1

Random Walks on Graphs

The theory of electrical networks is a fundamental tool for studying the recurrence of reversible Markov chains. The Kirchhoff laws and Thomson principle permit a neat proof of Pólya's theorem for random walk on a d-dimensional grid.

1.1 Random Walks and Reversible Markov Chains

A basic knowledge of probability theory is assumed in this volume. Readers keen to acquire this are referred to [150] for an elementary introduction, and to [148] for a somewhat more advanced account. We shall generally use the letter $\mathbb{P}$ to denote a generic probability measure, with more specific notation when helpful. The expectation of a random variable f will be written as either $\mathbb{P}(f)$ or $\mathbb{E}(f)$.

Only a little knowledge is assumed about graphs, and many readers will have sufficient acquaintance already. Others are advised to consult Section 1.6. Of the many books on graph theory, we mention [50].

Let $G = (V, E)$ be a finite or countably infinite graph, which we generally assume, for simplicity, to have neither loops nor multiple edges. If G is infinite, we shall usually assume in addition that every vertex-degree is finite. A particle moves around the vertex-set V. Having arrived at the vertex S_n at time n, its next position S_{n+1} is chosen uniformly at random from the set of neighbours of S_n. The trajectory of the particle is called a *symmetric random walk* (SRW) on G.

Two of the basic questions concerning symmetric random walk are:

1. Under what conditions is the walk *recurrent*, in that it returns (almost surely) to its starting point?
2. How does the distance between S_0 and S_n behave as $n \to \infty$?

The above SRW is symmetric in that the jumps are chosen *uniformly* from the set of available neighbours. In a more general process, we take a function $w : E \to (0, \infty)$, and we jump along the edge e with probability proportional to w_e.

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Any reversible Markov chain[1] on the set V gives rise to such a walk as follows. Let $Z = (Z_n : n \geq 0)$ be a Markov chain on V with transition matrix P, and assume that Z is reversible with respect to some positive function $\pi : V \to (0, \infty)$, which is to say that

$$\pi_u p_{u,v} = \pi_v p_{v,u}, \qquad u, v \in V. \tag{1.1}$$

With each distinct pair $u, v \in V$, we associate the weight

$$w_{u,v} = \pi_u p_{u,v}, \tag{1.2}$$

noting by (1.1) that $w_{u,v} = w_{v,u}$. Then

$$p_{u,v} = \frac{w_{u,v}}{W_u}, \qquad u, v \in V, \tag{1.3}$$

where

$$W_u = \sum_{v \in V} w_{u,v}, \qquad u \in V.$$

That is, given that $Z_n = u$, the chain jumps to a new vertex v with probability proportional to $w_{u,v}$. This may be set in the context of a random walk on the graph with vertex-set V and edge-set E containing all $e = \langle u, v \rangle$ such that $p_{u,v} > 0$. With edge $e \in E$ we associate the weight $w_e = w_{u,v}$.

In this chapter, we develop the relationship between random walks on G and electrical networks on G. There are some excellent accounts of this subject area, and the reader is referred to the books of Doyle and Snell [83], Lyons and Peres [221], and Aldous and Fill [19], amongst others. The connection between these two topics is made via the so-called 'harmonic functions' of the random walk.

1.4 Definition Let $U \subseteq V$, and let Z be a Markov chain on V with transition matrix P, that is reversible with respect to the positive function π. The function $f : V \to \mathbb{R}$ is *harmonic* on U (with respect to P) if

$$f(u) = \sum_{v \in V} p_{u,v} f(v), \qquad u \in U,$$

or, equivalently, if $f(u) = \mathbb{E}(f(Z_1) \mid Z_0 = u)$ for $u \in U$.

From the pair (P, π), we can construct the graph G as above, and the weight function w as in (1.2). We refer to the pair (G, w) as the weighted graph associated with (P, π). We shall speak of f as being harmonic (for (G, w)) if it is harmonic with respect to P.

[1] Accounts of Markov chain theory are found in [148, Chap. 6] and [150, Chap. 12].

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The so-called hitting probabilities are basic examples of harmonic functions for the chain Z. Let $U \subseteq V$, $W = V \setminus U$, and $s \in U$. For $u \in V$, let $g(u)$ be the probability that the chain, started at u, hits s before W. That is,

$$g(u) = \mathbb{P}_u(Z_n = s \text{ for some } n < T_W),$$

where

$$T_W = \inf\{n \geq 0 : Z_n \in W\}$$

is the first-passage time to W, and $\mathbb{P}_u(\cdot) = \mathbb{P}(\cdot \mid Z_0 = u)$ denotes the conditional probability measure given that the chain starts at u.

1.5 Theorem *The function g is harmonic on $U \setminus \{s\}$.*

Evidently, $g(s) = 1$, and $g(v) = 0$ for $v \in W$. We speak of these values of g as being the 'boundary conditions' of the harmonic function g. See Exercise 1.13 for the uniqueness of harmonic functions with given boundary conditions.

Proof. This is an elementary exercise using the Markov property. For $u \notin W \cup \{s\}$,

$$\begin{aligned} g(u) &= \sum_{v \in V} p_{u,v} \mathbb{P}_u\big(Z_n = s \text{ for some } n < T_W \,\big|\, Z_1 = v\big) \\ &= \sum_{v \in V} p_{u,v} g(v), \end{aligned}$$

as required. □

1.2 Electrical Networks

Throughout this section, $G = (V, E)$ is a finite graph with neither loops nor multiple edges, and $w : E \to (0, \infty)$ is a weight function on the edges. We shall assume further that G is connected.

We may build an electrical network with diagram G, in which the edge e has conductance w_e (or, equivalently, resistance $1/w_e$). Let $s, t \in V$ be distinct vertices termed *sources*, and write $S = \{s, t\}$ for the *source-set*. Suppose we connect a battery across the pair s, t. It is a physical observation that electrons flow along the wires in the network. The flow is described by the so-called Kirchhoff laws, as follows.

To each edge $e = \langle u, v \rangle$, there are associated (directed) quantities $\phi_{u,v}$ and $i_{u,v}$, called the *potential difference* from u to v, and the *current* from u to v, respectively. These are antisymmetric,

$$\phi_{u,v} = -\phi_{v,u}, \qquad i_{u,v} = -i_{v,u}.$$

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1.6 Kirchhoff's potential law The cumulative potential difference around any cycle $v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ of G is zero, that is,

$$\sum_{j=1}^{n} \phi_{v_j, v_{j+1}} = 0. \tag{1.7}$$

1.8 Kirchhoff's current law The total current flowing out of any vertex $u \in V$ other than the source-set is zero, that is,

$$\sum_{v \in V} i_{u,v} = 0, \qquad u \neq s, t. \tag{1.9}$$

The relationship between resistance/conductance, potential difference, and current is given by Ohm's law.

1.10 Ohm's law For any edge $e = \langle u, v \rangle$,

$$i_{u,v} = w_e \phi_{u,v}.$$

Kirchhoff's potential law is equivalent to the statement that there exists a function $\phi : V \to \mathbb{R}$, called a *potential function*, such that

$$\phi_{u,v} = \phi(v) - \phi(u), \qquad \langle u, v \rangle \in E.$$

Since ϕ is determined up to an additive constant, we are free to pick the potential of any single vertex. Note our convention that *current flows uphill*: $i_{u,v}$ has the same sign as $\phi_{u,v} = \phi(v) - \phi(u)$.

1.11 Theorem *A potential function is harmonic on the set of all vertices other than the source-set.*

Proof. Let $U = V \setminus \{s, t\}$. By Kirchhoff's current law and Ohm's law,

$$\sum_{v \in V} w_{u,v} [\phi(v) - \phi(u)] = 0, \qquad u \in U,$$

which is to say that

$$\phi(u) = \sum_{v \in V} \frac{w_{u,v}}{W_u} \phi(v), \qquad u \in U,$$

where

$$W_u = \sum_{v \in V} w_{u,v}.$$

That is, ϕ is harmonic on U. □

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We can use Ohm's law to express potential differences in terms of currents, and thus the two Kirchhoff laws may be viewed as concerning currents only. Equation (1.7) becomes

$$\sum_{j=1}^{n} \frac{i_{v_j,v_{j+1}}}{w_{\langle v_j,v_{j+1}\rangle}} = 0, \tag{1.12}$$

valid for any cycle $v_1, v_2, \dots, v_n, v_{n+1} = v_1$. With (1.7) written thus, each law is linear in the currents, and the superposition principle follows.

1.13 Theorem (Superposition principle) *If i^1 and i^2 are solutions of the two Kirchhoff laws with the same source-set then so is the sum $i^1 + i^2$.*

Next we introduce the concept of a 'flow' on a graph.

1.14 Definition Let $s, t \in V$, $s \neq t$. An *s/t-flow* j is a vector $j = (j_{u,v} : u, v \in V,\ u \neq v)$, such that:

(a) $j_{u,v} = -j_{v,u}$,
(b) $j_{u,v} = 0$ whenever $u \nsim v$,
(c) for any $u \neq s, t$, we have that $\sum_{v \in V} j_{u,v} = 0$.

The vertices s and t are called the 'source' and 'sink' of an s/t flow, and we usually abbreviate 's/t flow' to 'flow'. For any flow j, we write

$$J_u = \sum_{v \in V} j_{u,v}, \qquad u \in V,$$

noting by (c) above that $J_u = 0$ for $u \neq s, t$. Thus,

$$J_s + J_t = \sum_{u \in V} J_u = \sum_{u,v \in V} j_{u,v} = \tfrac{1}{2} \sum_{u,v \in V} (j_{u,v} + j_{v,u}) = 0.$$

Therefore, $J_s = -J_t$, and we call $|J_s|$ the *size* of the flow j, denoted $|j|$. If $|J_s| = 1$, we call j a *unit flow*. We shall normally take $J_s > 0$, in which case s is the *source* and t is the *sink* of the flow, and we say that j is a flow from s to t.

Note that any solution i to the Kirchhoff laws with source-set $\{s, t\}$ is an s/t flow.

1.15 Theorem *Let i^1 and i^2 be two solutions of the Kirchhoff laws with the same source and sink and equal size. Then $i^1 = i^2$.*

Proof. By the superposition principle, $j = i^1 - i^2$ satisfies the two Kirchhoff laws. Furthermore, under the flow j, no current enters or leaves the system. Therefore, $J_v = 0$ for all $v \in V$. Suppose $j_{u_1,u_2} > 0$ for some edge $\langle u_1, u_2 \rangle$. By the Kirchhoff current law, there exists u_3 such that

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$j_{u_2,u_3} > 0$. Since $|V| < \infty$, there exists by iteration a cycle $u_l, u_{l+1}, \dots, u_m, u_{m+1} = u_l$ such that $j_{u_k,u_{k+1}} > 0$ for $k = l, l+1, \dots, m$. By Ohm's law, the corresponding potential function satisfies

$$\phi(u_l) < \phi(u_{l+1}) < \cdots < \phi(u_{m+1}) = \phi(u_l),$$

a contradiction. Therefore, $j_{u,v} = 0$ for all u, v. □

For a given size of input current, and given source s and sink t, there can be no more than one solution to the two Kirchhoff laws, but is there a solution at all? The answer is of course affirmative, and the unique solution can be expressed explicitly in terms of counts of spanning trees.[2] Consider first the special case when $w_e = 1$ for all $e \in E$. Let N be the number of spanning trees of G. For any edge $\langle a, b\rangle$, let $\Pi(s, a, b, t)$ be the property of spanning trees that: the unique s/t path in the tree passes along the edge $\langle a, b\rangle$ in the direction from a to b. Let $\mathcal{N}(s, a, b, t)$ be the set of spanning trees of G with the property $\Pi(s, a, b, t)$, and let $N(s, a, b, t) = |\mathcal{N}(s, a, b, t)|$.

1.16 Theorem *The function*

$$i_{a,b} = \frac{1}{N}\big[N(s, a, b, t) - N(s, b, a, t)\big], \qquad \langle a, b\rangle \in E, \tag{1.17}$$

defines a unit flow from s to t satisfying the Kirchhoff laws.

Let T be a spanning tree of G chosen uniformly at random from the set $\mathcal{T}$ of all such spanning trees. By Theorem 1.16 and the previous discussion, the unique solution to the Kirchhoff laws with source s, sink t, and size 1 is given by

$$i_{a,b} = \mathbb{P}\big(T \text{ has } \Pi(s, a, b, t)\big) - \mathbb{P}\big(T \text{ has } \Pi(s, b, a, t)\big).$$

We shall return to uniform spanning trees in Chapter 2.

We prove Theorem 1.16 next. Exactly the same proof is valid in the case of general conductances w_e. In that case, we define the weight of a spanning tree T as

$$w(T) = \prod_{e \in T} w_e,$$

and we set

$$N^* = \sum_{T \in \mathcal{T}} w(T), \qquad N^*(s, a, b, t) = \sum_{T \text{ with } \Pi(s,a,b,t)} w(T). \tag{1.18}$$

The conclusion of Theorem 1.16 holds in this setting with

$$i_{a,b} = \frac{1}{N^*}\big[N^*(s, a, b, t) - N^*(s, b, a, t)\big], \qquad \langle a, b\rangle \in E.$$

[2]This was discovered in an equivalent form by Kirchhoff in 1847, [188].

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Proof of Theorem 1.16. We first check the Kirchhoff current law. In every spanning tree T, there exists a unique vertex b such that the s/t path of T contains the edge $\langle s, b\rangle$, and the path traverses this edge from s to b. Therefore,

$$\sum_{b\in V} N(s,s,b,t) = N, \qquad N(s,b,s,t) = 0 \text{ for } b \in V.$$

By (1.17),

$$\sum_{b\in V} i_{s,b} = 1,$$

and, by a similar argument, $\sum_{b\in V} i_{b,t} = 1$.

Let T be a spanning tree of G. The contribution towards the quantity $i_{a,b}$, made by T, depends on the s/t path π of T and equals

$$\begin{array}{rl} N^{-1} & \text{if } \pi \text{ passes along } \langle a,b\rangle \text{ from } a \text{ to } b, \\ -N^{-1} & \text{if } \pi \text{ passes along } \langle a,b\rangle \text{ from } b \text{ to } a, \\ 0 & \text{if } \pi \text{ does not contain the edge } \langle a,b\rangle. \end{array} \tag{1.19}$$

Let $v \in V$, $v \neq s, t$, and write $I_v = \sum_{w\in V} i_{v,w}$. If $v \in \pi$, the contribution of T towards I_v is $N^{-1} - N^{-1} = 0$ since π arrives at v along some edge of the form $\langle a, v\rangle$ and departs from v along some edge of the form $\langle v, b\rangle$. If $v \notin \pi$, then T contributes 0 to I_v. Summing over T, we obtain that $I_v = 0$ for all $v \neq s, t$, as required for the Kirchhoff current law.

We next check the Kirchhoff potential law. Let $v_1, v_2, \dots, v_n, v_{n+1} = v_1$ be a cycle C of G. We shall show that

$$\sum_{j=1}^{n} i_{v_j,v_{j+1}} = 0, \tag{1.20}$$

and this will confirm (1.12), on recalling that $w_e = 1$ for all $e \in E$. It is more convenient in this context to work with 'bushes' than spanning trees. A *bush* (or, more precisely, an *s/t bush*) is defined to be a forest on V containing exactly two trees, one denoted T_s and containing s, and the other denoted T_t and containing t. We write (T_s, T_t) for this bush. Let $e = \langle a, b\rangle$, and let $\mathcal{B}(s,a,b,t)$ be the set of bushes with $a \in T_s$ and $b \in T_t$. The sets $\mathcal{B}(s,a,b,t)$ and $\mathcal{N}(s,a,b,t)$ are in one–one correspondence, since the addition of e to $B \in \mathcal{B}(s,a,b,t)$ creates a unique member $T = T(B)$ of $\mathcal{N}(s,a,b,t)$, and vice versa.

By (1.19) and the above, a bush $B = (T_s, T_t)$ makes a contribution to $i_{a,b}$ of

$$\begin{array}{rl} N^{-1} & \text{if } B \in \mathcal{B}(s,a,b,t), \\ -N^{-1} & \text{if } B \in \mathcal{B}(s,b,a,t), \\ 0 & \text{otherwise.} \end{array}$$

Therefore, B makes a contribution towards the sum in (1.20) that is equal to $N^{-1}(F_+ - F_-)$, where F_+ (respectively, F_-) is the number of pairs v_j, v_{j+1} of C, $1 \le j \le n$, with $v_j \in T_s$, $v_{j+1} \in T_t$ (respectively, $v_{j+1} \in T_s$, $v_j \in T_t$). Since C is a cycle, we have $F_+ = F_-$, whence each bush contributes 0 to the sum and (1.20) is proved. □

1.3 Flows and Energy

Let $G = (V, E)$ be a connected graph as before. Let $s, t \in V$ be distinct vertices, and let j be an s/t flow. With w_e the conductance of the edge e, the (dissipated) *energy* of j is defined as

$$E(j) = \sum_{e=\langle u,v\rangle \in E} j_{u,v}^2 / w_e = \frac{1}{2} \sum_{\substack{u,v \in V \\ u \sim v}} j_{u,v}^2 / w_{\langle u,v \rangle}.$$

The following piece of linear algebra will be useful.

1.21 Proposition *Let $\psi : V \to \mathbb{R}$, and let j be an s/t flow. Then*

$$[\psi(t) - \psi(s)] J_s = \frac{1}{2} \sum_{u,v \in V} [\psi(v) - \psi(u)] j_{u,v}.$$

Proof. By the properties of a flow,

$$\begin{aligned} \sum_{u,v \in V} [\psi(v) - \psi(u)] j_{u,v} &= \sum_{v \in V} \psi(v)(-J_v) - \sum_{u \in V} \psi(u) J_u \\ &= -2[\psi(s) J_s + \psi(t) J_t] \\ &= 2[\psi(t) - \psi(s)] J_s, \end{aligned}$$

as required. □

Let ϕ and i satisfy the two Kirchhoff laws. We apply Proposition 1.21 with $\psi = \phi$ and $j = i$ to find by Ohm's law that

$$E(i) = [\phi(t) - \phi(s)] I_s. \tag{1.22}$$

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That is, the energy of the true current-flow i from s to t equals the energy dissipated in a (notional) single $\langle s, t\rangle$ edge carrying the same potential difference and total current. The conductance W_{eff} of such an edge would satisfy Ohm's law, that is,

$$I_s = W_{\text{eff}}[\phi(t) - \phi(s)], \tag{1.23}$$

and we define the *effective conductance* W_{eff} by this equation. The effective resistance is

$$R_{\text{eff}} = \frac{1}{W_{\text{eff}}}, \tag{1.24}$$

which, by (1.22) and (1.23), equals $E(i)/I_s^2$. We state this as a lemma.

1.25 Lemma *The effective resistance R_{eff} of the network between vertices s and t equals the dissipated energy when a unit flow passes from s to t.*

It is useful to be able to do calculations. Electrical engineers have devised a variety of formulaic methods for calculating the effective resistance of a network, of which the simplest are the series and parallel laws, illustrated in Figure 1.1.

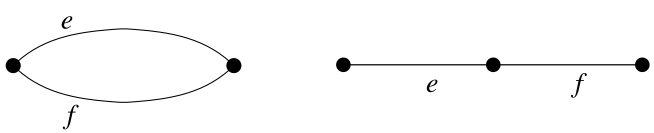

Figure 1.1 Two edges e and f in parallel and in series.

1.26 Series law Two resistors of size r_1 and r_2 in series may be replaced by a single resistor of size $r_1 + r_2$.

1.27 Parallel law Two resistors of size r_1 and r_2 in parallel may be replaced by a single resistor of size R, where $R^{-1} = r_1^{-1} + r_2^{-1}$.

A third such rule, the so-called 'star–triangle transformation', may be found at Exercise 1.5. The following 'variational principle' has many uses.

1.28 Theorem (Thomson principle) *Let $G = (V, E)$ be a connected graph and $(w_e : e \in E)$ strictly positive conductances. Let $s, t \in V$, $s \neq t$. Amongst all unit flows through G from s to t, the flow that satisfies the Kirchhoff laws is the unique s/t flow i that minimizes the dissipated energy. That is,*

$$E(i) = \inf\left\{E(j) : j \text{ a unit flow from } s \text{ to } t\right\}.$$

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Proof. Let j be a unit flow from source s to sink t, and set $k = j - i$, where i is the (unique) unit-flow solution to the Kirchhoff laws. Thus, k is a flow with zero size. Now, with $e = \langle u, v\rangle$ and $r_e = 1/w_e$,

$$\begin{aligned} 2E(j) &= \sum_{u,v\in V} j_{u,v}^2 r_e = \sum_{u,v\in V} (k_{u,v} + i_{u,v})^2 r_e \\ &= \sum_{u,v\in V} k_{u,v}^2 r_e + \sum_{u,v\in V} i_{u,v}^2 r_e + 2\sum_{u,v\in V} i_{u,v} k_{u,v} r_e. \end{aligned}$$

Let ϕ be the potential function corresponding to i. By Ohm's law and Proposition 1.21,

$$\begin{aligned} \sum_{u,v\in V} i_{u,v} k_{u,v} r_e &= \sum_{u,v\in V} [\phi(v) - \phi(u)] k_{u,v} \\ &= 2[\phi(t) - \phi(s)] K_s, \end{aligned}$$

which equals zero. Therefore, $E(j) \ge E(i)$, with equality if and only if $j = i$. □

The Thomson 'variational principle' leads to a proof of the 'obvious' fact that the effective resistance of a network is a non-decreasing function of the resistances of individual edges.

1.29 Theorem (Rayleigh principle) *The effective resistance R_{eff} of the network is a non-decreasing function of the edge-resistances $(r_e : e \in E)$.*

It is left as an exercise to show that R_{eff} is a concave function of the vector (r_e). See Exercise 1.6.

Proof. Consider two vectors $(r_e : e \in E)$ and $(r'_e : e \in E)$ of edge-resistances with $r_e \le r'_e$ for all e. Let i and i' denote the corresponding unit flows satisfying the Kirchhoff laws. By Lemma 1.25, with $r_e = r_{\langle u,v\rangle}$,

$$\begin{aligned} R_{\text{eff}} &= \tfrac{1}{2} \sum_{\substack{u,v\in V \\ u\sim v}} i_{u,v}^2 r_e \\ &\le \tfrac{1}{2} \sum_{\substack{u,v\in V \\ u\sim v}} (i'_{u,v})^2 r_e && \text{by the Thomson principle} \\ &\le \tfrac{1}{2} \sum_{\substack{u,v\in V \\ u\sim v}} (i'_{u,v})^2 r'_e && \text{since } r_e \le r'_e \\ &= R'_{\text{eff}}, \end{aligned}$$

as required. □

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1.4 Recurrence and Resistance

Let $G = (V, E)$ be an infinite connected graph with finite vertex-degrees, and let $(w_e : e \in E)$ be (strictly positive) conductances. We shall consider a reversible Markov chain $Z = (Z_n : n \geq 0)$ on the state space V with transition probabilities given by (1.3). Our purpose is to establish a condition on the pair (G, w) that is equivalent to the recurrence of Z.

Let 0 be a distinguished vertex of G, called the 'origin', and suppose that $Z_0 = 0$. The graph-theoretic distance between two vertices u, v is the number of edges in a shortest path between u and v, denoted $\delta(u, v)$. Let

$$\Lambda_n = \{u \in V : \delta(0, v) \leq n\},$$
$$\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1} = \{u \in V : \delta(0, v) = n\}.$$

We think of $\partial\Lambda_n$ as the 'boundary' of Λ_n. Let G_n be the subgraph of G induced by the vertex-set Λ_n. We let $\overline{G}_n$ be the graph obtained from G_n by identifying the vertices in $\partial\Lambda_n$ as a single composite vertex denoted I_n. The resulting finite graph $\overline{G}_n$ may be considered as an electrical network with sources 0 and I_n. Let $R_{\text{eff}}(n)$ be the effective resistance of this network. The graph $\overline{G}_n$ may be obtained from $\overline{G}_{n+1}$ by identifying all vertices lying in $\partial\Lambda_n \cup \{I_{n+1}\}$, and thus, by the Rayleigh principle, $R_{\text{eff}}(n)$ is non-decreasing in n. Therefore, the limit

$$R_{\text{eff}} = \lim_{n\to\infty} R_{\text{eff}}(n)$$

exists.

1.30 Theorem *The probability of ultimate return by Z to the origin* 0 *is given by*

$$\mathbb{P}_0(Z_n = 0 \text{ for some } n \geq 1) = 1 - \frac{1}{W_0 R_{\text{eff}}},$$

where $W_0 = \sum_{v:\, v\sim 0} w_{\langle 0,v\rangle}$.

The return probability is non-decreasing as $W_0 R_{\text{eff}}$ increases. By the Rayleigh principle, this can be achieved, for example, by removing an edge of E that is not incident to 0. The removal of an edge incident to 0 can have the opposite effect, since W_0 decreases while R_{eff} increases (see Figure 1.2).

A $0/\infty$ *flow* is a vector $j = (j_{u,v} : u, v \in V,\ u \neq v)$ satisfying (1.14)(a), (b) and also (c) for all $u \neq 0$. That is, it has source 0 but no sink.

1.31 Corollary

(a) *The chain Z is recurrent if and only if $R_{\text{eff}} = \infty$.*

(b) *The chain Z is transient if and only if there exists a non-zero $0/\infty$ flow j on G whose energy $E(j) = \sum_e j_e^2/w_e$ satisfies $E(j) < \infty$.*

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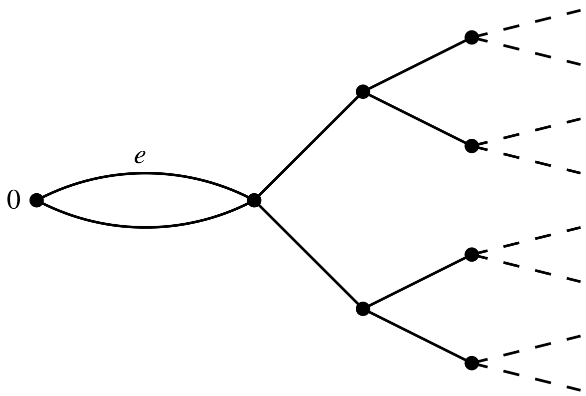

Figure 1.2 This is an infinite binary tree with two parallel edges joining the origin to the root. When each edge has unit resistance, it is an easy calculation that $R_{\mathrm{eff}} = \frac{3}{2}$, so the probability of return to 0 is $\frac{2}{3}$. If the edge e is removed, this probability becomes $\frac{1}{2}$.

It is left as an exercise to extend this to countable graphs G without the assumption of finite vertex-degrees.

Proof of Theorem 1.30. Let

$$g_n(v) = \mathbb{P}_v(Z \text{ hits } \partial\Lambda_n \text{ before } 0), \qquad v \in \Lambda_n.$$

By Theorem 1.5 and Exercise 1.13, g_n is the unique harmonic function on G_n with boundary conditions

$$g_n(0) = 0, \qquad g_n(v) = 1 \text{ for } v \in \partial\Lambda_n.$$

Therefore, g_n is a potential function on $\overline{G}_n$ viewed as an electrical network with source 0 and sink I_n.

By conditioning on the first step of the walk, and using Ohm's law,

$$\begin{aligned}
&\mathbb{P}_0(Z \text{ returns to } 0 \text{ before reaching } \partial\Lambda_n)\\
&\qquad = 1 - \sum_{v:\, v\sim 0} p_{0,v}\, g_n(v)\\
&\qquad = 1 - \sum_{v:\, v\sim 0} \frac{w_{0,v}}{W_0}[g_n(v) - g_n(0)]\\
&\qquad = 1 - \frac{|i(n)|}{W_0},
\end{aligned}$$

where $i(n)$ is the flow of currents in $\overline{G}_n$, and $|i(n)|$ is its size. By (1.23) and (1.24), $|i(n)| = 1/R_{\mathrm{eff}}(n)$. The theorem is proved on noting that

$$\mathbb{P}_0(Z \text{ returns to } 0 \text{ before reaching } \partial\Lambda_n) \to \mathbb{P}_0(Z_n = 0 \text{ for some } n \geq 1)$$

as $n \to \infty$, by the continuity of probability measures. □

Proof of Corollary 1.31. Part (a) is an immediate consequence of Theorem 1.30, and we turn to part (b). By Lemma 1.25, there exists a unit flow $i(n)$ in $\overline{G}_n$ with source 0 and sink I_n, and with energy $E(i(n)) = R_{\text{eff}}(n)$. Let i be a non-zero $0/\infty$ flow; by dividing by its size, we may take i to be a unit flow. When restricted to the edge-set E_n of $\overline{G}_n$, i forms a unit flow from 0 to I_n. By the Thomson principle, Theorem 1.28,

$$E(i(n)) \le \sum_{e \in E_n} i_e^2 / w_e \le E(i),$$

whence

$$E(i) \ge \lim_{n\to\infty} E(i(n)) = R_{\text{eff}}.$$

Therefore, by part (a), $E(i) = \infty$ if the chain is recurrent.

Suppose, conversely, that the chain is transient. By diagonal selection,[3] there exists a subsequence (n_k) along which $i(n_k)$ converges to some limit j (that is, $i(n_k)_e \to j_e$ for every $e \in E$). Since each $i(n_k)$ is a unit flow from the origin, j is a unit $0/\infty$ flow. Now,

$$\begin{aligned} E(i(n_k)) &= \sum_{e\in E} i(n_k)_e^2 / w_e \\ &\ge \sum_{e\in E_m} i(n_k)_e^2 / w_e \\ &\to \sum_{e\in E_m} j(e)^2 / w_e && \text{as } k \to \infty \\ &\to E(j) && \text{as } m \to \infty. \end{aligned}$$

Therefore,

$$E(j) \le \lim_{k\to\infty} R_{\text{eff}}(n_k) = R_{\text{eff}} < \infty,$$

and j is a flow with the required properties. □

[3] *Diagonal selection*: Let $(x_m(n) : m, n \ge 1)$ be a bounded collection of reals. There exists an increasing sequence $n_1, n_2, \dots$ of positive integers such that, for every m, the limit $\lim_{k\to\infty} x_m(n_k)$ exists.

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1.5 Pólya's Theorem

The d-dimensional cubic lattice $\mathbb{L}^d$ has vertex-set $\mathbb{Z}^d$ and edges between any two vertices that are Euclidean distance one apart. The following celebrated theorem can be proved by estimating effective resistances.[4]

1.32 Pólya's theorem [242] *Symmetric random walk on the lattice $\mathbb{L}^d$ in d dimensions is recurrent if $d = 1, 2$ and transient if $d \geq 3$.*

The advantage of the following proof of Pólya's theorem over more standard arguments is its robustness with respect to the underlying graph. Similar arguments are valid for graphs that are, in broad terms, comparable to $\mathbb{L}^d$ when viewed as electrical networks.

Proof. For simplicity, and with only little loss of generality (see Exercise 1.10), we shall concentrate on the cases $d = 2, 3$. Let $d = 2$, for which case we aim to show that $R_{\text{eff}} = \infty$. This is achieved by finding an infinite lower bound for R_{eff}, and lower bounds can be obtained by decreasing individual edge-resistances. The identification of two vertices of a network amounts to the addition of a resistor with 0 resistance, and, by the Rayleigh principle, the effective resistance of the network can only decrease.

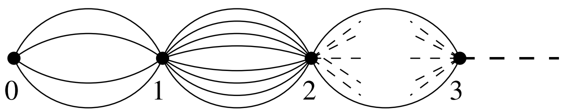

Figure 1.3 The vertex labelled i is a composite vertex obtained by identifying all vertices with distance i from 0. There are $8i - 4$ edges of $\mathbb{L}^2$ joining vertices $i - 1$ and i.

From $\mathbb{L}^2$, we construct a new graph in which, for each $k = 1, 2, \ldots,$ the set $\partial\Lambda_k = \{v \in \mathbb{Z}^2 : \delta(0, v) = k\}$ is identified as a singleton. This transforms $\mathbb{L}^2$ into the graph shown in Figure 1.3. By the series/parallel laws and the Rayleigh principle,

$$R_{\text{eff}}(n) \geq \sum_{i=1}^{n-1} \frac{1}{8i - 4},$$

whence $R_{\text{eff}}(n) \geq c \log n \to \infty$ as $n \to \infty$.

Suppose now that $d = 3$. There are at least two ways of proceeding. We shall present one such route, taken from [222], and we shall then sketch

[4] An amusing story is told in [243] about Pólya's inspiration for this theorem.

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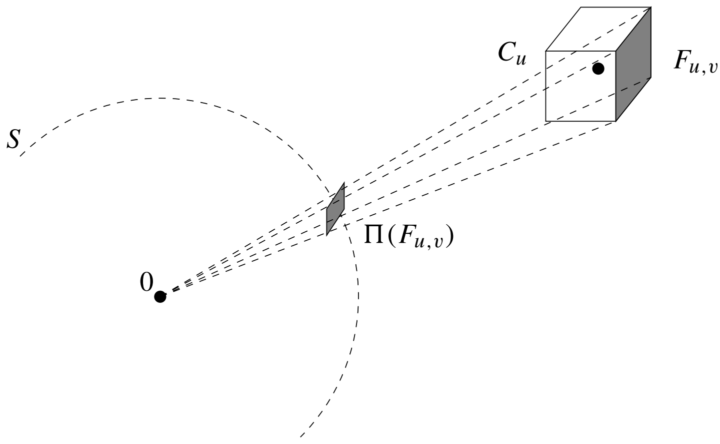

Figure 1.4 The flow along the edge $\langle u, v\rangle$ is equal to the area of the projection $\Pi(F_{u,v})$ on the unit sphere centred at the origin, with a suitable convention for its sign.

the second, which has its inspiration in [83]. By Corollary 1.31, it suffices to construct a non-zero $0/\infty$ flow with finite energy. Let S be the surface of the unit sphere of $\mathbb{R}^3$ with centre at the origin 0. Take $u \in \mathbb{Z}^3$, $u \neq 0$, and position a unit cube C_u in $\mathbb{R}^3$ with centre at u and edges parallel to the axes (see Figure 1.4). For each neighbour v of u, the directed edge $[u, v\rangle$ intersects a unique face, denoted $F_{u,v}$, of C_u.

For $x \in \mathbb{R}^3$, $x \neq 0$, let $\Pi(x)$ be the point of intersection with S of the straight line segment from 0 to x. Let $j_{u,v}$ be equal in absolute value to the surface measure of $\Pi(F_{u,v})$. The sign of $j_{u,v}$ is taken to be positive if and only if the scalar product of $\frac{1}{2}(u+v)$ and $v-u$, viewed as vectors in $\mathbb{R}^3$, is positive. Let $j_{v,u} = -j_{u,v}$. We claim that j is a $0/\infty$ flow on $\mathbb{L}^3$. Parts (a) and (b) of Definition 1.14 follow by construction, and it remains to check (c).

The surface of C_u has projection $\Pi(C_u)$ on S. The sum $J_u = \sum_{v\sim u} j_{u,v}$ is the integral over $\mathbf{x} \in \Pi(C_u)$, with respect to surface measure, of the number of neighbours v of u (counted with sign) for which $\mathbf{x} \in \Pi(F_{u,v})$. Almost every $\mathbf{x} \in \Pi(C_u)$ is counted twice, with signs $+$ and $-$. Thus the integral equals 0, whence $J_u = 0$ for all $u \neq 0$.

It is easily seen that $J_0 \neq 0$, so j is a non-zero flow. Next, we estimate its energy. By an elementary geometric consideration, there exist $c_i < \infty$

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such that:

(i) $|j_{u,v}| \leq c_1/|u|^2$ for $u \neq 0$, where $|u| = \delta(0, u)$ is the length of a shortest path from 0 to u,

(ii) the number of $u \in \mathbb{Z}^3$ with $|u| = n$ is smaller than $c_2 n^2$.

It follows that

$$E(j) \leq \sum_{u \neq 0} \sum_{v \sim u} j_{u,v}^2 \leq \sum_{n=1}^{\infty} 6c_2 n^2 \left(\frac{c_1}{n^2}\right)^2 < \infty,$$

as required. □

Another way of showing $R_{\text{eff}} < \infty$ when $d = 3$ is to find a finite upper bound for R_{eff}. Upper bounds can be obtained either by increasing individual edge-resistances or by removing edges. The idea is to embed a tree with finite resistance in $\mathbb{L}^3$. Consider a binary tree T_ρ in which each connection between generation $n-1$ and generation n has resistance ρ^n, where $\rho > 0$. It is an easy exercise using the series/parallel laws that the effective resistance between the root and infinity is

$$R_{\text{eff}}(T_\rho) = \sum_{n=1}^{\infty} \left(\frac{\rho}{2}\right)^n,$$

which we make finite by choosing $\rho < 2$. We proceed to embed T_ρ in $\mathbb{Z}^3$ in such a way that a connection between generation $n - 1$ and generation n is a lattice-path with length of order ρ^n. There are 2^n vertices of T_ρ in generation n, and their lattice-distance from 0 is of order $\sum_{k=1}^{n} \rho^k$, that is, order ρ^n. The surface of the k-ball in $\mathbb{R}^3$ is of order k^2, and thus it is necessary that

$$c(\rho^n)^2 \geq 2^n,$$

which is to say that $\rho > \sqrt{2}$.

Let $\sqrt{2} < \rho < 2$. It is now fairly simple to check that $R_{\text{eff}} < c' R_{\text{eff}}(T_\rho)$. This method was used in [138] to prove the transience of the infinite open cluster of percolation on $\mathbb{L}^3$. It is related to, but different from, the tree embeddings of [83].

1.6 Graph Theory

A graph $G = (V, E)$ comprises a finite or countably infinite vertex-set V and an associated edge-set E. Each element of E is an unordered pair u, v of vertices, written $\langle u, v\rangle$. Two edges with the same vertex-pairs are said to be in *parallel*, and edges of the form $\langle u, u\rangle$ are called *loops*. The graphs discussed in this text will generally contain neither parallel edges nor loops,

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and this is assumed henceforth. Two vertices u, v are said to be joined (or connected) by an edge if $\langle u, v\rangle \in E$. In this case, u and v are the *endvertices* of e, and we write $u \sim v$ and say that u is *adjacent* to v. An edge e is said to be *incident* to its endvertices. The number of edges incident to vertex u is called the *degree* of u, denoted $\deg(u)$. The negation of the relation $\sim$ is written $\nsim$.

Since the edges are unordered pairs, we call such a graph *undirected* (or *unoriented*). If some or all of its edges are *ordered* pairs, written $[u, v\rangle$, the graph is called *directed* (or *oriented*).

A *path* of G is defined as an alternating sequence $v_0, e_0, v_1, e_1, \dots, e_{n-1}, v_n$ of distinct vertices v_i and edges $e_i = \langle v_i, v_{i+1}\rangle$. Such a path has *length* n; it is said to connect v_0 to v_n, and is called a v_0/v_n path. A *cycle* or *circuit* of G is an alternating sequence $v_0, e_0, v_1, \dots, e_{n-1}, v_n, e_n, v_0$ of vertices and edges such that $v_0, e_0, \dots, e_{n-1}, v_n$ is a path and $e_n = \langle v_n, v_0\rangle$. Such a cycle has length $n + 1$. The (graph-theoretic) distance $\delta(u, v)$ from u to v is defined to be the number of edges in a shortest path of G from u to v.

We write $u \leftrightsquigarrow v$ if there exists a path connecting u and v. The relation $\leftrightsquigarrow$ is an equivalence relation, and its equivalence classes are called *components* (or *clusters*) of G. The components of G may be considered either as sets of vertices or as graphs. The graph G is *connected* if it has a unique component. It is a *forest* if it contains no cycle, and a *tree* if in addition it is connected.

A *subgraph* of the graph $G = (V, E)$ is a graph $H = (W, F)$ with $W \subseteq V$ and $F \subseteq E$. The subgraph H is a *spanning tree* of G if $V = W$ and H is a tree. A subset $U \subseteq V$ of the vertex-set of G has *boundary* $\partial U = \{u \in U : u \sim v \text{ for some } v \in V \setminus U\}$.

Lattice-graphs are the most important type of graph for applications in areas such as statistical mechanics. Lattices are sometimes termed 'crystalline' since they are periodic structures of crystal-like units. A general definition of a lattice may confuse readers more than help them, and instead we describe some principal examples.

Let d be a positive integer. We write $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ for the set of all integers, and $\mathbb{Z}^d$ for the set of all d-vectors $v = (v_1, v_2, \dots, v_d)$ with integral coordinates. For $v \in \mathbb{Z}^d$, we generally write v_i for the ith coordinate of v, and we define

$$\delta(u, v) = \sum_{i=1}^{d} |u_i - v_i|.$$

The *origin* of $\mathbb{Z}^d$ is denoted by 0. We turn $\mathbb{Z}^d$ into a graph, called the *d-dimensional (hyper)cubic lattice*, by adding edges between all pairs u, v of points of $\mathbb{Z}^d$ with $\delta(u, v) = 1$. This graph is denoted as $\mathbb{L}^d$, and its edge-set

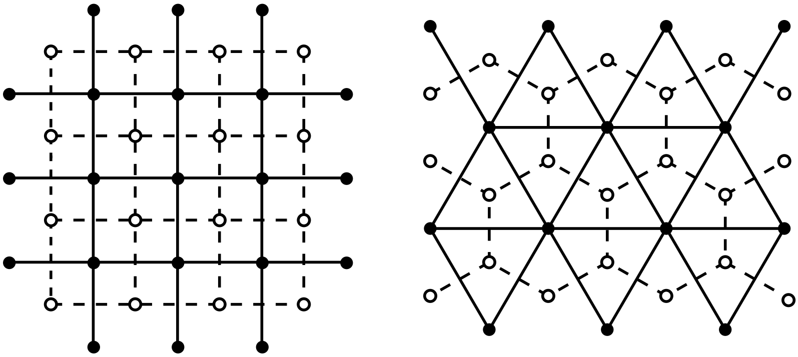

Figure 1.5 The square, triangular, and hexagonal (or 'honeycomb') lattices. The solid and dashed lines illustrate the concept of 'planar duality' discussed after (3.7).

as $\mathbb{E}^d$: thus, $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$. We often think of $\mathbb{L}^d$ as a graph embedded in $\mathbb{R}^d$, the edges being straight line-segments between their endvertices. The *edge-set* $\mathbb{E}_V$ of $V \subseteq \mathbb{Z}^d$ is the set of all edges of $\mathbb{L}^d$ both of whose endvertices lie in V.

The two-dimensional cubic lattice $\mathbb{L}^2$ is called the *square lattice* and is illustrated in Figure 1.5. Two other lattices in two dimensions that feature in this text are drawn there also.

1.7 Exercises

1.1 Let $G = (V, E)$ be a finite connected graph with unit edge-weights. Show that the effective resistance between two distinct vertices s, t of the associated electrical network may be expressed as B/N, where B is the number of s/t bushes of G, and N is the number of its spanning trees. (See the proof of Theorem 1.16 for an explanation of the term 'bush'.)

Extend this result to general edge-weights $w_e > 0$.

1.2 Let $G = (V, E)$ be a finite connected graph with strictly positive edge-weights $(w_e : e \in E)$, and let N^* be given by (1.18). Show that

$$i_{a,b} = \frac{1}{N^*}\big[N^*(s, a, b, t) - N^*(s, b, a, t)\big]$$

constitutes a unit flow through G from s to t satisfying Kirchhoff's laws.

1.3 (continuation) Let $G = (V, E)$ be finite and connected with given conductances $(w_e : e \in E)$, and let $(x_v : v \in V)$ be reals satisfying $\sum_v x_v = 0$. To G we append a notional vertex labelled ∞, and we join ∞ to each $v \in V$. Show that there exists a solution i to Kirchhoff's laws on the expanded graph, viewed as two laws concerning current flow, such that the current along the edge $\langle v, \infty\rangle$ is x_v.

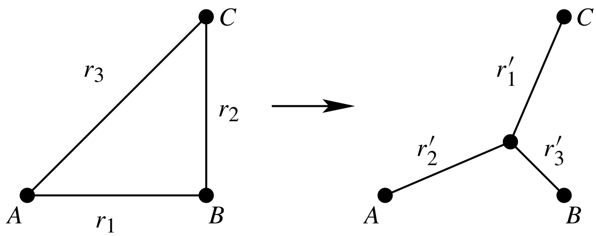

Figure 1.6 Edge-resistances in the star–triangle transformation. The triangle T on the left is replaced by the star S on the right, and the corresponding resistances are denoted as marked.

1.4 Prove the series and parallel laws for electrical networks.

1.5 *Star–triangle transformation.* The triangle T is replaced by a star S in an electrical network, as illustrated in Figure 1.6. Explain the sense in which the two networks are the same when the resistances are chosen such that $r_j r'_j = c$ for $j = 1, 2, 3$ and some $c = c(r_1, r_2, r_3)$ to be determined.

Note. The star–triangle transformation and its derivatives find many important applications in probability theory and mathematical physics. The transformation was discovered first in 1899, in the above form, by Kennelly [185].

1.6 Let $R(r)$ be the effective resistance between two given vertices of a finite network with edge-resistances $r = (r(e) : e \in E)$. Show that R is concave, in that

$$\tfrac{1}{2}\big[R(r_1) + R(r_2)\big] \le R\big(\tfrac{1}{2}(r_1 + r_2)\big).$$

1.7 *Maximum principle.* Let $G = (V, E)$ be a finite or infinite network with finite vertex-degrees and associated conductances $(w_e : e \in E)$. Let $H = (W, F)$ be a connected subgraph of G, and write

$$\Delta W = \{v \in V \setminus W : v \sim w \text{ for some } w \in W\}$$

for the 'external boundary' of W. Let $\phi : V \to [0, \infty)$ be harmonic on the set W, and suppose the supremum of ϕ on W is achieved and satisfies

$$\sup_{w \in W} \phi(w) = \|\phi\|_\infty := \sup_{v \in V} \phi(v).$$

Show that ϕ is constant on $W \cup \Delta W$, where it takes the value $\|\phi\|_\infty$.

1.8 Let G be an infinite connected graph, and let $\partial\Lambda_n$ be the set of vertices at distance n from the vertex labelled 0. With E_n the number of edges joining $\partial\Lambda_n$ to $\partial\Lambda_{n+1}$, show that a random walk on G is recurrent if $\sum_n E_n^{-1} = \infty$.

1.9 (continuation) Assume that G is 'spherically symmetric' in that: for all n, for all $x, y \in \partial\Lambda_n$, there exists a graph automorphism that fixes 0 and maps x to y. Show that a random walk on G is transient if $\sum_n E_n^{-1} < \infty$.

1.10 Let G be a countably infinite connected graph with finite vertex-degrees and with a nominated vertex 0. Let H be a connected subgraph of G containing

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0. Show that a simple random walk, starting at 0, is recurrent on H whenever it is recurrent on G, but that the converse need not hold.

1.11 Let G be a finite connected network with strictly positive conductances $(w_e : e \in E)$, and let a, b be distinct vertices. Let $i_{x,y}$ denote the current along an edge from x to y when a unit current flows from the source vertex a to the sink vertex b. Run the associated Markov chain, starting at a, until it reaches b for the first time, and let $u_{x,y}$ be the mean of the total number of transitions of the chain between x and y. Transitions from x to y count as positive, and from y to x as negative, so that $u_{x,y}$ is the mean number of transitions from x to y, minus the mean number from y to x. Show that $i_{x,y} = u_{x,y}$.

1.12 [83] Let G be an infinite connected graph with bounded vertex-degrees. Let $k \geq 1$, and let G_k be obtained from G by adding an edge between any pair of vertices that are non-adjacent (in G) but separated by a graph-theoretic distance k or less. (The graph G_k is sometimes called the *k-fuzz* of G.) Show that a simple random walk is recurrent on G_k if and only if it is recurrent on G.

1.13 *Uniqueness theorem.* Let $G = (V, E)$ be a finite or infinite connected network with finite vertex-degrees, and let W be a proper subset of V. Let $f, g : V \to \mathbb{R}$ be harmonic on W and equal on $V \setminus W$. Show, by the maximum principle or otherwise, that $f \equiv g$.