theorem 6, and the remark immediately above it, we see that a differential operator from $\mathcal{D}_1^1(M)$ can be uniquely written in the form $i_K + \mathcal{L}_L + \mu_\alpha$. Theorem 4 says that an isomorphism is established between $\mathcal{D}_1^1(M)$ and $\mathcal{D}_2^{-1}(M)$ upon conjugation by $*$. Therefore, for each $D \in \mathcal{D}_2^{-1}(M)$, there exist unique $K \in \Omega^2(M;TM)$, $L \in \Omega^1(M;TM)$, and $\alpha \in \Omega^1(M)$, such that, $(-1)^p *^{-1} \circ D \circ *|_{\Omega^p(M)} = i_K + \mathcal{L}_L + \mu_\alpha$ □

We are now in position of determining when does an operator $D \in \mathcal{D}_2^{-1}(M)$ has the property of being Jacobi. Note that,

$$\begin{aligned} D^2 &= -(* \circ (i_K + \mathcal{L}_L + \mu_\alpha) \circ *^{-1})^2|_{\Omega^p(M)} \\ &= - * \circ (i_K + \mathcal{L}_L)^2 \circ *^{-1}|_{\Omega^p(M)} \quad (\mathrm{mod}\,\mathcal{D}_2^{-2}(M)), \end{aligned}$$

whereas,

$$-(*^{-1} \circ D \circ *)^2|_{\Omega^p(M)} = (i_K + \mathcal{L}_L)^2 \quad (\mathrm{mod}\,\mathcal{D}_0^2(M)).$$

THEOREM 8. *Under the correspondence of Proposition* 7, *the differential operator* $D \in \mathcal{D}_2^{-1}(M)$ *is a graded Jacobi operator if and only if the derivation* $i_K + \mathcal{L}_L \in \mathrm{Der}^1\Omega(M)$ *it comes from, has null square.*

Proof. Supose $D \in \mathcal{D}_2^{-1}(M)$ is Jacobi. Then, $D^2 \in \mathcal{D}_2^{-2}(M)$ implies $- *^{-1} \circ D^2 \circ *|_{\Omega^p(M)} \in \mathcal{D}_0^2(M)$. Therefore, $(i_K + \mathcal{L}_L)^2 \in \mathcal{D}_0^2(M)$ must be (multiplication by) a 2-form. But $(i_K + \mathcal{L}_L)^2$ is a derivation, since it is equal to $\frac{1}{2}[i_K + \mathcal{L}_L, i_K + \mathcal{L}_L]$. Therefore, $(i_K + \mathcal{L}_L)^2 = 0$. The converse is trivial. □

2. Relationship with differentials of $\Omega(M)$

This section collects some properties of nondegenerate differentials defined on $\Omega(M)$ and their relationship with nondegenerate graded Jacobi operators. The main conclusion to be drawn is that there exist one-to-one correspondences between nondegenerate graded Jacobi operators, bundle isomorphisms $TM \to TM$, and nondegenerate differentials of $\Omega(M)$.

DEFINITION 1. An odd derivation D, with no component of $\mathbb{Z}$-degree -1, is a nondegenerate differential of $\Omega(M)$ if $D^2 = 0$, and for any coordinate system $(U, \{x^i\})$ the system of differential forms $\{Dx^i\}$ generates $\Omega(U)$.

Our next result characterizes the nondegenerate differentials of $\Omega(M)$ having $\mathbb{Z}$-degree 1:

THEOREM 2. *There is a one-to-one correspondence between nondegenerate differentials of* $\mathbb{Z}$*-degree* 1 *and bundle isomorphisms* $L : TM \to TM$.

Proof. Given a bundle isomorphism $L : TM \to TM$, we consider $L^* : T^*M \to T^*M$, and its extension to an algebra isomorphism $\Omega(M) \to \Omega(M)$ (still denoted by L^*). The conjugation $L^* \circ \mathrm{d} \circ (L^*)^{-1}$ of the exterior differential is clearly a nondegenerate differential of $\mathbb{Z}$-degree 1. We now show that this correspondence is one-to-one. The injectivity is obvious. To prove the surjectivity, let D be any nondegenerate differential of $\mathbb{Z}$-degree 1, and write it in the form $D = i_K + \mathcal{L}_L$ where, $K \in \Omega^2(M; TM)$, and $L \in \Omega^1(M; TM)$. The nondegeneracy of the differential implies that L defines a bundle isomorphism. Use this isomorphism to get $L^* \circ \mathrm{d} \circ (L^*)^{-1}$. Note that the derivation $D - L^* \circ \mathrm{d} \circ (L^*)^{-1}$ acts on smooth functions as zero: Indeed,

$$(D - L^* \circ \mathrm{d} \circ (L^*)^{-1}) f = \mathcal{L}_L(f) - L^*(\mathrm{d}f) = \mathrm{d}f \circ L - L^*(\mathrm{d}f) = 0.$$

This now implies that $(D - L^* \circ \mathrm{d} \circ (L^*)^{-1})(Df) = 0$ for any $f \in \Omega^0(M) = C^\infty(M)$. Since D is nondegenerate, it follows that $D - L^* \circ \mathrm{d} \circ (L^*)^{-1}$ vanishes on generators of $\Omega(U)$ for any open coordinate neighborhood U. Since derivations are local operators, we conclude that $D = L^* \circ \mathrm{d} \circ (L^*)^{-1}$. □

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Remark. Note that the derivation $L^* \circ \mathrm{d} \circ (L^*)^{-1}$, according to the Frölicher–Nijenhuis decomposition, is given by: $L^* \circ \mathrm{d} \circ (L^*)^{-1} = \mathcal{L}_L + i_{-(1/2)L^{-1}\circ[L,L]_{FN}}$, where $[L, L]_{FN}$ stands for the Frölicher–Nijenhuis bracket of L with itself (cf. [FN]). This follows, since the value of $L^* \circ \mathrm{d} \circ (L^*)^{-1}$ on a smooth function f is $L^*(\mathrm{d}f) = \mathcal{L}_L f$. Thus, writing the differential as $i_K + \mathcal{L}_L$, one may easily obtain the conditions needed to satisfy $(i_K + \mathcal{L}_L)^2 = 0$: They are,

$$\text{(a)} \quad \begin{array}{l} \frac{1}{2}[L, L]_{FN} + i_K L = 0 \quad \text{and,} \\ [L, K]_{FN} + \frac{1}{2}[K, K]_{RN} = 0, \end{array}$$

where now $[K, K]_{RN}$ stands for the Richardson–Nijenhuis bracket of K with itself (cf. [NR]). Now, when $L \in \Omega^1(M; TM) \simeq \Gamma(\mathrm{End}\, TM)$ is a bundle isomorphism, the first equation fully determines K as,

$$\begin{aligned} K(X, Y) &= -\tfrac{1}{2} L^{-1}([L, L]_{FN}(X, Y)) \\ &= -L^{-1}[LX, LY] + [LX, Y] + [X, LY] - L[X, Y], \end{aligned}$$

and, the second equation in (a) always holds for such a K.

As it was mentioned in the introduction, a graded Jacobi operator $D \in \mathcal{D}_2^{-1}(M)$ is nondegenerate whenever the graded Poisson structure it defines on $\Omega(M)$ is nondegenerate. In view of this remark, Proposition 1.7, and Theorem 2 above, we may now conclude the following

THEOREM 3. *There exists a one-to-one correspondence between (equivalence classes of) nondegenerate graded Jacobi operators D of degree -1, and bundle isomorphisms, $L: TM \to TM$.*

In the following section we shall indicate how an alternative proof of this fact may be obtained. We shall close this section, however, by looking at the action of the automorphism group of the algebra $\Omega(M)$ on the nondegenerate differentials of odd degree

PROPOSITION 4. *Let D be a nondegenerate differential of $\Omega(M)$ of odd degree. Then, there exists an algebra isomorphism $\varphi \in \mathrm{Aut}\, \Omega(M)$, that restricts to the identity on $C^\infty(M) = \Omega^0(M)$, and such that $D = \varphi \circ \mathrm{d} \circ \varphi^{-1}$.*

Proof. The derivation D can be uniquely written as $D_1 + D_{\geqslant 3}$, where D_1 is a derivation of degree 1 and $D_{\geqslant 3} = D - D_1$. Condition $D^2 = 0$ implies $D_1^2 = 0$. Nondegeneracy of D implies nondegeneracy of D_1. Therefore D_1 is a nondegenerate differential of $\Omega(M)$ of degree 1 and then $D_1 = L_1^* \circ \mathrm{d} \circ (L_1^*)^{-1}$, where $L_1 \in \Omega^1(M; TM)$ is an isomorphism. Note that the composition $(L_1^*)^{-1} \circ D \circ L_1^*$ is a derivation of the form $\mathrm{d} + D_3 + D_{\geqslant 5}$, and its square is zero. Also note that D_3 satisfies $[\mathrm{d}, D_3] = 0$. Therefore, D_3 is a derivation of the form $\mathcal{L}_{L_3}$ where $L_3 \in \Omega^3(M; TM)$. Now, it is easy to check that $\exp(-i_{L_3}) \circ (\mathrm{d} + D_3 + D_{\geqslant 5}) \circ$

$\exp(i_{L_3}) = \mathrm{d} + D_5 + D_{\geqslant 7}$. By repeating this process we get, after a finite number of steps, that if $\varphi = \exp(i_{L_\ell}) \circ \cdots \circ \exp(i_{L_3}) \circ L_1^*$, then $\varphi \circ \mathrm{d} \circ \varphi^{-1} = D$. □

3. Jacobi operators under the action of $\mathrm{Aut}\,\Omega(M)$

The aim of this section is to complete our description of the graded Jacobi operators by proving the following analog of Proposition 2.4.

PROPOSITION 1. *Let M be an oriented Riemannian manifold, and let g be its metric tensor. Let $*$ be the Hodge operator associated to g, and let $\delta = (-1)^{n-p} * \circ \mathrm{d} \circ *^{-1}|_{\Omega^p(M)}$ be its corresponding codifferential operator. Given a nondegenerate graded Jacobi operator D, there exists an automorphism φ of the algebra $\Omega(M)$, such that φ restricts to the identity on $\Omega^0(M) = C^\infty(M)$, and $D = \varphi \circ \delta \circ \varphi^{-1}$* $(\mathrm{mod}\ \mathrm{Der}\,\Omega(M))$.

The proof of this result makes use of some graded-geometry techniques. Since nondegenerate Jacobi operators come from nondegenerate graded Poisson brackets of odd degree, which in turn come from graded symplectic forms on $\Omega(M)$ of odd degree, the problem can be traced down to the structure of the latter. This was understood in [BM2] by looking first at the graded symplectic structures of $\mathbb{Z}$-degree $+1$ (i.e., graded Poisson brackets of $\mathbb{Z}$-degree -1), and studying afterwards their orbits under the $\mathbb{Z}_2$-graded automorphisms of the algebra $\Omega(M)$

PROPOSITION 2. ([BM2: Prop. 3.3 and Cor. 3.4]). *There is a one-to-one correspondence between graded symplectic forms of $\mathbb{Z}$-degree $+1$ and bundle isomorphisms* $\mathbf{L}\colon T^*M \to TM$.

Remark. Note how our previous Theorem 2.3 can also be recovered as a corollary of this result, with $L = \mathbf{L} \circ g^\flat$. A few more comments about this Proposition are in order: First of all, graded symplectic forms of odd degree on $\Omega(M)$ are necessarily exact. This is a consequence of the fact that the cohomology defined by the graded exterior differential is isomorphic to the cohomology of the base manifold. It is in this way that a bijective correspondence is set between graded symplectic forms on $\Omega(M)$ of degree $+1$, and bundle isomorphisms $\mathbf{L}\colon T^*M \to TM$: Any graded symplectic form of $\mathbb{Z}$-degree $+1$ can be written as $\omega_{\mathbf{L}} = \mathrm{d}^G \lambda_{\mathbf{L}}$, where $\lambda_{\mathbf{L}}$ is the graded 1-form of degree $+1$ defined by the linear isomorphism $\mathbf{L}\colon T^*M \to TM$ (that graded 1-forms of degree $+1$ are in one-to-one correspondence with bundle isomorphisms $T^*M \to TM$, is easy). The $\mathbb{Z}_2$-graded description is then completed by looking at the orbits through the graded symplectic form $\omega_{\mathbf{L}}$, under the action of the group of $\mathbb{Z}_2$-graded algebra automorphisms of $\Omega(M)$

PROPOSITION 3. ([BM2]). *Any graded symplectic form on $\Omega(M)$ of odd $\mathbb{Z}_2$-degree is of the form $\varphi^*(\omega_{\mathbf{L}})$, where φ is an automorphism that induces the identity on M.*

Thus, any nondegenerate graded Poisson structure (resp., Jacobi operator) of odd $\mathbb{Z}_2$-degree is reached from a graded Poisson structure (resp., Jacobi operator) of $\mathbb{Z}$-degree -1, by the corresponding action of $\mathrm{Aut}\,\Omega(M)$. Therefore we conclude the following

THEOREM 4. *Any nondegenerate graded Jacobi operator is the conjugation by an automorphism of* $\Omega(M)$ *that induces the identity on* M, *of a nondegenerate graded Jacobi operator of degree* -1.

Proof. Let D be a nondegenerate graded Jacobi operator. Its nondegenerate odd Poisson bracket is associated to an odd graded symplectic form ω_D, which is of the form $\varphi^*(\omega_{\mathbf{L}})$, and $\omega_{\mathbf{L}}$ is the graded symplectic form of $\mathbb{Z}$-degree 1 defined by a linear isomorphism $\mathbf{L}\colon T^*M \to TM$; φ being an automorphism of $\Omega(M)$ that induces the identity on M. Let D_{-1} be the graded Jacobi operator of degree -1 corresponding to $\mathbf{L} = L \circ g^\sharp$. Then, it is easy to check that $D = \varphi \circ D_{-1} \circ \varphi^{-1}$ and,

$$[\![\alpha, \beta]\!]_D = \varphi(\,[\![\varphi^{-1}(\alpha), \varphi^{-1}(\beta)]\!]_{D_{-1}}\,). \qquad \square$$

We may now give the proof of Proposition 1

Proof of Proposition 1. According to Theorem 4, there exists an automorphism φ of $\Omega(M)$, restricting to the identity on $\Omega^0(M)$, such that, $\varphi^{-1} \circ D \circ \varphi = D_{-1}$ is a graded Jacobi operator of degree -1. By Theorem 2.3 its class defines a unique bundle isomorphism $L\colon TM \to TM$. By Theorem 2.2, the operator $L^* \circ \mathrm{d} \circ (L^*)^{-1}$ is a nondegenerate differential of $\Omega(M)$, and $D_{-1} = (-1)^{n-p} * \circ L^* \circ \mathrm{d} \circ (L^*)^{-1} \circ *^{-1}|_{\Omega^p(M)}$, by Theorem 1.4. We may then consider the isomorphism $\bar{L}^* : \Omega(M) \to \Omega(M)$ defined by means of $\bar{L}^* = * \circ L^* \circ *^{-1}$, so that,

$$\begin{aligned} D &= \varphi \circ D_{-1} \circ \varphi^{-1} = \varphi \circ ((-1)^{n-p} * \circ L^* \circ \mathrm{d} \circ (L^*)^{-1} \circ *^{-1}|_{\Omega^p(M)}) \circ \varphi^{-1} \\ &= \varphi \circ \bar{L}^* \circ \delta \circ (\bar{L}^*)^{-1} \circ \varphi^{-1}. \end{aligned}$$

Whence, $D = \varphi' \circ \delta \circ \varphi'^{-1}$, where, $\varphi' = \varphi \circ \bar{L}^*$, and $\delta = (-1)^{n-p} * \circ \mathrm{d} \circ *^{-1}|_{\Omega^p(M)}$. $\square$

4. Main examples of Jacobi operators

From [Kz], and [BM1], we know that a good source of nondegenerate graded Jacobi operators is found amongst the nondegenerate Poisson bivectors $P \in \Gamma(\Lambda^2 TM)$; namely, such a P yields the Jacobi operator $D = \mathcal{L}_P \in \mathcal{D}_2^{-1}(M)$. To exhibit this family of general examples we need to recall the following alternative decomposition for differential operators from $\mathcal{D}_2^{-1}(M)$.

PROPOSITION 1. *Let* M *be an oriented Riemannian manifold, and let* g *be its metric tensor. Let* $*$ *be the Hodge operator associated to* g, *and let* δ *be its corresponding codifferential operator. If* $D \in \mathcal{D}_2^{-1}(M)$ *is a differential operator on*

$\Omega(M)$, then there exist unique sections $P \in \Gamma(\Lambda^2 TM)$, $Q \in \Gamma^{g\text{-sym}}(T^*M \otimes TM)$, $C \in \Gamma(T^*M \otimes \Lambda^2 TM)$, and $X \in \Gamma(TM)$, such that,

$$D = \mathcal{L}_P + \delta_Q + i_C + i_X,$$

where $\delta_Q = [i_Q, \delta]$, and $\Gamma^{g\text{-sym}}(T^*M \otimes TM)$ is the space of TM-valued 1-forms Q satisfying $g(Q(X), Y) = g(X, Q(Y))$.

Proof. For any $f \in C^\infty(M)$, $[D, f] \in \mathcal{D}_1^{-1}(M)$; i.e., it is, a derivation of degree -1. Thus, $[D, f] = i_{H_f}$ for some vector field $H_f \in \Gamma(TM)$ depending on f. In fact, it depends on df because the map $C^\infty(M) \ni f \mapsto H_f \in \Gamma(TM)$ is easily seen to be a derivation; i.e., $H_{fh} = fH_h + hH_f$. This follows by applying $[[[D, f], h], \alpha] = 0$, for any $\alpha \in \Omega(M)$, and any $f, h \in \Omega^0(M)$, as $D \in \mathcal{D}_2^{-1}(M)$. So we write $H_f = \hat{H}_{\mathrm{d}f}$, and the map $\Gamma(T^*M) \ni \mathrm{d}f \mapsto \hat{H}_{\mathrm{d}f} \in \Gamma(TM)$ defines a section $L_{\hat{H}} \in \Gamma(TM \otimes TM)$, by letting $L_{\hat{H}}(\mathrm{d}f; \mathrm{d}h) = \mathrm{d}f(\hat{H}_{\mathrm{d}h})$. Conversely, each section $L \in \Gamma(TM \otimes TM)$ uniquely defines a $C^\infty(M)$-linear map $H^L\colon \Gamma(T^*M) \to \Gamma(TM)$, through $H^L_{\mathrm{d}f} = L(\mathrm{d}f;\ \ )$, where $L(\mathrm{d}f;\ \ )h = L(\mathrm{d}f; \mathrm{d}h)$.

Now given $L \in \Gamma(TM \otimes TM)$, we may decompose it in the form $L = L_s + L_a$, where $L_s \in \Gamma(S^2 TM)$, and $L_a \in \Gamma(\Lambda^2 TM)$. Now, L_s is used to construct a g-symmetric endomorphism $Q = L_s^{(g)} \in \Gamma(\mathrm{End}TM)$, for a given Riemannian metric g, by letting $\alpha(L_s^{(g)}(X)) = L_s(g(X,\ \ ); \alpha)$, for any 1-form α. It follows that $g(L_s^{(g)}(X), Y) = g(X, L_s^{(g)}(Y))$. Now consider the operator $D' = D - \mathcal{L}_{L_a} - \delta_{L_s^{(g)}}$, where $\mathcal{L}_{L_a} = [i_{L_a}, \mathrm{d}]$, and $\delta_{L_s^{(g)}} = [i_{L_s^{(g)}}, \delta]$ for the codifferential operator δ associated to g. Thus, $P = L_a$, and $Q = L_s^{(g)}$. A straightforward computation shows that D' is $C^\infty(M)$-linear, so it is uniquely of the form $i_C + i_X$ for some $C \in \Gamma(T^*M \otimes \Lambda^2 TM)$, and some $X \in \Gamma(TM)$. □

Given an isomorphism $L\colon TM \to TM$, and a Riemannian metric g on M as in Section 1, we shall write $\mathbf{L} = L \circ g^\sharp\colon T^*M \to TM$. Since, each Jacobi operator $D \in \mathcal{D}_2^{-1}(M)$ defines a Poisson bracket $[\![\ ,\ ]\!]_D$ on $\Omega(M)$ of degree -1, we may now relate our results in Section 2 and Section 3, with the Poisson brackets previously studied in [BM2] (see also [Kr]). We therefore start with the following

DEFINITION 2. Let $\mathbf{L}\colon \Lambda T^*M \to \Lambda TM$ be the algebra isomorphism defined by the universal extension of the bundle isomorphism $\mathbf{L}: T^*M \to TM$. The Poisson bracket of degree -1 on $\Omega(M)$, $[\![\ ,\ ]\!]_{\mathbf{L}}$, associated to $\mathbf{L}$ is defined by,

$$[\![\alpha, \beta]\!]_{\mathbf{L}} = -\mathbf{L}^{-1}\big([\mathbf{L}(\alpha), \mathbf{L}(\beta)]_{SN}\big),$$

for $\alpha, \beta \in \Omega(M)$. Here $[\ ,\ ]_{SN}$, denotes the Schouten-Nijenhuis bracket on multivectors.

To further explicit the correspondence between the isomorphisms $\mathbf{L}$ and graded Jacobi operators D of degree -1, we compare the brackets $[\![\ ,\ ]\!]_{\mathbf{L}}$ and $[\![\ ,\ ]\!]_D$. In

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particular, from Definition 2 above we may compute its value on functions and on exact 1-forms

$$[\![f,h]\!]_{\mathbf{L}} = 0, \qquad [\![\mathrm{d}f,h]\!]_{\mathbf{L}} = -\mathrm{d}h(\mathbf{L}(\mathrm{d}f)),$$
$$[\![\mathrm{d}f,\mathrm{d}h]\!]_{\mathbf{L}} = -\mathbf{L}^{-1}(\,[\mathbf{L}\,\mathrm{d}f,\mathbf{L}\,\mathrm{d}h]\,),$$

for $f,h \in C^\infty(M)$. These fromulae completely determine $[\![\ ,\ ]\!]_{\mathbf{L}}$ on $\Omega(M)$. On the other hand, Definition 1.2 yields:

$$[\![f,h]\!]_D = 0,$$
$$[\![\mathrm{d}f,h]\!]_D = -D(h\,\mathrm{d}f) + hD(\mathrm{d}f),$$
$$[\![\mathrm{d}f,\mathrm{d}h]\!]_D = -(D(\mathrm{d}f\wedge\mathrm{d}h) - D(\mathrm{d}f)\mathrm{d}h + D(\mathrm{d}h)\,\mathrm{d}f),$$

for $f,h \in C^\infty(M)$. Now, to establish the desired correspondence note that the following formula completely determines $\mathbf{L}$ when D is known

$$\begin{aligned} L(\mathrm{d}f;\mathrm{d}h) &= \mathrm{d}f(\mathbf{L}(\mathrm{d}h)) = [\![f,\mathrm{d}h]\!]_{\mathbf{L}} = [\![f,\mathrm{d}h]\!]_D \\ &= D(f\,\mathrm{d}h) - fD(\mathrm{d}h) = [D,f](\mathrm{d}h). \end{aligned}$$

That is, $\hat{H}_{\mathrm{d}f} = \mathbf{L}^*(\mathrm{d}f) = \mathrm{d}f\circ\mathbf{L}$ is the unique vector field such that $i_{\hat{H}_{\mathrm{d}f}} = [D,f]$ (see the proof of Prop. 1 above). Conversely, if $\mathbf{L}$ is a linear isomorphism, then the corresponding differential operator of degree -1 and order $\leqslant 2$ is determined by the following conditions: First, for any smooth function f,

$$\text{(b)} \quad [D,f] = [\![f,\ ]\!]_D = [\![f,\ ]\!]_{\mathbf{L}} = i_{\mathbf{L}^*(\mathrm{d}f)}.$$

This determines the nontensorial part of D, and therefore, its value on 1-forms (see the proof of Prop. 1 above). The value of D on 2-forms is determined by the formula

$$\begin{aligned} &-(D(\mathrm{d}f\wedge\mathrm{d}h) - D(\mathrm{d}f)\,\mathrm{d}h + D(\mathrm{d}h)\,\mathrm{d}f) \\ \text{(c)} \qquad &= [\![\mathrm{d}f,\mathrm{d}h]\!]_D \\ &= [\![\mathrm{d}f,\mathrm{d}h]\!]_{\mathbf{L}} = -\mathbf{L}^{-1}(\,[\mathbf{L}\,\mathrm{d}f,\mathbf{L}\,\mathrm{d}h]\,). \end{aligned}$$

Now, it is very easy to check that the bracket generated by D is in fact the graded Poisson bracket that $\mathbf{L}$ defines because they agree on generators.

Since an isomorphism $\mathbf{L}\colon T^*M\to TM$ defines a tensor field $L\in\Gamma(TM\otimes TM)$, we may now obtain two main classes of nondegenerate graded Jacobi operators: those associated to linear isomorphisms $T^*M\to TM$ coming from symmetric tensor fields $g\in\Gamma(S^2TM)$; i.e., *Riemannian metrics* on M, and those coming from skew-symmetric tensor fields $P\in\Gamma(\Lambda^2TM)$; i.e., nondegenerate bivectors (not necessarily Poisson bivectors!). We shall denote by $\mathbf{P}\colon T^*M\to TM$ the linear isomorphism corresponding to P.

PROPOSITION 3. *Under the correspondence between nondegenerate graded Jacobi operators of degree* -1 *and linear isomorphisms* $T^*M \to TM$, *we have:*

(1) *A Riemannian metric* $g \in \Gamma(S^2TM)$ *corresponds to the Jacobi operator* $-\delta^g$, *where* δ^g *is the codifferential associated to* g.

(2) *A nondegenerate bivector* $P \in \Gamma(\Lambda^2TM)$ *corresponds to the Jacobi operator* $\mathcal{L}_P + i_C$, *where* $C \in \Gamma(T^*M \otimes \Lambda^2TM)$ *is determined by the condition*

$$[P, P]_{SN} = -2\,C(\mathbf{P}\,;\ ,\ ).$$

Proof. (1) According to Proposition 1, a symmetric tensor $L \in \Gamma(TM \otimes TM)$ gives rise to a differential operator of the form $D = \delta_Q + i_C$ for some Riemannian metric g and a g-symmetric tensor $Q \in \Gamma(T^*M \otimes TM) \simeq \Gamma(\mathrm{End}\,TM)$. In fact, D and L are related by $[D, f] = i_{\mathbf{L}^* \mathrm{d}f}$, where, $L(\alpha, \beta) = \alpha(\mathbf{L}\beta)$ for $\alpha, \beta \in \Omega^1(M)$. Since L is symmetric and nondegenerate, it defines a Riemannian metric. So we choose g so that $L(\alpha, \beta) = g^{-1}(\alpha, \beta)$. Now, Proposition 1 says that Q is defined through $\alpha(Q(X)) = L(g(X,\ \ ), \alpha)$. So, our choice of g immediately implies $Q(X) = X$. That is, $\delta_Q = \delta_{\mathrm{Id}} = [i_{\mathrm{Id}}, \delta^g]$. In particular, the value of $D = [i_{\mathrm{Id}}, \delta^g] + i_C$ on exact 1-forms is

$$D(\mathrm{d}f) = [i_{\mathrm{Id}}, \delta^g](\mathrm{d}f) + i_C(\mathrm{d}f) = -\delta^g(\mathrm{d}f),$$

where we have used the fact that $i_C(\mathrm{d}f) = 0$, $i_{\mathrm{Id}}(\mathrm{d}f) = \mathrm{d}f$, and $i_{\mathrm{Id}}(h) = 0$ for any smooth functions f, and h. On the other hand, using eqn. (c) above, we know what the value of D on 2-forms $\mathrm{d}f \wedge \mathrm{d}h$ has to be

$$\begin{aligned} D(\mathrm{d}f \wedge \mathrm{d}h) &= D(\mathrm{d}f)\mathrm{d}h - \mathrm{d}f\, D(\mathrm{d}h) + \mathbf{L}^{-1}(\,[\mathbf{L}\,\mathrm{d}f, \mathbf{L}\,\mathrm{d}h]\,) \\ &= -\delta^g(\mathrm{d}f)\,\mathrm{d}h + \mathrm{d}f\,\delta^g(\mathrm{d}h) + g^\flat([g^\sharp(\mathrm{d}f), g^\sharp(\mathrm{d}h)]) \\ &= -\delta^g(\mathrm{d}f \wedge \mathrm{d}h), \end{aligned}$$

where in the last step we have used the formula (see for example [Va]):

$$\text{(d)} \quad \delta^g(\alpha \wedge \beta) = \delta^g(\alpha) \wedge \beta + (-1)^p \alpha \wedge \delta^g(\beta) - g^\flat([g^\sharp\alpha, g^\sharp\beta]_{SN}),$$

for $\alpha \in \Omega^p(M)$. We therefore conclude that, $D = -\delta^g$ since they both coincide on 1-forms and 2-forms, thus proving (1).

(2) Similarly, skew-symmetric tensors produce differential operators of the form $D = \mathcal{L}_P + i_C$ for some bivector P, and some $\wedge^2TM$-valued 1-form C. The graded Poisson bracket associated to D is determined by

$$\begin{aligned} &[\![f_1, f_2]\!]_D = 0, && [\![\mathrm{d}f_1, f_2]\!]_D = P(\mathrm{d}f_2, \mathrm{d}f_1), \\ &[\![f_1, \mathrm{d}f_2]\!]_D = P(\mathrm{d}f_2, \mathrm{d}f_1), && [\![\mathrm{d}f_1, \mathrm{d}f_2]\!]_D = \mathrm{d}P(\mathrm{d}f_2, \mathrm{d}f_1) - C(\ ; \mathrm{d}f_2, \mathrm{d}f_1), \end{aligned}$$

for $f_1, f_2 \in C^\infty(M)$. Let us suposse that D is Jacobi. Then it comes from the linear isomorphism $\mathbf{L}$ determined by (b). Then,

$$[\![f_1, \mathrm{d}f_2]\!]_D = \mathrm{d}f_2\mathbf{P}(\mathrm{d}f_1) = P(\mathrm{d}f_2, \mathrm{d}f_1) = \mathrm{d}f_1(-\mathbf{P}(\mathrm{d}f_2)) = [\![f_1, \mathrm{d}f_2]\!]_{-\mathbf{P}},$$

and it follows that $\mathbf{L} = -\mathbf{P}$. In particular, P is nondegenerate. Also note that

(e) $\quad \mathrm{d}P(\mathrm{d}f_2, \mathrm{d}f_1) - C(\ ;\mathrm{d}f_2, \mathrm{d}f_1) = \mathbf{P}^{-1}([\mathbf{P}\,\mathrm{d}f_1, \mathbf{P}\,\mathrm{d}f_2])$.

Both sides are differential forms of degree 1. Computing their value on a vector field of the form $\mathbf{P}(\mathrm{d}f_3)$, we obtain the desired condition

$$[P, P]_{SN}(\mathrm{d}f_3, \mathrm{d}f_1, \mathrm{d}f_2) = -2\,C(\mathbf{P}\,\mathrm{d}f_3; \mathrm{d}f_1, \mathrm{d}f_2).$$

On the other hand, if P is a skew-symmetric, nondegenerate section of $\Lambda^2 TM$, and,

$$[P, P]_{SN}(\mathrm{d}f_3, \mathrm{d}f_1, \mathrm{d}f_2) = -2\,C(\mathbf{P}\,\mathrm{d}f_3; \mathrm{d}f_1, \mathrm{d}f_2)$$

for all smooth functions f_1, f_2, f_3, then $[\![\ ,\ ]\!]_D = [\![\ ,\ ]\!]_{-\mathbf{P}}$, because both brackets yield the same value on generators. Therefore, $[\![\ ,\ ]\!]_D$ is a graded Poisson bracket, since $[\![\ ,\ ]\!]_{-\mathbf{P}}$ is; whence D is a graded Jacobi operator. □

The following proposition now computes some Hamiltonian graded vector fields associated to the graded Poisson brackets of our main examples of graded Jacobi operators. Note that it suffices to compute Hamiltonian fields for 0-forms and exact 1-forms, since any other Hamiltonian graded vector field may then be computed using the graded derivation property of the Poisson bracket $[\![\ ,\ ]\!]_D$.

PROPOSITION 4. *Let D be a nondegenerate graded Jacobi operator of degree -1.*

(1) *If D is of the form $\mathcal{L}_P + i_C$ for some bivector P, and C is its corresponding $\wedge^2 TM$-valued 1-form, then,*

$$[\![f,\ ]\!]_D = i_{\mathbf{P}\mathrm{d}f}, \qquad [\![\mathrm{d}f,\ ]\!]_D = \mathcal{L}_{\mathbf{P}\mathrm{d}f} - i_{C(\ ;\ ,\mathrm{d}f)},$$

for any $f \in C^\infty(M)$. Moreover, if Ω is the differential 2-form associated to the nondegenerate bivector P, then,

$$[\![\Omega,\ ]\!]_D = \mathrm{d} + i_{\widetilde{C}},$$

where $\widetilde{C} \in \Omega^2(M; TM)$ is defined by $\widetilde{C}(X, Y; \alpha) = \frac{1}{2}[P, P]_{SN}(\mathbf{P}^{-1}X, \mathbf{P}^{-1}Y, \alpha)$, for $X, Y \in \Gamma(TM)$ and $\alpha \in \Omega^1(M)$.

(2) *If D is of the form $-\delta^g$ for some Riemannian metric g, then,*

$$[\![f,\ ]\!]_D = i_{g^\sharp(\mathrm{d}f)}, \qquad [\![\mathrm{d}f,\ ]\!]_D = -\mathcal{L}_{g^\sharp(\mathrm{d}f)} + i_{\hat{g}(\mathrm{d}f)},$$

where $\hat{g}(\mathrm{d}f) \in \Omega^1(M; TM) \simeq \Gamma(\mathrm{End}\,TM)$ is the TM-valued 1-form defined by the map $TM \to TM$, which is the composition of the map $TM \to T^*M$, given by $X \mapsto (\mathcal{L}_{g^\sharp(\mathrm{d}f)}\,g)(\ , X)$, and the map $g^\sharp \colon T^*M \to TM$.

Proof. A direct consequence of the fact that the operator D is Jacobi of degree -1, is that $[\![\alpha,\ ]\!]_D$ is in fact a derivation of degree $k-1$ whenever $\alpha \in \Omega^k(M)$. In particular, we shall use this formula in the cases $\alpha = f \in \Omega^0(M)$, and $\alpha = \mathrm{d}f \in \Omega^1(M)$. In any case $[\![\ ,\ ]\!]_D = [\![\ ,\ ]\!]_{\mathbf{L}}$ for the linear isomorphism $\mathbf{L}$ corresponding to D, as it was shown in the proof of Proposition 3. Thus, $[\![f,\ ]\!]_{\mathbf{L}}$ is a derivation of degree -1 (of the form i_{X_f} for some $X_f \in \Gamma(TM)$), which will be completely determined by its value on exact 1-forms

$$\text{(f1)} \qquad [\![f,\ ]\!]_{\mathbf{L}}(\mathrm{d}h) = [\![f, \mathrm{d}h]\!]_{\mathbf{L}} = -[f, \mathbf{L}\,\mathrm{d}h]_{SN} = (\mathbf{L}\,\mathrm{d}h)\, f = L(\mathrm{d}f; \mathrm{d}h).$$

On the other hand, $[\![\mathrm{d}f,\ ]\!]_{\mathbf{L}}$ is a derivation of degree 0. It is therefore, completely determined by its value on 0-forms, and exact 1-forms:

$$\text{(f2)} \qquad \begin{aligned} &[\![\mathrm{d}f,\ ]\!]_{\mathbf{L}}(h) = [\![\mathrm{d}f, h]\!]_{\mathbf{L}} = -[\![h, \mathrm{d}f]\!]_{\mathbf{L}} = -L(\mathrm{d}h; \mathrm{d}f),\\ &[\![\mathrm{d}f,\ ]\!]_{\mathbf{L}}(\mathrm{d}h) = [\![\mathrm{d}f, \mathrm{d}h]\!]_{\mathbf{L}} = -\mathbf{L}^{-1}([\mathbf{L}\,\mathrm{d}f, \mathbf{L}\,\mathrm{d}h]). \end{aligned}$$

Now, if D is of the form $\mathcal{L}_P + i_C$, the isomorphism $\mathbf{L}$ is equal to $-\mathbf{P}$, and using formula (e) we get

$$\begin{aligned} [\![f, \mathrm{d}h]\!]_{-\mathbf{P}} &= -P(\mathrm{d}f, \mathrm{d}h) = i_{\mathbf{P}\,\mathrm{d}f}\, \mathrm{d}h,\\ [\![\mathrm{d}f, \mathrm{d}h]\!]_{-\mathbf{P}} &= \mathbf{P}^{-1}([\mathbf{P}\,\mathrm{d}f, \mathbf{P}\,\mathrm{d}h]) = \mathrm{d}P(\mathrm{d}h, \mathrm{d}f) - C(\ ; \mathrm{d}h, \mathrm{d}f)\\ &= \mathcal{L}_{\mathbf{P}\,\mathrm{d}f} - i_{C(\ ;\ ,\mathrm{d}f)}(\mathrm{d}h). \end{aligned}$$

Finally, note that the Hamiltonian $[\![\Omega,\ ]\!]_D$ is a derivation of degree 1. So, it is determined by its value on differentiable functions, and on exact one forms:

$$\begin{aligned} [\![\Omega, f]\!]_D &= [\![f, \Omega]\!]_{-\mathbf{P}} = i_{\mathbf{P}\,\mathrm{d}f}\Omega = \mathrm{d}f,\\ [\![\Omega, \mathrm{d}f]\!]_D &= -[\![\mathrm{d}f, \Omega]\!]_{-\mathbf{P}} = -i_{\mathbf{P}\,\mathrm{d}f}(\mathrm{d}\Omega) + i_{C(\ ;\ ,\mathrm{d}f)}\Omega. \end{aligned}$$

A straightforward calculation using (e), and,

$$\mathrm{d}\Omega(\mathbf{P}\,\mathrm{d}f, X, Y) = -\tfrac{1}{2}[P, P]_{SN}(\mathbf{P}^{-1}X, \mathbf{P}^{-1}Y, \mathrm{d}f),$$

determines the value of the differential 2-form $[\![\Omega, \mathrm{d}f]\!]_D$ on the pair of vector fields (X, Y):

$$\begin{aligned} [\![\Omega, \mathrm{d}f]\!]_D(X, Y) &= -\mathrm{d}\Omega(\mathbf{P}\,\mathrm{d}f, X, Y) - C(X; \mathbf{P}^{-1}Y, \mathrm{d}f)\\ &\quad + C(Y; \mathbf{P}^{-1}X, \mathrm{d}f)\\ &= \tfrac{1}{2}[P, P]_{SN}(\mathbf{P}^{-1}X, \mathbf{P}^{-1}Y, \mathrm{d}f) = \widetilde{C}(X, Y; \mathrm{d}f). \end{aligned}$$

We now prove (2): If D is of the form $-\delta^g$ the isomorphism $\mathbf{L}$ is equal to $g^\sharp$ (which we now write as $\mathbf{g}^{-1}$, for consistency with the notation used in the first part of the proof). Then, formulae (f1) and (f2) yield

$$\llbracket f,\ \rrbracket_{\mathbf{g}^{-1}}(\mathrm{d}h) = \llbracket f, \mathrm{d}h\rrbracket_{\mathbf{g}^{-1}} = g^{-1}(\mathrm{d}f, \mathrm{d}h) = i_{\mathbf{g}^{-1}(\mathrm{d}f)}\,\mathrm{d}h,$$
$$\llbracket \mathrm{d}f,\ \rrbracket_{\mathbf{g}^{-1}}(\mathrm{d}h) = \llbracket \mathrm{d}f, \mathrm{d}h\rrbracket_{\mathbf{g}^{-1}} = -\mathbf{g}([\mathbf{g}^{-1}(\mathrm{d}f), \mathbf{g}^{-1}(\mathrm{d}h)]).$$

The nontensorial part of $\llbracket \mathrm{d}f,\ \rrbracket_{\mathbf{g}^{-1}}$ (which is a derivation of degree 0), is determined by its value on 0-forms:

$$\llbracket \mathrm{d}f,\ \rrbracket_{\mathbf{g}^{-1}}(h) = \llbracket \mathrm{d}f, h\rrbracket_{\mathbf{g}^{-1}} = -\llbracket h, \mathrm{d}f\rrbracket_{\mathbf{g}^{-1}} = -\mathcal{L}_{\mathbf{g}^{-1}(\mathrm{d}f)}h.$$

Therefore, its tensorial part is,

$$(\llbracket \mathrm{d}f,\ \rrbracket_{\mathbf{g}^{-1}} + \mathcal{L}_{\mathbf{g}^{-1}(\mathrm{d}f)})(\mathrm{d}h) = -\mathbf{g}([\mathbf{g}^{-1}(\mathrm{d}f), \mathbf{g}^{-1}(\mathrm{d}h)]) + \mathcal{L}_{\mathbf{g}^{-1}(\mathrm{d}f)}(\mathrm{d}h).$$

Now note that, for any vector field X,

$$\begin{aligned}
&\mathbf{g}\left(\mathcal{L}_{\mathbf{g}^{-1}(\mathrm{d}f)}\,\mathbf{g}^{-1}(\mathrm{d}h)\right)(X) = g\left(\mathcal{L}_{\mathbf{g}^{-1}(\mathrm{d}f)}\,\mathbf{g}^{-1}(\mathrm{d}h)\,, X\right)\\
&\quad = \mathcal{L}_{\mathbf{g}^{-1}(\mathrm{d}f)}\left(g(\mathbf{g}^{-1}(\mathrm{d}h), X)\right) - (\mathcal{L}_{\mathbf{g}^{-1}(\mathrm{d}f)}\,g)(\mathbf{g}^{-1}(\mathrm{d}h), X)\\
&\qquad -g(\mathbf{g}^{-1}(\mathrm{d}h), \mathcal{L}_{\mathbf{g}^{-1}(\mathrm{d}f)}\,X)\\
&\quad = \mathcal{L}_{\mathbf{g}^{-1}(\mathrm{d}f)}(\mathrm{d}h(X)) - (\mathcal{L}_{\mathbf{g}^{-1}(\mathrm{d}f)}\,g)(\mathbf{g}^{-1}(\mathrm{d}h), X) - \mathrm{d}h\,(\mathcal{L}_{\mathbf{g}^{-1}(\mathrm{d}f)}\,X).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(\llbracket \mathrm{d}f,\ \rrbracket_{\mathbf{g}^{-1}} + \mathcal{L}_{\mathbf{g}^{-1}(\mathrm{d}f)})(\mathrm{d}h)(X)\\
&\quad = (\mathcal{L}_{\mathbf{g}^{-1}(\mathrm{d}f)}\,\mathrm{d}h)(X) - \mathcal{L}_{\mathbf{g}^{-1}(\mathrm{d}f)}(\mathrm{d}h(X))\\
&\qquad +(\mathcal{L}_{\mathbf{g}^{-1}(\mathrm{d}f)}\,g)(\mathbf{g}^{-1}(\mathrm{d}h), X) + \mathrm{d}h\,(\mathcal{L}_{\mathbf{g}^{-1}(\mathrm{d}f)}\,X)\\
&\quad = (\mathcal{L}_{\mathbf{g}^{-1}(\mathrm{d}f)}\,g)(\mathbf{g}^{-1}(\mathrm{d}h), X) = (i_{\hat{g}(\mathrm{d}f)}\,\mathrm{d}h)(X),
\end{aligned}$$

where the last step defines the TM-valued 1-form $\hat{g}(\mathrm{d}f)$ of the statement. □

COROLLARY 5. *Let Ω be the differential 2-form associated to the nondegenerate bivector P as in Proposition 4, and let $H_\Omega = \mathrm{d} + i_{\widetilde{C}}$ be its Hamiltonian graded vector field. Then $H_\Omega \circ H_\Omega = 0$ if and only if P is a Poisson bivector.*

Proof. The nontensorial part of the derivation $H_\Omega^2 = \frac{1}{2}[H_\Omega, H_\Omega]$ is $\mathcal{L}_{\widetilde{C}}$. Therefore, $H_\Omega^2 = 0$ implies $\widetilde{C} = 0$, but then, $[P, P]_{SN} = 0$, i.e., P is a Poisson bivector. □

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