J. Inst. Math. Jussieu (2023), **22**(6), 2619–2644
doi:10.1017/S1474748022000081

REDUCTIONS OF 2-DIMENSIONAL SEMISTABLE REPRESENTATIONS WITH LARGE $\mathcal{L}$-INVARIANT

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(Received 15 December 2020; revised 31 January 2022; accepted 2 February 2022; first published online 1 April 2022)

Abstract We determine reductions of 2-dimensional, irreducible, semistable, and non-crystalline representations of $\mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right)$ with Hodge–Tate weights $0 < k-1$ and with $\mathcal{L}$-invariant whose p-adic norm is sufficiently large, depending on k. Our main result provides the first systematic examples of the reductions for$k \geq p$.

Key words and phrases: p-adic Hodge theory; semistable representations; local Galois representations modulo p

2020 Mathematics Subject Classification: Primary 11F80
Secondary 11F85

Contents

1. Introduction

Let p be a prime number and $\overline{\mathbb{Q}}_p$ be an algebraic closure of the p-adic numbers $\mathbb{Q}_p$. The goal of this article is to determine the reductions of certain 2-dimensional p-adic representations of $G_{\mathbb{Q}_p} = \mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right)$ that are semistable and not crystalline in the sense of Fontaine [15]. Examples of such representations arise from local p-adic representations associated with eigenforms with $\Gamma_0(p)$-level.

1.1. Main result

Write v_p for the p-adic valuation on $\overline{\mathbb{Q}}_p$, normalized so that $v_p(p) = 1$. Choose $\varpi \in \overline{\mathbb{Q}}_p$ such that $\varpi^2 = p$. Then for each integer $k \geq 2$ and each $\mathcal{L} \in \overline{\mathbb{Q}}_p$, there is a 2-dimensional filtered (φ, N)-module $D_{k,\mathcal{L}} = \overline{\mathbb{Q}}_p e_1 \oplus \overline{\mathbb{Q}}_p e_2$ where, in the basis (e_1, e_2), we have

$$\varphi = \begin{pmatrix} \varpi^k & 0 \\ 0 & \varpi^{k-2} \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{Fil}^i D_{k,\mathcal{L}} = \begin{cases} D_{k,\mathcal{L}} & \text{if } i \leq 0, \\ \overline{\mathbb{Q}}_p \cdot (e_1 + \mathcal{L} e_2) & \text{if } 1 \leq i \leq k-1, \\ \{0\} & \text{if } k \leq i. \end{cases} \tag{1.1}$$

Each $D_{k,\mathcal{L}}$ is weakly admissible, so a theorem of Colmez and Fontaine implies there is a unique 2-dimensional $\overline{\mathbb{Q}}_p$-linear representation $V_{k,\mathcal{L}}$ of $G_{\mathbb{Q}_p}$ such that $D_{k,\mathcal{L}} = D_{\mathrm{st}}^*(V_{k,\mathcal{L}})$. Up to a twist by a crystalline character, the representations $V_{k,\mathcal{L}}$ enumerate all $\overline{\mathbb{Q}}_p$-linear 2-dimensional semistable and non-crystalline representations of $G_{\mathbb{Q}_p}$. They are irreducible except if $k = 2$.

We aim to determine the semisimple mod p reductions $\overline{V}_{k,\mathcal{L}}$ of $V_{k,\mathcal{L}}$. Twenty years ago, Breuil and Mézard determined $\overline{V}_{k,\mathcal{L}}$ for even $k < p$ and any $\mathcal{L}$ [7, Théorème 4.2.4.7]. Guerberoff and Park recently studied odd $k < p$ [17, Theorem 5.0.5]. The reader who takes a moment to examine the cited theorems should be left with an impression of the complicated dependence of $\overline{V}_{k,\mathcal{L}}$ on $\mathcal{L}$, and that is just for $k < p$.

Prior results are limited by their ambition to determine $\overline{V}_{k,\mathcal{L}}$ for all $\mathcal{L}$. Here, we focus on determining $\overline{V}_{k,\mathcal{L}}$ for any k while restricting to $\mathcal{L}$ that place $V_{k,\mathcal{L}}$ in a p-adic neighborhood of a crystalline representation (see §1.2). Write $\mathbb{Q}_{p^2}$ for the unramified quadratic extension of $\mathbb{Q}_p$, χ for its quadratic character modulo p, and ω_2 for a niveau 2 fundamental character on $G_{\mathbb{Q}_{p^2}}$.

Theorem 1.1 (Theorem 4.1). *Assume $k \geq 4$ and $p \neq 2$. Then, if*

$$v_p(\mathcal{L}) < 2 - \frac{k}{2} - v_p((k-2)!),$$

we have $\overline{V}_{k,\mathcal{L}} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \left(\omega_2^{k-1}\chi\right)$.

To be accurate, our method proves Theorem 1.1 when $k \geq 5$ or $p = 3$ and $k = 4$. The theorem holds for $k = 4$ and $p \geq 5$ by the work of Breuil and Mézard, and it is consistent with their work and the work of Guerberoff and Park for $5 \leq k < p$. Our method also directly obtains a result for $k = 3$ and $k = 4$ with a weaker bound (see Remark 4.8 for a more detailed discussion). Our exclusion of $p = 2$ is more fundamental (see Remark 1.4).

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Remark 1.2. When $k < p$ and k is even, the bound in Theorem 1.1 is optimal by the results of Breuil and Mézard. The same can be said if $5 \le k < p$ and k is odd, by the work of Guerberoff and Park. We do not know to what extent the bound is optimal for higher weights (see §1.3).

Theorem 1.1 is a natural analogue of widely studied theorems that determine reductions of 2-dimensional, irreducible, crystalline representations of $G_{\mathbb{Q}_p}$. For instance, Buzzard and Gee [9] developed a strategy to determine reductions of certain crystalline representations, with unbounded Hodge–Tate weights, using the p-adic local Langlands correspondence. We do not know whether a direct analogue for semistable but non-crystalline representations has been tried, or even whether such an approach would be feasible.

Another approach in the crystalline case is via integral p-adic Hodge theory. Berger [4] and Berger, Li, and Zhu [5] proved local constancy results for reductions of crystalline representations using Wach modules. Recently, the first two authors of this article improved the Berger–Li–Zhu result using Kisin modules [3]. Those are what we will use here also. One incentive to write the previous article was as training to conduct the current research.

Finally, an indirect approach to calculating $\overline{V}_{k,\mathcal{L}}$ is explained in a recent preprint by Chitrao, Ghate, and Yasuda [10], though their investigation heads in a interesting separate direction from ours.

1.2. Overview of strategy

We now describe our strategy, first recontextualizing Theorem 1.1 through the lens of local constancy of reductions as in [3, 4, 5].

The parametrization of semistable and non-crystalline representations by $\mathcal{L} \in \overline{\mathbb{Q}}_p$ extends to a $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$-parametrization with a crystalline representation at ∞. Namely, for $\mathcal{L} \neq 0$ we consider $D_{k,\mathcal{L}}$ with basis $(e_1', e_2') = (e_1, \mathcal{L}e_2)$ – in which case, rather than equation (1.1), we have

$$\varphi = \begin{pmatrix} \varpi^k & 0 \\ 0 & \varpi^{k-2} \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ \mathcal{L}^{-1} & 0 \end{pmatrix}, \quad \mathrm{Fil}^i D_{k,\mathcal{L}} = \begin{cases} D_{k,\mathcal{L}} & \text{if } i \le 0, \\ \overline{\mathbb{Q}}_p \cdot (e_1' + e_2') & \text{if } 1 \le i \le k-1, \\ \{0\} & \text{if } k \le i. \end{cases} \tag{1.2}$$

Thus, $D_{k,\mathcal{L}} \to D_{k,\infty}$ as $\mathcal{L}^{-1} \to 0$, where $D_{k,\infty}$ is the filtered (φ, N)-module with the same φ and filtration as equation (1.2) but with $N = 0$. In fact, $D_{k,\infty} \cong D^*_{\mathrm{crys}}(V_{k,\infty})$, where $V_{k,\infty}$ is a 2-dimensional *crystalline* representation of $G_{\mathbb{Q}_p}$ whose Frobenius trace is $a_p = \varpi^{k-2} + \varpi^k$. Replacing the filtered (φ, N)-modules with Galois representations, we have $V_{k,\mathcal{L}} \to V_{k,\infty}$ as $\mathcal{L}^{-1} \to 0$ (see the description in [12, §§4.5, 4.6] in terms of the space of trianguline representations, for instance). Thus, $\overline{V}_{k,\mathcal{L}} \cong \overline{V}_{k,\infty}$ for $\mathcal{L}^{-1} \to 0$. Furthermore, $v_p(a_p) = \frac{k-2}{2}$ and so $\left\lfloor \frac{k-1}{p} \right\rfloor < v_p(a_p)$, except if $p = 2$ or k is small, and

https://doi.org/10.1017/S1474748022000081 Published online by Cambridge University Press

so $\overline{V}_{k,\infty} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}\left(\omega_2^{k-1}\chi\right)$ by [3, Corollary 5.2.3]. We have reduced the theorem to the question: At which point as $\mathcal{L}^{-1} \to 0$ do we have $\overline{V}_{k,\mathcal{L}} \cong \overline{V}_{k,\infty}$?

We now recall the relationship between reductions and Kisin modules. To ease notation, assume for the remainder of this subsection that k is even and $\mathcal{L} \in \mathbb{Q}_p$, so $V_{k,\mathcal{L}}$ and $V_{k,\infty}$ are defined over $\mathbb{Q}_p$. Let $\mathfrak{S} = \mathbb{Z}_p[\![u]\!]$, and write $\varphi : \mathfrak{S} \to \mathfrak{S}$ for the Frobenius map $\varphi(u) = u^p$. Then consider the category $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\leq k-1}$ of φ-modules over $\mathfrak{S}$ with height $\leq k-1$ [18]. Objects in this category, which are called Kisin modules, are finite free $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a φ-semilinear operator $\varphi : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of the linearization $\varphi^*\mathfrak{M} \to \mathfrak{M}$ is annihilated by $E(u)^{k-1}$, where $E(u) = u+p$. When $\mathfrak{M}$ satisfies the *monodromy condition*, Kisin's theory constructs a canonical semistable representation $V_{\mathfrak{M}}$ such that $D_{\mathrm{st}}^*(V_{\mathfrak{M}}) \cong \mathfrak{M}/u\mathfrak{M}[1/p]$, for a certain filtration and monodromy on the right-hand side. Furthermore, $\overline{V}_{\mathfrak{M}}$ is determined by $\mathfrak{M}/p\mathfrak{M}\left[u^{-1}\right]$ as a φ-module over $\mathbb{F}_p((u))$. The challenge in calculating $\overline{V}_{\mathfrak{M}}$ this way is determining $\mathfrak{M}$ from $V_{\mathfrak{M}}$ or, equivalently, $D_{\mathrm{st}}^*(V_{\mathfrak{M}})$. That task was carried out for $V_{k,\infty}$ in [3, Theorem 5.2.1].

The heart of this article is a two-step argument to do the same for $V_{k,\mathcal{L}}$ as $\mathcal{L}^{-1} \to 0$. The difficulty presented by nontrivial monodromy on $D_{k,\mathcal{L}}$ requires us to develop a new technique to pass from filtered (φ,N)-modules to Kisin modules. We make use of a category intermediate between filtered (φ,N)-modules and Kisin modules. Namely, write $\mathrm{Mod}_{S_{\mathbb{Q}_p}}^{\varphi,\leq k-1}$ for the category of φ-modules over $S_{\mathbb{Q}_p} = \mathbb{Z}_p\left[\!\left[u, \frac{E^p}{p}\right]\!\right]\left[\frac{1}{p}\right]$ with height $\leq k-1$. This category is close to certain filtered (φ,N)-modules considered by Breuil [6]. Adapting Breuil's work, we explicitly construct a canonical object $\mathcal{M}_{k,\mathcal{L}} \in \mathrm{Mod}_{S_{\mathbb{Q}_p}}^{\varphi,\leq k-1}$ such that if $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,\leq k-1}$ and $\mathcal{M}_{k,\mathcal{L}} \cong \mathfrak{M} \otimes_{\mathfrak{S}} S_{\mathbb{Q}_p}$, then $V_{\mathfrak{M}} \cong V_{k,\mathcal{L}}$. 'Explicit' means that for any (nonzero) $\mathcal{L}$, we determine a basis of $\mathcal{M}_{k,\mathcal{L}}$ and an exact formula for φ in that basis. This is where we overcome the difficulty of nontrivial monodromy on $D_{k,\mathcal{L}}$.

The second step is to descend $\mathcal{M}_{k,\mathcal{L}}$ from $S_{\mathbb{Q}_p}$ to $\mathfrak{S}$ when $\mathcal{L}^{-1} \to 0$, thus producing an $\mathfrak{M}$ for $V_{k,\mathcal{L}}$. Here we view $S_{\mathbb{Q}_p}$ as a subring of R_2, where R_2 is the ring of p-adic rigid analytic functions on $|u| \leq p^{-1/2}$ (using $p \neq 2$). In [3, §4], a row-reduction algorithm is presented for semilinear operators that, under certain conditions, can descend from R_2 to $\mathfrak{S}$. Specifically, the main theorem there gives a sufficient condition to descend $\mathcal{M}_{k,\mathcal{L}} \otimes_{S_{\mathbb{Q}_p}} R_2$ to $\mathfrak{S}$. Saving the details for later, we use the explicit calculation of $\mathcal{M}_{k,\mathcal{L}}$ to check that those conditions are met when $v_p(\mathcal{L}) < 2 - \frac{k}{2} + v_p((k-2)!)$.

Remark 1.3. As already discussed, our approach in the first step is more general than [3], as it applies in the semistable, non-crystalline case. In fact, the method is quite general and can be used (with a suitable descent process) to compute reductions for higher-dimensional semistable representations. For example, the third author has used the strategy here to compute reductions of irreducible 3-dimensional crystalline representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $\{0,r,s\}$ satisfying $2 \leq r \leq p-2$ and $p+2 \leq s \leq r+p-2$ [22].

Remark 1.4. We exclude $p=2$ twice. The second time, when we embed $S_{\mathbb{Q}_p}$ into R_2, is likely technical. However, we also exclude $p=2$ when referencing the calculation of $\overline{V}_{k,\infty}$ in [3], and that seems crucial: our strategy is based on knowing not just $\overline{V}_{k,\infty}$ but also how to construct a Kisin module for $V_{k,\infty}$. Including $p=2$ here would necessarily require calculating $\overline{V}_{k,\infty}$ when $p=2$ as well. We note that the formula $\overline{V}_{k,\infty} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \left(\omega_2^{k-1}\chi\right)$ should still be true, but we cannot justify it.

1.3. Global context

We end this introduction with a discussion of the global situation. Suppose $N \geq 1$ and $f = \sum a_n(f)q^n$ is a cuspidal (normalized) eigenform of level $\Gamma_1(N)$, weight $k \geq 2$, and nebentype character ψ_f. Eichler, Shimura, and Deligne famously associated with f a 2-dimensional, irreducible, continuous representation V_f of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$. We normalize V_f so that for $\ell \nmid Np$ the restriction $V_f|_{D_\ell}$ to D_ℓ, a decomposition group at ℓ, is unramified and the characteristic polynomial of a geometric Frobenius element is $t^2 - a_\ell(f)t + \psi_f(\ell)\ell^{k-1}$. The representation $V_f|_{D_p}$ is semistable when $p^2 \nmid N$ and the conductor of ψ_f is prime-to-p; it is crystalline when $p \nmid N$ [24].

We assume now that $V_f|_{D_p}$ is semistable and non-crystalline, in which case we define the $\mathcal{L}$-invariant of f to be the unique $\mathcal{L}_f \in \overline{\mathbb{Q}}_p$ such that $V_f|_{D_p} \cong V_{k,\mathcal{L}_f}$. The $\mathcal{L}$-invariant defined this way is called the Fontaine–Mazur $\mathcal{L}$-invariant (it agrees with [23, §12] up to a sign). It is a local quantity, but it famously arises in global situations. Examining how it arises allows us to provide global examples where Theorem 1.1 applies and to connect $\mathcal{L}$-invariants to global phenomena on p-adic families.

Theorem 1.1 determines $\left(\overline{V_f}|_{D_p}\right)^{\mathrm{ss}}$ in arbitrary weights $k \geq p$ as long as $v_p(\mathcal{L}_f)$ is sufficiently negative, but it is not immediately obvious that eigenforms exist with $v_p(\mathcal{L}_f)$ so negative. Recent research, however, sheds light on the situation. For instance, Gräf [16] and Anni, Böckle, Gräf, and Troya (see [1], which builds on [11]) have developed the theory and practice needed to calculate the multiset of valuations of $\mathcal{L}$-invariants in a fixed weight and level. Pollack has also developed computer code to calculate $\mathcal{L}$-invariants. His method, which dates to the early 2000s, uses the appearance of $\mathcal{L}$-invariants in exceptional zero phenomena for p-adic L-functions. That method is being written up as part of a joint investigation by Pollack and the first author.

Using their works, both Pollack and Gräf kindly calculated some $\mathcal{L}$-invariants for us. In Table 1, we partially list the p-adic valuations found when $p=3$ and $N = 51 = 3 \cdot 17$. Note that the bound in Theorem 1.1 is $v_3(\mathcal{L}_f) < 0$ in weight $k=4$ and $v_3(\mathcal{L}_f) < -2$ in weight $k=6$, so Table 1 provides two examples of Theorem 1.1 in weight $k=4$ and one example in weight $k=6$, though none in weight $k=8$.

Let us look further at $p=3$ and $k=6$ and the boundary case $v_3(\mathcal{L}) = -2$ in Theorem 1.1. Pollack's code, in fact, reports not just $v_3(\mathcal{L}_f)$ for each newform f but also $\overline{V}_f$. This refined data shows that the eigenforms with weight $k=6$ and $v_3(\mathcal{L}_f)$ equal to -3 and

TABLE 1. 3-adic valuations of some $\mathcal{L}$-invariants.

k	$v_3(\mathcal{L}_f)$ for newforms $f \in S_k(\Gamma_0(51))$
4	$-2, -1, 0, 0, \ldots$
6	$-3, -2, -1, -1, -1, \ldots$
8	$-3, -3, -\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}, -1, \ldots$

https://doi.org/10.1017/S1474748022000081 Published online by Cambridge University Press

-2 have isomorphic global Galois representations modulo 3. Since Theorem 1.1 applies to $v_3(\mathcal{L}) = -3$, we see that there exist $\mathcal{L}$-invariants with $v_3(\mathcal{L}) = -2$ for which the conclusion of Theorem 1.1 continues to hold. More numerical data is required before we can theorize about the sharpness of the bound in Theorem 1.1.

The $\mathcal{L}$-invariants also arise, globally, from p-adic families. Namely, f lives in a p-adic family of eigenforms parametrized by weights $k \in \mathbb{Z}_p$ and $\mathcal{L}_f = -2\,\mathrm{dlog}\,a_p(k) = -2\frac{a_p'(k)}{a_p(f)}$ [13, Corollaire 0.7]. This appearance reveals an obstruction to the 'radius' of the largest 'constant slope' family through f. Indeed, for $p \neq 2$, [2, Theorem 4.3] implies $v_p\left(\mathcal{L}_f^{-1}\right) \leq m(f)$, where $m(f)$ is the least positive integer such that f lives in a p-adic family of eigenforms f' with $v_p(a_p(f')) = v_p(a_p(f))$ and weight $k' \equiv k \bmod (p-1)p^{m(f)}$.

So, ruling out exceptions to Theorem 1.1, $v_p(\mathcal{L}_f) < 2 - \frac{k}{2} - v_p((k-2)!)$ implies

- $\left(\overline{V}_f|_{D_p}\right)^{\mathrm{ss}} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}\left(\omega_2^{k-1}\chi\right)$ and
- $m(f) > \frac{k}{2} - 2 + v_p((k-2)!) \approx \frac{k-2}{2} + \frac{k}{p-1}$.

To connect these, if $k \not\equiv 1 \bmod p+1$, then $\overline{V}_f|_{D_p}$ is irreducible. On the other hand, the second implication generically implies $m(f) > \frac{k-2}{2} = v_p(a_p(f))$. The fact that $m(f) > v_p(a_p(f))$ occurs in a situation where $\overline{V}_f|_{D_p}$ is irreducible is not a coincidence. It follows a pattern of counter examples found by Buzzard and Calegari, to a conjecture of Gouvêa and Mazur, which is related to the $m(f)$. The counter-examples were found by Buzzard and Calegari [8]. See [2, §9] for more discussion.

2. Theoretical background

In this section, we recall filtered (φ,N)-modules and Breuil and Kisin modules. We explain, in theory, how to calculate a finite-height φ-module, over a ring larger than $\mathfrak{S}$, associated with a filtered (φ,N)-module (Theorem 2.7). In §3 we carry this out in practice in a special case.

2.1. Notations

Let k be a finite field and $W(k)$ be the Witt vectors over k. Set $K_0 = W(k)[1/p]$ and assume K/K_0 is a totally ramified extension of degree e. Let Λ_K be the ring of integers of K, $\pi \in \Lambda_K$ a uniformizer, and $E = E(u) \in W(k)[u]$ its Eisenstein polynomial. Choosing $\pi_0 = \pi$ and $\pi_1, \pi_2, \ldots$ a sequence in $\overline{K}$ such that $\pi_{i+1}^p = \pi_i$, we let G_∞ be the absolute Galois group of $\varinjlim K(\pi_i)$. Let $\mathcal{O} \subseteq K_0[\![u]\!]$ be the rigid analytic functions on $|u| < 1$ and $\mathfrak{S} = W(k)[\![u]\!] \subseteq \mathcal{O}$. The action of φ on $K_0[\![u]\!]$, by the Frobenius on K_0 and $\varphi(u) = u^p$, preserves $\mathfrak{S} \subseteq \mathcal{O} \subseteq K_0[\![u]\!]$.

We also choose $F/\mathbb{Q}_p$ a finite extension, which will play the role of linear coefficients. In §2.4 we assume that F contains a subfield isomorphic the Galois closure of K. We write $\Lambda \subseteq F$ for the ring of integers, $\mathfrak{m}_F \subseteq \Lambda$ for the maximal ideal, and $\mathbb{F}$ for the residue field. Define $\mathfrak{S}_\Lambda = \mathfrak{S} \otimes_{\mathbb{Z}_p} \Lambda$ and $\mathcal{O}_F = \mathcal{O} \otimes_{K_0} F$. Extending φ linearly, we have φ-stable subrings of $\mathfrak{S}_\Lambda \subseteq S_F \subseteq \left(K_0 \otimes_{\mathbb{Q}_p} F\right)[\![u]\!]$, where $S_F = \mathfrak{S}\left[\!\left[\frac{E^p}{p}\right]\!\right] \otimes_{\mathbb{Q}_p} F$.

https://doi.org/10.1017/S1474748022000081 Published online by Cambridge University Press

2.2. Kisin modules

Let $R \subseteq \left(K_0 \otimes_{\mathbb{Q}_p} F\right)[\![u]\!]$ be a φ-stable subring containing E. A φ-module over R is a finite free R-module M equipped with an injective φ-semilinear operator $\varphi_M : M \to M$. Let Mod_R^{φ} be the category of φ-modules over R with morphisms being R-linear maps that commute with φ. For a φ-module M, write $\varphi^* M = R \otimes_{\varphi,R} M$, so $1 \otimes \varphi_M$ defines an R-linear map $\varphi^* M \to M$ called the linearization of φ. For $h \geq 0$, an element $M \in \mathrm{Mod}_R^{\varphi}$ has (E)-height $\leq h$ if its linearization has cokernel annihilated by E^h. The subcategory of φ-modules over R with height $\leq h$ is denoted $\mathrm{Mod}_R^{\varphi,\leq h}$. A *Kisin module* over $\mathfrak{S}_\Lambda$ with height $\leq h$ is an object in $\mathrm{Mod}_{\mathfrak{S}_\Lambda}^{\varphi,\leq h}$.

Let $\mathrm{MF}_F^{\varphi,N}$ be the category of *positive* filtered (φ,N,K,F)-modules, which we shorten to just filtered (φ,N)-modules over F (see [7, §3.1.1]). For $D \in \mathrm{MF}_F^{\varphi,N}$, set $D_K = K \otimes_{K_0} D$; here, 'positive' means $\mathrm{Fil}^0 D_K = D_K$. Let $\mathrm{Rep}_F^{\mathrm{st},h}$ be the category of F-linear semistable representations V of G_K whose Hodge–Tate weights lie in $\{0,\dots,h\}$. Then there exists a fully faithful, contravariant functor

$$D_{\mathrm{st}}^* : \mathrm{Rep}_F^{\mathrm{st},h} \to \mathrm{MF}_F^{\varphi,N}$$

whose image is the subcategory of weakly admissible filtered (φ,N)-modules over F (see [14, 15] and [7, Corollaire 3.1.1.3]). For $V \in \mathrm{Rep}_F^{\mathrm{st},h}$ and $T \subseteq V$ a G_∞-stable and Λ-linear lattice, there exists, by [20, Theorem 5.4.1], a canonical Kisin module $\mathfrak{M} = \mathfrak{M}(T)$ over $\mathfrak{S}_\Lambda$ with height $\leq h$. Naturally, we say a Kisin module $\mathfrak{M}$ is associated with V if $\mathfrak{M} = \mathfrak{M}(T)$ for some T. By [3, Corollary 2.3.2], the semisimple mod p representation $\overline{V}$ can be determined from any associated Kisin module.

One category that intervenes in determining an $\mathfrak{M}$ associated with $V \in \mathrm{Rep}_F^{\mathrm{st},h}$ is the category of (φ,N_∇)-modules over $\mathcal{O}_F$ [18]. Let $\lambda = \prod_{n\geq 0} \varphi^n(E(u)/E(0)) \in \mathcal{O}_F$. An object $\mathcal{M}_{\mathcal{O}_F} \in \mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N_\nabla}$ is a finite-height φ-module over $\mathcal{O}_F$ equipped with a differential operator N_∇ lying over $-u\lambda \frac{d}{du}$ on $\mathcal{O}_F$ and satisfying $N_\nabla \varphi = p\frac{E(u)}{E(0)}\varphi N_\nabla$. By [18, Theorem 1.2.15], we have quasi-inverse equivalences of categories

$$\mathrm{MF}_F^{\varphi,N} \underset{\underline{\mathcal{M}}_{\mathcal{O}_F}}{\overset{\underline{D}_{\mathcal{O}_F}}{\rightleftarrows}} \mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N_\nabla}. \tag{2.1}$$

For $s > 0$, write $\mathcal{O}_s$ for the $\mathcal{O}_F$-algebra of rigid analytic functions converging on $|u| < p^{-s}$.

Proposition 2.1. *Suppose $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_\Lambda}^{\varphi,\leq h}$, $V \in \mathrm{Rep}_F^{\mathrm{st},h}$, and s is such that $1/pe < s < 1/e$ and $\mathfrak{M} \otimes_{\mathfrak{S}_\Lambda} \mathcal{O}_s \cong \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{\mathrm{st}}^*(V)) \otimes_{\mathcal{O}_F} \mathcal{O}_s$ in $\mathrm{Mod}_{\mathcal{O}_s}^{\varphi,\leq h}$. Then, $\mathfrak{M} = \mathfrak{M}(T)$ for some $T \subseteq V$ as before.*

Proof. Since $s < 1/e$, π lies in the disc $|u| < p^{-s}$. Since $\mathfrak{M} \otimes_{\mathfrak{S}_\Lambda} \mathcal{O}_s \cong \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{\mathrm{st}}^*(V)) \otimes_{\mathcal{O}_F} \mathcal{O}_s$, [3, Corollary 2.2.5] implies that $\mathcal{M}_{\mathcal{O}_F} := \mathfrak{M} \otimes_{\mathfrak{S}_\Lambda} \mathcal{O}_F$ is canonically an object in $\mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N_\nabla}$. Then [20, Theorem 5.4.1] implies that there exists a $V' \in \mathrm{Rep}_F^{\mathrm{st},h}$ such that $\mathfrak{M} = \mathfrak{M}(T)$ for a lattice $T \subseteq V'$ for some T. We claim that $V \cong V'$. Indeed, since

$1/pe < s < 1/e$, the definition of $\underline{D}_{\mathcal{O}_F}(\mathcal{M}_{\mathcal{O}_F})$ in [18, §§1.2.5–1.2.7] depends on only the finite-height φ-module $\mathcal{M}_{\mathcal{O}_F} \otimes_{\mathcal{O}_F} \mathcal{O}_s$ over $\mathcal{O}_s$. Thus we have

$$D^*_{\mathrm{st}}(V') \cong \underline{D}_{\mathcal{O}_F}(\mathcal{M}_{\mathcal{O}_F}) \cong \underline{D}_{\mathcal{O}_F}\left(\underline{\mathcal{M}}_{\mathcal{O}_F}(D^*_{\mathrm{st}}(V))\right) \cong D^*_{\mathrm{st}}(V).$$

Since D^*_{st} is fully faithful, we have $V \cong V'$, completing the proof. □

Remark 2.2. To be accurate, the equivalence (2.1) is constructed in [18] only when $F = \mathbb{Q}_p$. We use multiple references with the same technical limitation. We pause to detail one approach to resolving the issue. Later, we omit details for other functors.

First, we may define the functors $\underline{D}_{\mathcal{O}_F}$ and $\underline{\mathcal{M}}_{\mathcal{O}_F}$ using the same formulas as (2.1), or equivalently, we can define them by forcing the diagram

$$\begin{array}{ccc} \mathrm{MF}_F^{\varphi,N} & \underset{\underline{\mathcal{M}}_{\mathcal{O}_F}}{\overset{\underline{D}_{\mathcal{O}_F}}{\rightleftarrows}} & \mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N\nabla} \\ {\scriptstyle \mathrm{forget}}\downarrow & & \downarrow{\scriptstyle \mathrm{forget}} \\ \mathrm{MF}_{\mathbb{Q}_p}^{\varphi,N} & \underset{\underline{\mathcal{M}}_{\mathcal{O}}}{\overset{\underline{D}_{\mathcal{O}}}{\rightleftarrows}} & \mathrm{Mod}_{\mathcal{O}}^{\varphi,N\nabla} \end{array}$$

to commute. Here, the vertical arrows are the natural forgetful functors and the bottom arrows are as in [18], where they are proved to be quasi-inverses. If $\mathcal{M}_{\mathcal{O}_F} \in \mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N\nabla}$, we thus have a natural isomorphism $\alpha : \underline{\mathcal{M}}_{\mathcal{O}_F}\left(\underline{D}_{\mathcal{O}_F}(\mathcal{M}_{\mathcal{O}_F})\right) \cong \mathcal{M}_{\mathcal{O}_F}$ in $\mathrm{Mod}_{\mathcal{O}}^{\varphi,N\nabla}$. Since multiplication by $x \in F$ defines an endomorphism of $\mathcal{M}_{\mathcal{O}_F}$ in $\mathrm{Mod}_{\mathcal{O}}^{\varphi,N\nabla}$ and α is natural, we see that α is an isomorphism in $\mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N\nabla}$. Thus, $\underline{\mathcal{M}}_{\mathcal{O}_F}$ is a left quasi-inverse to $\underline{D}_{\mathcal{O}_F}$. Proving that $\underline{D}_{\mathcal{O}_F}$ is a right quasi-inverse to $\underline{\mathcal{M}}_{\mathcal{O}_F}$ is analogous.

2.3. Breuil modules

Let S_{Br} be the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to the ideal generated by E. Breuil [6] classically identified $\mathrm{MF}_{\mathbb{Q}_p}^{\varphi,N}$ with a category of filtered (φ,N)-modules over $S_{\mathrm{Br}}\left[\frac{1}{p}\right]$. We recall this, replacing S_{Br} with a simpler ring.

One extends the Frobenius φ to $K_0[\![u]\!]$ via $\varphi(u) = u^p$. We define $N = -u\frac{d}{du}$ on $K_0[\![u]\!]$. Let $\widehat{S}_E$ be the E-completion of $W(k)[u]\left[\frac{1}{p}\right]$. For a subring $R \subseteq \widehat{S}_E$ and $j \geq 0$, set $\mathrm{Fil}^j R = R \cap E^j \widehat{S}_E$. In particular, we can take $R = S := W(k)\left[\!\left[u, \frac{E^p}{p}\right]\!\right]$. As a subring of $K_0[\![u]\!]$, S is closed under φ and N. We define $S_\Lambda = S \otimes_{\mathbb{Z}_p} \Lambda$ and $S_F = S \otimes_{\mathbb{Z}_p} F$, extending φ, N, and $\mathrm{Fil}^\bullet$ linearly.

Clearly $S \subseteq S_{\mathrm{Br}} \subseteq \widehat{S}_E$, which are compatible with the $\left(u, \frac{E^p}{p}\right)$-topology on S, the p-adic topology on $S_{\mathrm{B}r}$, and the (E)-topology on $\widehat{S}_E$. One advantage S enjoys over S_{Br} is that $\mathrm{Fil}^j S_F = E^j S_F$. To see this, note that any element $f \in \mathrm{Fil}^j S_F$ can be uniquely written in the form $f = \sum_i a_i(u)\frac{E^{pi}}{p^i}$, with $a_i(u) \in K_0[u]$ a polynomial of degree strictly less than ep (e is the degree of E). Then, when $j < pi$, we have $\frac{E^{pi-j}}{p^i} = \frac{1}{p^{i-l}} E^{pi-pl}\left(\frac{E^p}{p}\right)^l$, with $l = \left\lfloor \frac{pi-j}{p} \right\rfloor$. In this situation, $i - l$ depends only on j, so factoring E^j out of the expression

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for f and examining the leftover summation, one sees at once that $f \in E^j S_F$. Note as well: S_F is an $\mathcal{O}_F$-algebra, and $\varphi(E) = p\mathfrak{c}$ with $\mathfrak{c} \in S^\times$. In particular, $\varphi(\lambda) \in S^\times \subseteq S_F^\times$.

The category $\mathrm{MF}_{S_F}^{\varphi,N}$ of filtered (φ,N)-modules over S_F, or *Breuil modules* over S_F, are objects $(\mathcal{D},\varphi_{\mathcal{D}}) \in \mathrm{Mod}_{S_F}^{\varphi}$ such that the linearization of $\varphi_{\mathcal{D}}$ is an isomorphism, and $\mathcal{D}$ is equipped with the following:

- a decreasing filtration $\mathrm{Fil}^\bullet \mathcal{D}$ by S_F-submodules such that $\mathrm{Fil}^0 \mathcal{D} = \mathcal{D}$ and
$$\mathrm{Fil}^i S_F \cdot \mathrm{Fil}^j \mathcal{D} \subseteq \mathrm{Fil}^{i+j} \mathcal{D}$$
for all $i,j \geq 0$;
- an operator $N_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ that acts as a derivation over N, with
  - $N_{\mathcal{D}}\varphi_{\mathcal{D}} = p\varphi_{\mathcal{D}}N_{\mathcal{D}}$ and
  - $N_{\mathcal{D}}\left(\mathrm{Fil}^i \mathcal{D}\right) \subseteq \mathrm{Fil}^{i-1} \mathcal{D}$ for all $i \geq 1$.

A morphism in $\mathrm{MF}_{S_F}^{\varphi,N}$ is an S_F-linear map equivariant for φ, N, and $\mathrm{Fil}^\bullet$.

We define a functor $\underline{\mathcal{D}} : \mathrm{MF}_F^{\varphi,N} \to \mathrm{MF}_{S_F}^{\varphi,N}$ as follows:

- $\mathcal{D} := \underline{\mathcal{D}}(D) = S_F \otimes_{K_0 \otimes_{\mathbb{Q}_p} F} D$ as an S_F-module,
- $\varphi_{\mathcal{D}} = \varphi \otimes \varphi_D$,
- $N_{\mathcal{D}} = N \otimes 1 + 1 \otimes N_D$,
- $\mathrm{Fil}^0(\mathcal{D}) = \mathcal{D}$, and
$$\mathrm{Fil}^i(\mathcal{D}) = \left\{x \in \mathcal{D} \mid N_{\mathcal{D}}(x) \in \mathrm{Fil}^{i-1} \mathcal{D} \text{ and } (\mathrm{ev}_\pi \otimes 1)(x) \in \mathrm{Fil}^i D_K\right\}$$
for $i \geq 1$.

Here, $\mathrm{ev}_\pi : S_F \to F \otimes_{\mathbb{Q}_p} K$ is the scalar extension of $\mathrm{ev}_\pi : W(k)[u] \twoheadrightarrow \Lambda_K$, the evaluation-at-$\pi$ map.

Theorem 2.3 (Breuil). *The functor $\underline{\mathcal{D}} : \mathrm{MF}_F^{\varphi,N} \to \mathrm{MF}_{S_F}^{\varphi,N}$ is an equivalence of categories.*

Breuil proves in [6, §6] that $\underline{\mathcal{D}}$ is an equivalence of categories when $F = \mathbb{Q}_p$ and S is replaced by S_{Br}. That one can replace S_{Br} by S is known to some, but there does not appear to be a reference. The only step in Breuil's proof that requires honestly new justification is the following analogue of [6, Proposition 6.2.1.1] (this version is even easier to prove):

Lemma 2.4. *Set $\mathcal{D} \in \mathrm{MF}_{S_F}^{\varphi,N}$ and $D = \mathcal{D}/u\mathcal{D}$. Then there exists a unique $F \otimes_{\mathbb{Q}_p} K_0$-linear φ-equivariant section $s : D \to \mathcal{D}$ of the reduction map.*

Proof. First, suppose $F = \mathbb{Q}_p$ and let $(\widehat{e}_1,\ldots,\widehat{e}_d)$ be an $S\left[\frac{1}{p}\right]$-basis of $\mathcal{D}$. Write $\varphi_{\mathcal{D}}(\widehat{e}_1,\ldots,\widehat{e}_d) = (\widehat{e}_1,\ldots,\widehat{e}_d)X$ and set $X_0 = X \bmod u$. Then $X \in p^k \mathrm{Mat}_d(S)$, $X_0^{-1} \in p^\ell \mathrm{Mat}_d(W(k))$, and $XX_0^{-1} \in I + up^m \mathrm{Mat}_d(S)$ for some $k,\ell,m \in \mathbb{Z}$. As in the proof of [6, Proposition 6.2.1.1], we need to show that
$$Y_n := X\varphi(X)\cdots\varphi^n(X)\varphi^n\left(X_0^{-1}\right)\cdots\varphi\left(X_0^{-1}\right)X_0^{-1}$$
converges in $\mathrm{Mat}_d\left(S\left[\frac{1}{p}\right]\right)$ as $n \to \infty$. But, in the notation already used,
$$Y_n - Y_{n-1} \in \varphi^n(u)p^{n(k+\ell)+m}\mathrm{Mat}_d(S).$$

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Since $\varphi^n(u)p^{nr} \to 0$ in $S\left[\frac{1}{p}\right]$ for any fixed r, we see that $Y_n - Y_{n-1} \to 0$ in $\mathrm{Mat}_d\left(S\left[\frac{1}{p}\right]\right)$, as needed.

If $F \neq \mathbb{Q}_p$, the proof already given implies that there exists a unique K_0-linear φ-equivariant section $s : D \to \mathcal{D}$. If $x \in F^\times$, then $x^{-1}sx$ is also K_0-linear and φ-equivariant, and thus s is F-linear. □

Proof of Theorem 2.3. Define $\underline{D}_{S_F} : \mathrm{MF}_{S_F}^{\varphi,N} \to \mathrm{MF}_F^{\varphi,N}$ as follows. Set $D = \underline{D}_{S_F}(\mathcal{D}) = \mathcal{D}/u\mathcal{D}$ with its induced action of φ and N. For s in Lemma 2.4, $(\mathrm{ev}_\pi \otimes 1) \circ s : D \to \mathcal{D}/E\mathcal{D}$ induces a canonical isomorphism $D_K \cong \mathcal{D}/E\mathcal{D}$. The filtration $\mathrm{Fil}^i(D_K)$ is the pullback of the filtration on $\mathcal{D}/E\mathcal{D}$ defined as the image $\mathrm{Fil}^i(\mathcal{D}) \to \mathcal{D}/E\mathcal{D}$. The arguments in [6], with Lemma 2.4 replacing its Proposition 6.2.1.1, show that $\underline{D}_{S[\frac{1}{p}]}$ and $\underline{\mathcal{D}}$ are quasi-inverses when $F = \mathbb{Q}_p$. In general, see Remark 2.2.

2.4. Comparison

We now assume that F contains a subfield isomorphic to the Galois closure of K (see Lemma 2.5). In practice, as in §§3 and 4, we take $K = \mathbb{Q}_p$, so this is no hindrance.

In the prior sections, we have described equivalences

$$\mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N\nabla} \xrightarrow{\cong} \mathrm{MF}_F^{\varphi,N} \xrightarrow{\cong} \mathrm{MF}_{S_F}^{\varphi,N}. \tag{2.2}$$

An analog of [21, Corollary 3.2.3] allows for a description of the composition that, unfortunately, is not practical for calculations. In the following, though, we explain how to determine $\underline{\mathcal{M}}_{\mathcal{O}_F}(D) \otimes_{\mathcal{O}_F} S_F$ as a φ-module over S_F from D, up to determining $\mathcal{D} = \underline{\mathcal{D}}(D)$. A key technical point, which follows from the next lemma, is that filtrations on Breuil modules over S_F are always free, in contrast to the filtrations on objects in $\mathrm{MF}_F^{\varphi,N}$ (compare with [7, Exemple 3.1.1.4]).

Lemma 2.5. *Suppose that $\mathcal{N}$ is a finite free S_F-module and $\mathcal{H} \subseteq \mathcal{N}$ is an S_F-submodule such that $E^j\mathcal{N} \subseteq \mathcal{H}$ for some $j \geq 0$. Then $\mathcal{H}$ is finite free over S_F.*

Proof. We may assume $j = 1$. Indeed, consider the nested sequence $\mathcal{H}_i = \mathcal{H} + E^i\mathcal{N}$ of S_F-modules, which satisfy $E\mathcal{H}_i \subseteq \mathcal{H}_{i+1} \subseteq \mathcal{H}_i$. By the $j = 1$ case we deduce that $\mathcal{H}_1 \subseteq \mathcal{N}$ is free, and then $\mathcal{H}_2$, and so on until $\mathcal{H}_j = \mathcal{H}$ is free. We may also assume $\mathcal{N} \cong S_F$. Indeed, if $0 \to \mathcal{N}'' \to \mathcal{N} \xrightarrow{f} \mathcal{N}' \to 0$ is an exact sequence of finite free S_F-modules, then $\mathcal{H}' = f(\mathcal{H})$ and $\mathcal{H}'' = \ker(f) \cap \mathcal{H}$ satisfy $E\mathcal{N}'' \subseteq \mathcal{H}''$ and $E\mathcal{N}' \subseteq \mathcal{H}'$. So if both $\mathcal{H}''$ and $\mathcal{H}'$ are free, then $\mathcal{H} \cong \mathcal{H}'' \oplus \mathcal{H}'$ is free as well.

We have reduced to proving that if $I \subseteq S_F$ is an ideal containing E, then I is free. Since F contains a subfield isomorphic to the Galois closure of K, we may decompose $S_F = \prod_{\sigma \in \mathrm{Hom}(K_0,F)} S_{F,\sigma}$ where $S_{F,\sigma} = \Lambda\left[\left[u, \frac{\sigma(E)^p}{p}\right]\right]\left[\frac{1}{p}\right]$ is a domain. The ideal I decomposes as a product of ideals I_σ such that $\sigma(E)S_{F,\sigma} \subseteq I_\sigma$. Since $\sigma(E)$ is nonzero, it suffices to show that each I_σ is principal. Write $\mathrm{Hom}_\sigma(K,F)$ for the embeddings $\tau : K \to F$ lifting σ. Then we have a canonical isomorphism

$$S_{F,\sigma}/\sigma(E)S_{F,\sigma} \cong K \otimes_{K_0,\sigma} F \cong F^{\mathrm{Hom}_\sigma(K,F)}.$$

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So $I_\sigma/\sigma(E)S_{F,\sigma} \cong F^T$ for some subset $T \subseteq \mathrm{Hom}_\sigma(K,F)$. But $J_T = \prod_{\tau\in T}(u-\tau(\pi))\cdot S_F$ also contains $\sigma(E)S_{F,\sigma}$ and $J_T/\sigma(E)S_{F,\sigma} \cong F^T$. Thus $I_\sigma = J_T$ is principal, completing the proof. $\square$

We now consider an ad hoc category of 'Breuil modules without monodromy'. Let $\mathrm{MF}^{\varphi,h}_{S_F}$ denote the category whose objects are $(\mathcal{D},\varphi_{\mathcal{D}}) \in \mathrm{Mod}^{\varphi}_{S_F}$ such that the linearization of $\varphi_{\mathcal{D}}$ is an isomorphism, and $\mathcal{D}$ is equipped with a finite *free* S_F-submodule $\mathrm{Fil}^h\mathcal{D} \subseteq \mathcal{D}$ such that $\mathrm{Fil}^h S_F \cdot \mathcal{D} \subseteq \mathrm{Fil}^h\mathcal{D}$. By Lemma 2.5 there is a natural forgetful functor $\mathrm{MF}^{\varphi,N}_{S_F} \to \mathrm{MF}^{\varphi,h}_{S_F}$.

Now define $\underline{\mathcal{D}}' : \mathrm{Mod}^{\varphi,\le h}_{S_F} \to \mathrm{MF}^{\varphi,h}_{S_F}$ by declaring $\underline{\mathcal{D}}'(\mathcal{M}) = S_F \otimes_{\varphi,S_F} \mathcal{M}$ as an S_F-module, and

- $\varphi_{\underline{\mathcal{D}}'(\mathcal{M})} = \varphi\otimes\varphi_{\mathcal{M}}$ and
- $\mathrm{Fil}^h\underline{\mathcal{D}}'(\mathcal{M}) = \left\{x \in \underline{\mathcal{D}}'(\mathcal{M}) \mid (1\otimes\varphi_{\mathcal{M}})(x) \in \mathrm{Fil}^h S_F\cdot\mathcal{M}\right\}$.

Since $E^h\underline{\mathcal{D}}'(\mathcal{M}) \subseteq \mathrm{Fil}^h\underline{\mathcal{D}}'(\mathcal{M})$, Lemma 2.5 implies that $\mathrm{Fil}^h\underline{\mathcal{D}}'(\mathcal{M})$ is finite free over S_F.

Proposition 2.6. *The functor $\underline{\mathcal{D}}'$ is an equivalence.*

Proof. We first show that $\underline{\mathcal{D}}'$ is fully faithful. Suppose $\mathcal{M}$ and $\mathcal{M}'$ are in $\mathrm{Mod}^{\varphi,\le h}_{S_F}$. Write $\mathcal{D} := \underline{\mathcal{D}}'(\mathcal{M})$ and $\mathcal{D}' := \underline{\mathcal{D}}'(\mathcal{M}')$. Choose a basis $(e_1,\dots,e_d)$ of $\mathcal{M}$ and write $\varphi_{\mathcal{M}}(e_1,\dots,e_d) = (e_1,\dots,e_d)A$ with $A \in \mathrm{Mat}_d(S_F)$. Since $\mathcal{M}$ has height $\le h$, there exists a matrix $B \in \mathrm{Mat}_d(S_F)$ such that $AB = BA = E^hI_d$. By assumption, $\mathrm{Fil}^h\mathcal{D}$ has basis $(\alpha_1,\dots,\alpha_d) = (\widetilde{e}_1,\dots,\widetilde{e}_d)B$ where $\widetilde{e}_i = 1\otimes e_i \in \mathcal{D}$ compose a basis of $\mathcal{D}$. Similarly, we get A',B', and $\widetilde{e}'_i$ from a basis $(e'_1,\dots,e'_{d'})$ of $\mathcal{M}'$.

Now suppose $f : \mathcal{D} \to \mathcal{D}'$ is a morphism in $\mathrm{MF}^{\varphi,h}_{S_F}$. We write $f(\widetilde{e}_1,\dots,\widetilde{e}_d) = (\widetilde{e}'_1,\dots,\widetilde{e}'_{d'})X$ for $X \in \mathrm{Mat}_d(S_F)$. Since f is φ-equivariant, we have $X\varphi(A) = \varphi(A')\varphi(X)$, and since $f\left(\mathrm{Fil}^h\mathcal{D}\right) \subseteq \mathrm{Fil}^h\mathcal{D}'$, we have $XB = B'Y$ for some $Y \in \mathrm{Mat}_d(S_F)$. Using $AB = BA = E^hI_d$ and $A'B' = B'A' = E^hI_{d'}$, we see that $\varphi(Y)\varphi\left(E^h\right) = X\varphi\left(E^h\right)$, and so $X = \varphi(Y)$ because $\varphi(E) \in S_F^\times$. It follows that $YA = A'\varphi(Y)$. Define $\mathfrak{f} : \mathcal{M} \to \mathcal{M}'$ by $\mathfrak{f}(e_1,\dots,e_d) = (e'_1,\dots,e'_{d'})Y$. Then $\mathfrak{f}$ is φ-equivariant and $f = \underline{\mathcal{D}}'(\mathfrak{f})$ since $X = \varphi(Y)$. This shows that $\underline{\mathcal{D}}'$ is full, and since Y determines X, we also see that $\underline{\mathcal{D}}'$ is faithful.

Now we prove that $\underline{\mathcal{D}}'$ is essentially surjective. Given a $\mathcal{D} \in \mathrm{MF}^{\varphi,h}_{S_F}$, choose bases $(e_1,\dots,e_d)$ of $\mathcal{D}$ and $(\alpha_1,\dots,\alpha_d)$ of $\mathrm{Fil}^h\mathcal{D}$. Write $(\alpha_1,\dots,\alpha_d) = (e_1,\dots,e_d)B$ and $\varphi_{\mathcal{D}}(e_1,\dots,e_d) = (e_1,\dots,e_d)X$ with $\det(X) \in S_F^\times$. Since $E^h\mathcal{D} \subseteq \mathrm{Fil}^h\mathcal{D}$, there exists $A \in \mathrm{Mat}_d(S_F)$ such that $AB = BA = E^hI_d$. Since $\varphi(E) = p\mathfrak{c} \in S_F^\times$, we see that $X\varphi(B) \in \mathrm{GL}_d(S_F)$, whereas $\varphi_{\mathcal{D}}(\alpha_1,\dots,\alpha_d) = (e_1,\dots,e_d)X\varphi(B)$. Thus $(f_1,\dots,f_d) = (e_1,\dots,e_d)X\varphi(B)p^{-h}\mathfrak{c}^{-h}$ is a basis of $\mathcal{D}$ and $\varphi_{\mathcal{D}}(\alpha_1,\dots,\alpha_d) = (f_1,\dots,f_d)p^h\mathfrak{c}^h$. Finally, $(\alpha_1,\dots,\alpha_d) = (f_1,\dots,f_d)B'$ where $B' = YB$ and $Y = \left(X\varphi(B)p^{-h}\mathfrak{c}^{-h}\right)^{-1}$, so there exists an A' such that $A'B' = B'A' = E^hI_d$. Now define $\mathcal{M} = \bigoplus_{i=1}^d S_F\mathfrak{f}_i$ and set $\varphi_{\mathcal{M}}(\mathfrak{f}_1,\dots,\mathfrak{f}_d) = (\mathfrak{f}_1,\dots,\mathfrak{f}_d)A'$. Then $\mathcal{M} \in \mathrm{Mod}^{\varphi,\le h}_{S_F}$ and $\underline{\mathcal{D}}'(\mathcal{M}) = \mathcal{D}$ (set $f_i = 1\otimes\mathfrak{f}_i$). $\square$

We now reach the main theorem of this section, which provides a mechanism to calculate a finite-height φ-module over S_F explicitly from $D \in \mathrm{MF}^{\varphi,N}_F$. We write $\varphi(E) = p\mathfrak{c}$ with $\mathfrak{c} \in S^\times$ as before.

Theorem 2.7. *Suppose* $D \in \mathrm{MF}_F^{\varphi,N}$. *Write* $\mathcal{D}' \in \mathrm{MF}_{S_F}^{\varphi,h}$ *for the image of* $\underline{\mathcal{D}}(D)$ *under the natural forgetful functor and* $\mathcal{M} = \underline{\mathcal{M}}_{\mathcal{O}_F}(D) \otimes_{\mathcal{O}_F} S_F$. *Then there is a natural isomorphism* $\underline{\mathcal{D}}'(\mathcal{M}) \cong \mathcal{D}'$.

In particular, $\mathcal{M}$ *is recovered from* D *via the following steps:*

(1) *Select* S_F*-bases* $(e_1,\dots,e_d)$ *of* $\mathcal{D} = \underline{\mathcal{D}}(D)$ *and* $(\alpha_1,\dots,\alpha_d)$ *of* $\mathrm{Fil}^h \mathcal{D}$.

(2) *Let* $\varphi_{\mathcal{D}}(e_1,\dots,e_d) = (e_1,\dots,e_d)X$ *and* $(\alpha_1,\dots,\alpha_d) = (e_1,\dots,e_d)B$ *with* $X, B \in \mathrm{Mat}_d(S_F)$.

(3) *Then* $\mathcal{M}$ *has an* S_F*-basis* $(\mathfrak{f}_1,\dots,\mathfrak{f}_d)$ *in which* $\varphi_{\mathcal{M}}(\mathfrak{f}_1,\dots,\mathfrak{f}_d) = (\mathfrak{f}_1,\dots,\mathfrak{f}_d)A$, *where*

$$A = E^h B^{-1} X \varphi(B) p^{-h} \mathfrak{c}^{-h}.$$

Proof. To start, once the isomorphism $\underline{\mathcal{D}}'(\mathcal{M}) \cong \mathcal{D}'$ is justified, the 'in particular' follows by tracing through the second half of the proof of Proposition 2.6.

For $\mathcal{M}_{\mathcal{O}_F} \in \mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N_\nabla}$ we define $\mathcal{D} = \underline{\mathcal{D}}_{\mathcal{O}_F}(\mathcal{M}_{\mathcal{O}_F}) = S_F \otimes_{\varphi,\mathcal{O}_F} \mathcal{M}_{\mathcal{O}_F}$, which is a finite free S_F-module, and equip it with the following structure of a Breuil module over S_F:

- $\varphi_{\mathcal{D}} = \varphi \otimes \varphi_{\mathcal{M}}$,
- $N_{\mathcal{D}} = N \otimes 1 + \frac{p}{\varphi(\lambda)} \otimes N_\nabla$,
- $\mathrm{Fil}^i(\mathcal{D}) = \left\{ x \in \mathcal{D} \mid (1 \otimes \varphi_{\mathcal{M}})(x) \in \mathrm{Fil}^i S_F \otimes_{\mathcal{O}_F} \mathcal{M}_{\mathcal{O}_F} \right\}$.

Following the proof of [20, Proposition 3.2.1], replacing S by S_{Br}, and adding linear F-coefficients, we see that $\underline{\mathcal{D}}_{\mathcal{O}_F} : \mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N_\nabla} \to \mathrm{MF}_{S_F}^{\varphi,N}$ defines a functor. Moreover, if $\mathcal{M}_{\mathcal{O}_F}$ has height $\leq h$, then

$$\underline{\mathcal{D}}_{\mathcal{O}_F}(\mathcal{M}_{\mathcal{O}_F}) \cong \underline{\mathcal{D}}'(\mathcal{M}_{\mathcal{O}_F} \otimes_{\mathcal{O}_F} S_F)$$

in the category $\mathrm{MF}_{S_F}^{\varphi,h}$. Thus it remains to show that $\underline{\mathcal{D}}_{\mathcal{O}_F}$ makes the diagram of functors

$$\begin{array}{ccc} \mathrm{MF}_F^{\varphi,N} & \xrightarrow{\underline{\mathcal{D}}} & \mathrm{MF}_{S_F}^{\varphi,N} \\ \uparrow{\scriptstyle \underline{D}_{\mathcal{O}_F}} & \nearrow{\scriptstyle \underline{\mathcal{D}}_{\mathcal{O}_F}} & \\ \mathrm{Mod}_{\mathcal{O}_F}^{\varphi,N_\nabla} & & \end{array} \tag{2.3}$$

commute as well. (In particular, $\underline{D}_{\mathcal{O}_F}$ is an equivalence.) It is enough to check this when $F = \mathbb{Q}_p$ (by Remark 2.2). In that case, if S is replaced by S_{Br}, this is the statement of [20, Corollary 3.2.3]. The proof there goes through here with only one adjustment. Namely, the isomorphism $S_{\mathrm{Br}}[\frac{1}{p}] \otimes_{K_0} \underline{D}_{\mathcal{O}}(\mathcal{M}_{\mathcal{O}}) \cong S_{\mathrm{Br}}[\frac{1}{p}] \otimes_{\varphi,\mathcal{O}} \mathcal{M}_{\mathcal{O}}$ implicit in the first two displayed equations there needs to have S_{Br} replaced by S. To make this adjustment, consider the map $\xi : \mathcal{O} \otimes_{K_0} \underline{D}(\mathcal{M}_{\mathcal{O}}) \to \mathcal{M}_{\mathcal{O}}$ constructed in [18, Lemma 1.2.6]. Thus ξ

is a φ-equivariant injection with cokernel annihilated by λ^h for some $h \geq 0$. From the diagram in the middle of the proof of [18, Lemma 1.2.6], we have ξ factors

$$\begin{array}{ccc} \mathcal{O}\otimes_{K_0}\underline{D}(\mathcal{M}_{\mathcal{O}}) & \xrightarrow{\ \xi\ } & \mathcal{M}_{\mathcal{O}}. \\ \downarrow & \nearrow_{1\otimes\varphi} & \\ \mathcal{O}\otimes_{\varphi,\mathcal{O}}\mathcal{M}_{\mathcal{O}}. & & \end{array} \tag{2.4}$$

We deduce that the vertical arrow has cokernel annihilated by $\varphi(\lambda)^h$. Since $\varphi(\lambda) \in S^{\times}$, we have

$$S[1/p]\otimes_{K_0}\underline{D}(\mathcal{M}_{\mathcal{O}}) \overset{1\otimes\xi}{\cong} S[1/p]\otimes_{\varphi,\mathcal{O}}\mathcal{M}_{\mathcal{O}}.$$

This completes the proof. □

Remark 2.8. The previous proof makes it clear to see that for $D \in \mathrm{MF}_F^{\varphi,N}$ and $\mathcal{D} = \underline{D}(D) \in \mathrm{MF}_{S_F}^{\varphi,N}$, the map ev_π induces an isomorphism $\mathrm{Fil}^{i+1}\mathcal{D}/E\,\mathrm{Fil}^i\mathcal{D} \cong \mathrm{Fil}^{i+1} D_K$. Indeed, since $\mathrm{ev}_\pi\left(\mathrm{Fil}^{i+1}\mathcal{D}\right) = \mathrm{Fil}^{i+1} D_K$, it suffices to show that $E\mathcal{D}\cap \mathrm{Fil}^{i+1}\mathcal{D} = E\,\mathrm{Fil}^i\mathcal{D}$. Pick $y = Ex \in \mathrm{Fil}^{i+1}\mathcal{D}$ with $x \in \mathcal{D}$. The proof of the theorem, especially the fact that diagram (2.3) commutes, shows that

$$\mathrm{Fil}^{i+1}(\mathcal{D}) = \left\{x \in \mathcal{D} \mid (1\otimes\varphi_{\mathcal{M}})(x) \in \mathrm{Fil}^{i+1} S_F \otimes_{\mathcal{O}_F} \mathcal{M}_{\mathcal{O}_F}\right\}.$$

Thus, we see that $(1\otimes\varphi_{\mathcal{M}})(Ex) = E(1\otimes\varphi_{\mathcal{M}})(x) \in \mathrm{Fil}^{i+1} S_F\otimes_{\mathcal{O}_F}\mathcal{M}_{\mathcal{O}_F}$. Since $\mathrm{Fil}^n S_F = E^n S_F$, it is clear that $(1\otimes\varphi_{\mathcal{M}})(x) \in \mathrm{Fil}^i S_F\otimes_{\mathcal{O}_F}\mathcal{M}_{\mathcal{O}_F}$, and hence $x \in \mathrm{Fil}^i\mathcal{D}$ as required. (Compare with the end of the proof of [21, Proposition 3.2.1].)

Example 2.9. Suppose $K = \mathbb{Q}_p$ and V is crystalline. By [19], $D = D^*_{\mathrm{st}}(V)$ admits a strongly divisible lattice $\left(M, \mathrm{Fil}^i M, \varphi_i\right)$. More precisely, there exist an F-basis $(e_1,\dots,e_d)$ of D and integers $0 = n_0 \leq n_1 \leq \cdots \leq n_h \leq d$ such that $\mathrm{Fil}^i D := \bigoplus_{j\geq n_i} Fe_j$, and $\varphi(e_1,\dots,e_d) = (e_1,\dots,e_d)XP$ where $X \in \mathrm{GL}_d(\Lambda)$ and P is a diagonal matrix whose iith entry is p^{s_i}, where $s_i = \max\{j \mid n_j \leq i\} = \max\left\{j \mid e_i \in \mathrm{Fil}^j D\right\}$. Since $N = 0$ on D, we easily compute that $\mathrm{Fil}^h\mathcal{D}$ admits a basis $(e_1,\dots,e_d)B$ where B is the diagonal matrix with (i,i)th entry E^{h-s_i} (compare §3.1). By the steps outlined in Theorem 2.7, using the basis $1\otimes e_i \in \mathcal{D}$, we see that the matrix of φ on $\mathcal{M}$ is given by $A = E^h B^{-1} X P \varphi(B) p^{-h}\mathfrak{c}^{-h}$, where $A = DXC$, D is a diagonal matrix with (i,i)th entry E^{s_i}, and C is a diagonal matrix with (i,i)th entry $\mathfrak{c}^{-s_i}$.

3. An explicit determination of a Breuil module

In this section, we assume $K = \mathbb{Q}_p$. We choose $\pi = -p$, so $E(u) = u + p$. We keep $F/\mathbb{Q}_p$ as a linear coefficient field and recall that Λ is its ring of integers. In §3.2, we explain the definition of the filtered (φ,N)-module $D_{h+1,\mathcal{L}} \in \mathrm{MF}_F^{\varphi,N}$, for $h \geq 1$ and $\mathcal{L} \in F$, discussed in the introduction. Let $\mathcal{M}_{h+1,\mathcal{L}} = \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{h+1,\mathcal{L}})\otimes_{\mathcal{O}_F} S_F \in \mathrm{Mod}_{S_F}^{\varphi,\leq h}$. The ultimate goal (Theorem 3.7) is to describe the matrix of φ in a certain trivialization $\mathcal{M}_{h+1,\mathcal{L}} \cong S_F^{\oplus 2}$, at least if $\mathcal{L} \neq 0$. We begin by describing the Breuil module $\mathcal{D}_{h+1,\mathcal{L}} = \underline{\mathcal{D}}(D_{h+1,\mathcal{L}})$.

https://doi.org/10.1017/S1474748022000081 Published online by Cambridge University Press

3.1. The filtration on some rank 2 Breuil modules

In order to minimize notation, in this subsection we let $D \in \mathrm{MF}_F^{\varphi,N}$ be any 2-dimensional filtered (φ,N)-module with Hodge–Tate weights $0 < h$. We also choose any basis (f_1,f_2) for D such that $\mathrm{Fil}^h D = Ff_2$. We write $N_D(f_1,f_2) = (f_1,f_2)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Mat}_2(F)$. (Compare with Lemma 3.6.)

Set $\mathcal{D} = \underline{\mathcal{D}}(D) = S_F \otimes_F D$. For $f \in D$ we write $\widehat{f} = 1 \otimes f \in \mathcal{D}$. In particular, $\mathcal{D}$ is a free S_F-module with basis $\left(\widehat{f}_1,\widehat{f}_2\right)$. Recall that $\mathrm{Fil}^i \mathcal{D}$ is defined by $\mathrm{Fil}^0 \mathcal{D} = \mathcal{D}$ and, for $i \geq 1$,

$$\mathrm{Fil}^i \mathcal{D} = \left\{x \in \mathcal{D} \mid N_{\mathcal{D}}(x) \in \mathrm{Fil}^{i-1} \mathcal{D} \text{ and } \mathrm{ev}_\pi(x) \in \mathrm{Fil}^i D\right\}.$$

When $i = 1$, the condition $N_{\mathcal{D}}(x) \in \mathrm{Fil}^0 \mathcal{D} = \mathcal{D}$ is a tautology. So $\mathrm{Fil}^1 \mathcal{D} = S_F \widehat{f}_2 + S_F E \widehat{f}_1$.

Proposition 3.1. *There exist $x_1,\dots,x_{h-1} \in F$ such that, if $0 \leq i \leq h$,*

$$\mathrm{Fil}^i \mathcal{D} = S_F \cdot \left(\widehat{f}_2 + \left(\sum_{j=1}^{i-1} x_j E^j\right)\widehat{f}_1\right) + S_F \cdot E^i \widehat{f}_1.$$

Proof. Assume by induction on $0 \leq i < h$ that there exist $x_1,\dots,x_{i-1} \in F$ such that for each $0 \leq j \leq i$ we have $\mathrm{Fil}^j \mathcal{D} = S_F \cdot \widehat{f}_2^{(j)} + S_F \cdot \widehat{f}_1$, where $\widehat{f}_2^{(j)} = \widehat{f}_2 + \left(\sum_{m=1}^{j-1} x_m E^m\right)\widehat{f}_1$. Setting $\widehat{f}_2^{(0)} = \widehat{f}_2^{(1)} = \widehat{f}_2$ handles the cases $i = 0$ and $i = 1$. So suppose $1 \leq i < h$.

For the $(i+1)$th case, we first define $x_i \in F$. By induction, $N_{\mathcal{D}}\left(\widehat{f}_2^{(i)}\right) \in \mathrm{Fil}^{i-1}\mathcal{D} = S_F \widehat{f}_2^{(i-1)} + S_F E^{i-1}\widehat{f}_1$. Since $\widehat{f}_2^{(i-1)} = \widehat{f}_2^{(i)} - x_{i-1}E^{i-1}\widehat{f}_1$, we can write

$$N_{\mathcal{D}}\left(\widehat{f}_2^{(i)}\right) = d_i \widehat{f}_2^{(i)} + b_i E^{i-1}\widehat{f}_1$$

for some $d_i, b_i \in S_F$ (compare Lemma 3.2). Set $x_i = b_i(\pi)/i\pi$, and then set $\widehat{f}_2^{(i+1)} = \widehat{f}_2^{(i)} + x_i E^i \widehat{f}_1$. Since $2 \leq i+1 \leq h$, we have $\mathrm{Fil}^{i+1} D = Ff_2$. Thus, $\mathrm{ev}_\pi\left(\widehat{f}_2^{(i+1)}\right) = \widehat{f}_2 \in \mathrm{Fil}^{i+1} D$. Further,

$$\begin{aligned} N_{\mathcal{D}}\left(\widehat{f}_2^{(i+1)}\right) &= N_{\mathcal{D}}\left(\widehat{f}_2^{(i)}\right) - x_i i u E^{i-1}\widehat{f}_1 + x_i E^i N_{\mathcal{D}}\left(\widehat{f}_1\right) \\ &= d_i \widehat{f}_2^{(i)} + (b_i - x_i i u)E^{i-1}\widehat{f}_1 + x_i E^i N_{\mathcal{D}}\left(\widehat{f}_1\right). \end{aligned} \tag{3.1}$$

Note that the last summand in equation (3.1) lies in $\mathrm{Fil}^i S_F \cdot \mathcal{D} \subseteq \mathrm{Fil}^i \mathcal{D}$, whereas the first lies in $\mathrm{Fil}^i \mathcal{D}$. By definition we have $\mathrm{ev}_\pi(b_i - x_i i u) = 0$, and so the middle summand also lies in $\mathrm{Fil}^i S_F \cdot \mathcal{D} \subseteq \mathrm{Fil}^i \mathcal{D}$. Thus $\widehat{f}_2^{(i+1)} \in \mathrm{Fil}^{i+1}\mathcal{D}$.

For a moment, define $F^{i+1}\mathcal{D} = S_F \widehat{f}_2^{(i+1)} + S_F E^{i+1}\widehat{f}_1 \subseteq \mathrm{Fil}^{i+1}\mathcal{D}$. We want to show equality. Since $E\widehat{f}_2^{(i)} = E\widehat{f}_2^{(i+1)} - x_i E^{i+1}\widehat{f}_1$, we in fact have

$$E\,\mathrm{Fil}^i \mathcal{D} \subseteq F^{i+1}\mathcal{D} \subseteq \mathrm{Fil}^{i+1}\mathcal{D}.$$

Since ev_π gives an isomorphism $\mathrm{Fil}^{i+1}\mathcal{D}/E\,\mathrm{Fil}^i \mathcal{D} \cong Ff_2$ by Remark 2.8, and $\mathrm{ev}_\pi\left(F^{i+1}\mathcal{D}\right) \neq 0$, we conclude that the natural map $F^{i+1}\mathcal{D}/E\,\mathrm{Fil}^i\mathcal{D} \to \mathrm{Fil}^{i+1}\mathcal{D}/E\,\mathrm{Fil}^i\mathcal{D}$ is an isomorphism. Thus, $F^{i+1}\mathcal{D} = \mathrm{Fil}^{i+1}\mathcal{D}$. □

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The proof of Proposition 3.1 allows for explicit control of the scalars x_j in terms of the monodromy matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. For the next two results, we explain this by reexamining the proof.

Lemma 3.2. *For $1 \le i \le h-1$, let $d_i, b_i \in S_F$ be such that $N_{\mathcal{D}}\left(\widehat{f}_2^{(i)}\right) = d_i \widehat{f}_2^{(i)} + b_i E^{i-1}\widehat{f}_1$. Then $d_1 = d$, $b_1 = b$, $x_1 = \frac{b}{\pi}$, and for $1 \le i < h-1$,*

$$\begin{aligned} d_{i+1} &= d_i + c x_i E^i \\ b_{i+1} &= x_i(a - c z_i - d_i) + (b_i - x_i i u)/E \\ x_{i+1} &= \frac{b_{i+1}(\pi)}{(i+1)\pi}, \end{aligned}$$

where $z_i = \sum_{j=1}^{i} x_j E^j$.

Proof. The values of d_1, b_1, and x_1 follow immediately from $\widehat{f}_2^{(1)} = \widehat{f}_2$ and $N_{\mathcal{D}}\left(\widehat{f}_2\right) = b\widehat{f}_1 + d\widehat{f}_2$. Next, by equation (3.1) and because $N_{\mathcal{D}}\left(\widehat{f}_1\right) = a\widehat{f}_1 + c\widehat{f}_2$, we have

$$N_{\mathcal{D}}\left(\widehat{f}_2^{(i+1)}\right) = d_i \widehat{f}_2^{(i)} + (b_i - x_i i u)E^{i-1}\widehat{f}_1 + x_i E^i\left(a\widehat{f}_1 + c\widehat{f}_2\right). \tag{3.2}$$

We can write $\widehat{f}_2^{(i)} = \widehat{f}_2^{(i+1)} - x_i E^i \widehat{f}_1$ and, separately, $\widehat{f}_2 = \widehat{f}_2^{(i+1)} - z_i \widehat{f}_1$. Thus equation (3.2) becomes

$$N_{\mathcal{D}}\left(\widehat{f}_2^{(i+1)}\right) = \left(d_i + c x_i E^i\right)\widehat{f}_2^{(i+1)} + \left(-d_i x_i E^i + (b_i - x_i i u)E^{i-1} + x_i E^i (a - c z_i)\right)\widehat{f}_1.$$

Factoring E^i out of the $\widehat{f}_1$-coefficient, the result is clear. □

Example 3.3. In Lemma 4.4, we will need an explicit calculation of the x_i and z. This can be done using the recursive formulas already given. The calculations we need, both of which are straightforward, are

$$\begin{aligned} x_2 &= \frac{b}{2\pi^2}(a - d - 1) \\ z_2(0) &= \frac{b}{2}(a - d - 3). \end{aligned}$$

(See also Example 3.9.)

Lemma 3.4. *Assume that $a - d \in \Lambda$ and $bc \in \Lambda$. Then for $1 \le i \le h-1$, we have*

$$v_p(x_i) + v_p(i!) + i \ge v_p(b).$$

Remark 3.5. The lemma is consistent with $b = 0$, since in that case $x_i = 0$ for all i.

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Proof of Lemma 3.4. Given $v \in \mathbb{R}$, we write

$$A_v = \left\{ \sum_{j\geq 0} y_j E^j \in F[u] \mid v_p(y_j) + v_p(j!) + j \geq v \right\}.$$

Note that A_v is a subgroup of $F[u]$. Since $v_p((j+k)!) \geq v_p(j!) + v_p(k!)$ for all nonnegative integers j,k (because binomial coefficients are integers), we have $A_v A_w \subseteq A_{v+w}$ as well. In particular, A_0 is a ring containing Λ as a subring, and each A_v is an A_0-module.

The lemma is equivalent to $x_i E^i \in A_{v_p(b)}$ for all $1 \leq i \leq h-1$, but to show $x_i E^i \in A_{v_p(b)}$ it suffices to show $b_i E^{i-1} \in A_{v_p(b)}$. Indeed, $b_i E^{i-1} \in b_i(\pi)E^{i-1} + E^i F[u]$, and so if $b_i E^{i-1} \in A_v$ (for any v), then $v_p(b_i(\pi)) + v_p((i-1)!) + i - 1 \geq v$. Since $b_i(\pi) = x_i i\pi$, by definition, we would clearly have $v_p(x_i) + v_p(i!) + i \geq v$ as well.

We have reduced to showing $b_i E^{i-1} \in A_{v_p(b)}$ for $1 \leq i \leq h-1$. For $i = 1$, by Lemma 3.2 we have $b_1 = b$, and so the claim is clear. Now assume that $b_j E^{j-1} \in A_{v_p(b)}$ for all $j \leq i$. By the previous paragraph, we have $x_j E^j \in A_{v_p(b)}$ for all $j \leq i$, and so $z_j \in A_{v_p(b)}$ for all $j \leq i$ (including z_0, which we define to be 0). By Lemma 3.2, we have

$$\begin{aligned} b_{i+1}E^i &= (a - cz_i - d_i)x_i E^i + (b_i - x_i iu)E^{i-1} \\ &= (a - d - c(z_i + z_{i-1}))x_i E^i + b_i E^{i-1} - x_i i\pi E^{i-1} - x_i iE^i. \end{aligned} \tag{3.3}$$

It is clear by induction that the final three summands are in $A_{v_p(b)}$. For the first summand, we know $z_i + z_{i-1} \in A_{v_p(b)}$. Since $v_p(c) + v_p(b) \geq 0$ and $a - d \in \Lambda$, we see that $a - d - c(z_i + z_{i-1}) \in A_0$. Since $x_i E^i \in A_{v_p(b)}$, by induction, the first summand also lies in $A_{v_p(b)}$. Thus, $b_{i+1}E^i \in A_{v_p(b)}$.

3.2. Explicit filtered (φ, N)-modules

Now assume F contains an element ϖ such that $\varpi^2 = p$. For $\mathcal{L} \in F$ and $h \geq 1$, we define $D_{h+1,\mathcal{L}} = Fe_1 \oplus Fe_2 \in \mathrm{MF}_F^{\varphi,N}$, where, in the basis (e_1, e_2),

$$\varphi = \begin{pmatrix} \varpi^{h+1} & 0 \\ 0 & \varpi^{h-1} \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{Fil}^i D_{h+1,\mathcal{L}} = \begin{cases} D_{h+1,\mathcal{L}} & \text{if } i \leq 0, \\ F \cdot (e_1 + \mathcal{L}e_2) & \text{if } 1 \leq i \leq h, \\ \{0\} & \text{if } h < i \end{cases}$$

(see [7, Exemple 3.1.2.2(iv)]). It is useful make a change of basis. Set $a_p = \varpi^{h-1} + \varpi^{h+1}$.

Lemma 3.6. *If $\mathcal{L} \neq 0$, then $(f_1, f_2) = (-\varphi(e_1 + \mathcal{L}e_2), e_1 + \mathcal{L}e_2)$ is a basis of $D_{h+1,\mathcal{L}}$ in which*

$$\varphi = \begin{pmatrix} a_p & -1 \\ p^h & 0 \end{pmatrix}, \quad N = \frac{p}{\mathcal{L}(1-p)} \begin{pmatrix} 1 & \varpi^{-h-1} \\ \varpi^{h+1} & -1 \end{pmatrix}, \quad \mathrm{Fil}^i D_{h+1,\mathcal{L}} = \begin{cases} D_{h+1,\mathcal{L}} & \textit{if } i \leq 0, \\ Ff_2 & \textit{if } 1 \leq i \leq h, \\ \{0\} & \textit{if } h < i. \end{cases}$$

Proof. If $\mathcal{L} \neq 0$, then $e_1 + \mathcal{L}e_2$ is *not* an eigenvector of φ, so (f_1, f_2) is a basis. We leave calculating the matrices for the reader. □

Now let $\mathcal{D}_{h+1,\mathcal{L}} = \underline{\mathcal{D}}(D_{h+1,\mathcal{L}})$ and $\mathcal{M}_{h+1,\mathcal{L}} = \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{h+1,\mathcal{L}}) \otimes_{\mathcal{O}_F} S_F \in \mathrm{Mod}_{S_F}^{\varphi,\leq h}$. Recall that $\mathfrak{c} = \varphi(E)/p \in S_F^\times$. Let $\lambda_- = \prod_{n\geq 0} \varphi^{2n+1}(E)/p$ and $\lambda_{++} = \varphi(\lambda_-)$.

Theorem 3.7. *If $\mathcal{L} \neq 0$, there exists a basis of $\mathcal{M}_{h+1,\mathcal{L}}$ in which the matrix of φ is given by*

$$A = \begin{pmatrix} (a_p - p^h z)\left(\frac{\lambda_-}{\lambda_{++}}\right)^h & -1 + \varphi(z)(a_p - p^h z) \\ E^h & E^h \varphi(z)\left(\frac{\lambda_{++}}{\lambda_-}\right)^h \end{pmatrix},$$

where $z = \sum_{j=1}^{h-1} x_j E^j \in F[E]$. Moreover, if $v_p(\mathcal{L}^{-1}) \geq -1$, then

$$v_p(x_j) \geq v_p(\mathcal{L}^{-1}) - \frac{h-1}{2} - v_p(j!) - j \tag{3.4}$$

for each $1 \leq j \leq h-1$.

Proof. Let (f_1, f_2) be the basis as in Lemma 3.6. Set $\widehat{f_1} = 1 \otimes f_1$ and $\widehat{f_2} = 1 \otimes f_2$, elements of $\mathcal{D}_{h+1,\mathcal{L}}$, as before. Then the matrix of φ in the basis $(\widehat{f_1}, \widehat{f_2})$ of $\mathcal{D}_{h+1,\mathcal{L}}$ is $X = \left(\begin{smallmatrix} a_p & -1 \\ p^h & 0 \end{smallmatrix}\right)$. Moreover, Proposition 3.1 implies that $\mathrm{Fil}^h \mathcal{D}_{h+1,\mathcal{L}} = S_F \alpha_1 \oplus S_F \alpha_2$, where

$$(\alpha_1, \alpha_2) = (\widehat{f_1}, \widehat{f_2}) \begin{pmatrix} E^h & z \\ 0 & 1 \end{pmatrix} =: (\widehat{f_1}, \widehat{f_2}) B$$

for $z = \sum_{j=1}^{h-1} x_j E^j$ and some $x_j \in F$. Theorem 2.7 implies that $\mathcal{M}_{h+1,\mathcal{L}}$ has a basis in which the matrix of φ is given by

$$A' = E^h B^{-1} X \varphi(B) p^{-h} \mathfrak{c}^{-h} = \begin{pmatrix} a - p^h z & p^{-h}\mathfrak{c}^{-h}\left(-1 + \varphi(z)(a_p - p^h z)\right) \\ E^h p^h & p^{-h}\mathfrak{c}^{-h} E^h p^h \varphi(z) \end{pmatrix}. \tag{3.5}$$

Since λ_- and λ_{++} are units in S_F, we can replace A' by $CA'\varphi(C^{-1})$ for $C = \left(\begin{smallmatrix} p^h \lambda_-^h & 0 \\ 0 & \lambda_{++}^h \end{smallmatrix}\right)$. A short calculation shows $A = CA'\varphi(C^{-1})$, completing the general proof.

Finally, if $v_p(\mathcal{L}^{-1}) \geq -1$, then the matrix of N in Lemma 3.6 satisfies the hypotheses of Lemma 3.4, so the estimates (3.4) follow from the b-entry of the monodromy matrix:

$$b = \frac{-p}{\varpi^{h+1}\mathcal{L}(1-p)} = \frac{-1}{\varpi^{h-1}\mathcal{L}(1-p)}.$$

This completes the proof. □

Remark 3.8. An analogous calculation in the crystalline case, where $z = 0$ (see Remark 3.5), was made in [3, §3]. The technique here, passing through the category $\mathrm{MF}_{S_F}^{\varphi,N}$, is different, but the descriptions are the same (compare Example 2.9).

Example 3.9. We need one ad hoc calculation in Lemma 4.4. Let $h = 3$. By Example 3.3, the element z in Theorem 3.7 satisfies $z(0) = \frac{b}{2}(a - d - 3)$, where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is the monodromy matrix in Lemma 3.6. For $p = h = 3$, plugging in the explicit matrix, we see $z(0) = \frac{1}{4\mathcal{L}}\left(\frac{1}{\mathcal{L}}+1\right)$.

4. Descent and reductions

The goal in this section is to prove the main theorem of the article. Given $h \geq 1$ and $\mathcal{L} \in F$, we write $V_{h+1,\mathcal{L}}$ for the unique 2-dimensional representation of $G_{\mathbb{Q}_p}$ such that $D^*_{\mathrm{st}}(V_{h+1,\mathcal{L}}) \cong D_{h+1,\mathcal{L}}$, where $D_{h+1,\mathcal{L}}$ is as in §3.2. Write $\overline{V}$ for the semisimple reduction modulo $\mathfrak{m}_F$ of V. Let $\mathbb{Q}_{p^2}$ be the unramified quadratic extension of $\mathbb{Q}_p$, χ the unramified quadratic character of $G_{\mathbb{Q}_{p^2}}$, and ω_2 a niveau 2 fundamental character of $\mathbb{Q}_{p^2}$. Note that $\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^h\chi)$ has determinant ω^h, where ω is the cyclotomic character, and its restriction to inertia is $\omega_2^h \oplus \omega_2^{ph}$.

Theorem 4.1. *Assume $h \geq 3$ and $p \neq 2$. Then if $\mathcal{L}$ satisfies*

$$v_p\left(\mathcal{L}^{-1}\right) > \frac{h-1}{2} - 1 + v_p((h-1)!),$$

we have $\overline{V}_{h+1,\mathcal{L}} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^h\chi)$.

Remark 4.2. Our contribution toward Theorem 4.1 is limited to $h \geq 4$ and $p = h = 3$. The case of $h = 3$ and $p \geq 5$ follows from the work of Breuil and Mézard. If we were to use the weaker bound $v_p\left(\mathcal{L}^{-1}\right) > \frac{h-1}{2} + v_p((h-1)!)$, then our calculation would also cover the cases of $h = 2$ and $h = 3$. See Remark 4.8 for further explanations.

We plan to take the matrix of φ acting on $\mathcal{M}_{h+1,\mathcal{L}} = \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{h+1,\mathcal{L}}) \otimes_{\mathcal{O}_F} S_F$ as in Theorem 3.7 and replace it with a φ-conjugate defined over $\mathfrak{S}_\Lambda$ when $v_p\left(\mathcal{L}^{-1}\right)$ satisfies the bound in the theorem. This defines a Kisin module $\mathfrak{M}$ for $V_{h+1,\mathcal{L}}$ that allows us to calculate the reduction $\overline{V}_{h+1,\mathcal{L}}$. Despite our theorem being limited to $h \geq 3$, we will present many calculations assuming only $h \geq 2$, in order to later justify Remark 4.2. So we assume without further comment that

$$p \neq 2 \text{ and } h \geq 2; \qquad (4.1)$$
$$v_p\left(\mathcal{L}^{-1}\right) > \frac{h-1}{2} - 1 + v_p((h-1)!).$$

We will clarify result by result where we need to limit to $h \geq 3$ or $h \geq 4$. Also, fix $z = \sum_{j=1}^{h-1} x_j E^j$ as in Theorem 3.7. Note that by formula (4.1), we have $v_p\left(\mathcal{L}^{-1}\right) \geq -1$, so the estimates (3.4) in Theorem 3.7 hold.

4.1. Preparing for descent

Consider the ring

$$R_2 = \left\{ f = \sum a_i u^i \in F[\![u]\!] \mid i + 2v_p(a_i) \to \infty \text{ as } i \to \infty \right\}.$$

Thus R_2 is the F-Banach algebra of series converging on $|u| \leq p^{-1/2}$. We equip R_2 with the valuation $v_{R_2}\left(\sum a_i u^i\right) = \inf_i \{i + 2v_p(a_i)\}$. The canonical map $\mathcal{O}_F \hookrightarrow R_2$ factors through

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S_F, since $v_{R_2}(E^p/p) = p-2 > 0$. Finally, given $v \in \mathbb{R}$, we define additive subgroups $H_v^\circ \subseteq H_v \subseteq R_2$ by

$$H_v = \{f \in R_2 \mid v_{R_2}(f) \geq v\}, \qquad H_v^\circ = \{f \in R_2 \mid v_{R_2}(f) > v\}.$$

For any v, H_v and H_v° are stable under φ. In fact, for any $j \geq 0$ we have

$$\varphi\left(H_v \cap u^j R_2\right) \subseteq H_{v+j(p-1)} \cap u^{pj} R_2, \tag{4.2}$$

and the same for H_v° replacing H_v (see, e.g., [3, Lemma 4.1.1]).

Our first lemma, concerning some entries of the matrix in Theorem 3.7, is straightforward, so we omit the proof (compare with [3, Lemma 5.1.1]).

Lemma 4.3. *Let $\lambda_- = \prod_{n\geq 0} \varphi^{2n+1}(E)/p$ and $\lambda_{++} = \varphi(\lambda_-)$ be as in Theorem 3.7. Then*

(a) $\lambda_- \in 1 + H_{p-2}$ *and* $\lambda_{++} \in 1 + H_{p^2-2}$,

(b) $\lambda_-, \lambda_{++} \in R_2^\times$, *and*

(c) $v_{R_2}\left(\lambda_-^{\pm 1}\right) = 0 = v_{R_2}\left(\lambda_{++}^{\pm 1}\right)$.

We also prepare estimates for z. Note that by formula (4.1), the estimate (3.4) becomes

$$v_p(x_j) > v_p((h-1)!) - v_p(j!) - j - 1 \geq -j-1. \tag{4.3}$$

Recall that we write $a_p = \varpi^{h-1} + \varpi^{h+1}$. Thus, $v_p(a_p) = \frac{h-1}{2}$.

Lemma 4.4. *For $z = \sum_{j=1}^{h-1} x_j E^j$ as before and $\nu = -1 + \varphi(z)\left(a_p - p^h z\right)$, we have the following:*

(a) $p^h z \in H_{h-1}^\circ$.

(b) $\varphi(z) \in H_{-2}^\circ$.

(c) $\nu \in -1 + H_{h-3}^\circ$.

(d) *If $h \geq 3$, then $\nu \in R_2^\times$.*

Furthermore, if $p = 3$ and $h = 3$, then $\varphi(z) \in H_{-1}^\circ$ and $\nu \in -1 + H_{h-2}^\circ = -1 + H_1^\circ$.

Proof. First, $v_{R_2}\left(E^j\right) = j$. By the ultrametric inequality and formula (4.3), we see

$$v_{R_2}(z) > \inf\{2(-j-1) + j \mid 1 \leq j \leq h-1\} = -1-h.$$

Part (a) follows because $v_{R_2}\left(p^h\right) = 2h$. For (b), note that $v_{R_2}\left(\varphi(E)^j\right) = 2j$. Thus, using formula (4.3),

$$v_{R_2}(\varphi(z)) > \inf\{2(-j-1) + 2j \mid 1 \leq j \leq h-1\} = -2.$$

Continuing, $\varphi(z)p^h z \in H_{h-3}^\circ$ by parts (a) and (b), and since $v_{R_2}(a_p) = h-1$, we have $\varphi(z)a_p \in H_{h-3}^\circ$. This proves (c). Finally, part (d) follows from the geometric series and part (c).

Finally, suppose $p = h = 3$. By the argument for (c), it suffices to show that $\varphi(z) \in H_{-1}^\circ$. We note that $v_{R_2}\left(\varphi(E)^j - E(0)^j\right) \geq p + 2j - 2$ for any j. Thus, by formula (4.3),

$$v_{R_2}(\varphi(z) - \varphi(z)(0)) > p + 2j - 2 - 2(j+1) = p - 4 = -1. \tag{4.4}$$

But by Example 3.9 we have $\varphi(z)(0) = z(0) = \frac{1}{4\mathcal{L}}\left(\frac{1}{\mathcal{L}} + 1\right)$. Since $v_p\left(\mathcal{L}^{-1}\right) > 0$, formula (4.4) implies $v_{R_2}(\varphi(z)) > -1$, as we wanted. ∎

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We now write $\mathcal{M}_2 = \mathcal{M}_{h+1,\mathcal{L}} \otimes_{S_F} R_2 \cong \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{h+1,\mathcal{L}}) \otimes_{\mathcal{O}_F} R_2$. Thus, $\mathcal{M}_2 \in \mathrm{Mod}_{R_2}^{\varphi,\leq h}$. We also introduce some notation. Given $A \in \mathrm{Mat}_d(R_2)$ and $C \in \mathrm{GL}_d(R_2)$, we write $C *_\varphi A = C \cdot A \cdot \varphi(C)^{-1}$. Thus, if (e_1,e_2) is a basis of $\mathcal{M}_2$ and A is the matrix of $\varphi_{\mathcal{M}_2}$ in that basis, $C *_\varphi A$ is the matrix of $\varphi_{\mathcal{M}_2}$ in the basis (e_1',e_2') given by $(e_1',e_2') = (e_1,e_2)C^{-1}$.

Proposition 4.5. *Assume $h \geq 4$ or $p = h = 3$. Then there exists a basis of $\mathcal{M}_2$ in which the matrix of $\varphi_{\mathcal{M}_2}$ is $\left(\begin{smallmatrix} G & -1 \\ E^h & 0 \end{smallmatrix}\right)$, where $G \in (a_p - p^h z)\left(\frac{\lambda_-}{\lambda_{++}}\right)^h + H_h^\circ$.*

Proof. By Theorem 3.7, there is a basis (e_1,e_2) of $\mathcal{M}_2$ such that $\varphi_{\mathcal{M}_2}(e_1,e_2) = (e_1,e_2)A$, where

$$A = \begin{pmatrix} (a_p - p^h z)\left(\frac{\lambda_-}{\lambda_{++}}\right)^h & -1 + \varphi(z)(a_p - p^h z) \\ E^h & E^h \varphi(z)\left(\frac{\lambda_{++}}{\lambda_-}\right)^h \end{pmatrix} = \begin{pmatrix} \mu & \nu \\ E^h & \eta \end{pmatrix},$$

with ν as in Lemma 4.4 and μ and η defined by the equality. Assume for now just that $h \geq 3$. Then, by Lemma 4.4(d), $\nu \in R_2^\times$. Making a change of basis on $\mathcal{M}_2$, we replace A (note that $\mu\eta = (1+\nu)E^h$) by

$$A' = \begin{pmatrix} 1 & 0 \\ -\eta/\nu & 1 \end{pmatrix} *_\varphi A = \begin{pmatrix} \mu + \frac{\nu\varphi(\eta)}{\varphi(\nu)} & \nu \\ -E^h \nu^{-1} & 0 \end{pmatrix}.$$

Since $v_{R_2}(\nu + 1) > 0$ by Lemma 4.4(c), we have $\nu(0) \in \Lambda^\times$. Thus $\nu_0 = \nu/\nu(0) \in 1 + (H_{h-3}^\circ \cap u R_2)$. By formula (4.2), we have $\varphi^k(\nu_0) \in 1 + H_{h-3+m_k}$, where $m_k \to \infty$ as $k \to \infty$. Thus, the infinite product $\nu_+ = \prod_{n \geq 0} \varphi^{2n}(\nu_0)$ converges in R_2. Set $\nu_- = \varphi(\nu_+)$, so $\nu_\pm \in 1 + H_{h-3}^\circ \subseteq R_2^\times$. We now change basis on $\mathcal{M}_2$ again to get a matrix A'' for $\varphi_{\mathcal{M}_2}$ given by

$$A'' = \begin{pmatrix} \frac{-1}{\nu(0)} \frac{\nu_-}{\nu_+} & 0 \\ 0 & \frac{\nu_+}{\nu_-} \end{pmatrix} *_\varphi A' = \begin{pmatrix} G & -1 \\ E^h & 0 \end{pmatrix},$$

where

$$G = \left(\mu + \frac{\nu\varphi(\eta)}{\varphi(\nu)}\right) \frac{\nu_-^2}{\nu_+ \nu_{++}} \tag{4.5}$$

and $\nu_{++} = \varphi(\nu_-)$.

To complete the argument, we justify $G \in \mu + H_h^\circ$. We already know $\nu_-^2/\nu_+\nu_{++} \in 1 + H_{h-3}^\circ$. The same is true for $\nu/\varphi(\nu)$. So

$$v_{R_2}\left(\frac{\nu\varphi(\eta)}{\varphi(\nu)}\right) \geq v_{R_2}(\varphi(\eta)) \geq v_{R_2}\left(\varphi(E)^h \varphi^2(z)\right), \tag{4.6}$$

where we used Lemma 4.3 to remove λ_- and λ_{++} from the estimate. We note that $v_{R_2}\left(\varphi(E)^h\right) = 2h$ and $v_{R_2}\left(\varphi^2(z)\right) \geq v_{R_2}(\varphi(z)) > -2$, by formula (4.2) and Lemma 4.4(b). Thus from formula (4.6) we deduce that $v_{R_2}(\nu\varphi(\eta)/\varphi(\nu)) > 2h - 2 = 2(h-1)$. We also note that $a_p - p^h z \in H_{h-1}$. Thus, $\mu \in H_{h-1}$, and so returning to the definition (4.5) of μ and G, we see

$$G \in \left(\mu + H^\circ_{2(h-1)}\right) \cdot (1 + H^\circ_{h-3}) \subseteq \mu + H^\circ_{2h-4} + H^\circ_{2(h-1)} = \mu + H^\circ_{2h-4}.$$

Now, if $h \geq 4$, then $2h-4 \geq h$ and so $G \in \mu + H^\circ_h$. This completes the proof except if $p = h = 3$. In that case, Lemma 4.4 shows that $\nu \in -1 + H^\circ_1$, rather than $-1 + H^\circ_0$, from which we deduce

$$G \in (\mu + H^\circ_4) \cdot (1 + H^\circ_1) \subseteq \mu + H^\circ_3 = \mu + H^\circ_h$$

anyway. This completes the proof. □

4.2. Descent

To descend to $\mathfrak{S}_\Lambda$, we use the algorithm from [3, §4]. Write $T_{\leq d} : R_2 \to F[u]$ for the 'truncation' operation $T_{\leq d}\left(\sum a_i u^i\right) = \sum_{i \leq d} a_i u^i$ and $T_{>d}(f) = f - T_{\leq d}(f)$. In the next two proofs, we will use the following principle: if $f \in R_2$ and $v_{R_2}(T_{\leq d}(f)) > d$ (for instance, if $v_{R_2}(f) > d$), then $T_{\leq d}(f) \in \mathfrak{m}_F[u]$.

Proposition 4.6. *Suppose that $G \in R_2$ such that*

(a) $G \in H_{h-1}$,
(b) $T_{>h}(G) \in H^\circ_{h-1}$, *and*
(c) $T_{\leq h}(G) \in \mathfrak{m}_F[u]$.

Then, given $A = \begin{pmatrix} G & -1 \\ E^h & 0 \end{pmatrix}$, there exist $C \in \mathrm{GL}_2(R_2)$ and $P \in \mathfrak{m}_F[u]$ such that $C *_\varphi A = \begin{pmatrix} P & -1 \\ E^h & 0 \end{pmatrix}$.

Proof. Since $E^h \in u^h + H_{h+1}$, (a) implies that

$$A \in \begin{pmatrix} 0 & -1 \\ u^h & 0 \end{pmatrix} + \begin{pmatrix} H_{h-1} & 0 \\ H_{h+1} & 0 \end{pmatrix}.$$

In the notation of [3, §4.3], set $a = 0$, $b = h$, $a' = \frac{h}{2} - \frac{p-1}{2}$, $b' = \frac{h}{2} + \frac{p-1}{2}$, and $(c_0, c_h) = (-1, 1)$. Since $h - 1 - a' = \frac{h}{2} - 1 + \frac{p-1}{2} \geq 1$, we see that A is γ-allowable with $\gamma = 1$ in the sense of [3, Definition 4.3.1]. The error of A, in the same definition, is $\varepsilon = v_{R_2}(T_{>h}(G)) - a'$. By [3, Theorem 4.3.7], with $R = R_2$, there exists $C \in \mathrm{GL}_2(R_2)$ such that $A' = C *_\varphi A$ satisfies the following:

(i) Evaluating at $u = 0$, we have $A'|_{u=0} = A|_{u=0}$.
(ii) The matrix A' is of the form $A' = \begin{pmatrix} P & -1 \\ f & 0 \end{pmatrix}$, with P and f polynomials of degree at most h.
(iii) We have an estimate $v_{R_2}(P - T_{\leq h}(G)) \geq \varepsilon + a' + 1$.

(For the reader checking references, note that the role of A versus C is reversed in [3].)

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We claim $P \in \mathfrak{m}_F[u]$ and $f = E^h$, which would finish the proof of the proposition. To see $P \in \mathfrak{m}_F[u]$, we start by combining the estimate (iii) and the assumption (b) in order to see that

$$v_{R_2}(P - T_{\leq h}(G)) \geq \varepsilon + a' + 1 = v_{R_2}(T_{>h}(G)) + 1 > h.$$

On the other hand, $P - T_{\leq h}(G)$ has degree at most h, by (ii), and so $P - T_{\leq h}(G) \in \mathfrak{m}_F[u]$, which implies $P \in \mathfrak{m}_F[u]$ by assumption (c).

To see $f = E^h$, we evidently have $f = \det(A') = rE^h$ for some $r \in R_2^\times$. In particular, f has a root of multiplicity h at $u = -p$. But f is a polynomial of degree at most h, by point (ii), and by point (i) we have $f(0) = E(0)^h$. It now follows quickly that $f = E^h$, since $F[\![u]\!]$ is a unique factorization domain. □

We now verify that the G from Proposition 4.5 satisfies the hypothesis of Proposition 4.6.

Lemma 4.7. *Set* $G \in (a_p - p^h z)\left(\frac{\lambda_-}{\lambda_{++}}\right)^h + H_h^\circ$. *Then*

(a) $G \in H_{h-1}$,

(b) $T_{>h}(G) \in H_{h-1}^\circ$, *and*

(c) $T_{\leq h}(G) \in \mathfrak{m}_F[u]$.

Proof. First, the conclusions depend only on $G \bmod H_h^\circ$, so we suppose $G = (a_p - p^h z)\left(\frac{\lambda_-}{\lambda_{++}}\right)^h$. Part (a) follows from Lemmas 4.3 and 4.4. For part (b), we first have, by Lemma 4.3(a), that $a_p\left(\frac{\lambda_-}{\lambda_{++}}\right)^h \in a_p + a_p H_{p-2}$, so $T_{>0}\left(a_p\left(\frac{\lambda_-}{\lambda_{++}}\right)^h\right) \in H_{h+p-3} \subseteq H_h$. On the other hand, by Lemma 4.4(a) we have $p^h z \in H_{h-1}^\circ$. Thus we have shown in fact that $T_{>0}(G) \in H_{h-1}^\circ$.

Finally, we consider part (c). Since $E = u + p$, any $f \in S_\Lambda$ can be written as $f = \sum_{n=0}^{\infty} \alpha_n \frac{u^n}{p^{\lfloor \frac{n}{p} \rfloor}}$ with $\alpha_n \in \Lambda$. Let $f = \frac{\lambda_-}{\lambda_{++}} \in S_\Lambda$ in particular. Since $v_p(a_p) = \frac{h-1}{2} > \left\lfloor \frac{h}{p} \right\rfloor$ unless $p = h = 3$ (or $p = 2$, which we have excluded in formula (4.1)), we see immediately that $T_{\leq h}\left(a_p f^h\right) \in \mathfrak{m}_F[u]$ except when $h = p = 3$. When $h = p$, however,

$$T_{\leq p}(f^p) = T_{\leq p}\left(\left(\sum_{n=0}^{p-1} \alpha_n u^n + \alpha_p \frac{u^p}{p}\right)^p\right) \in p \cdot \alpha_0^{p-1} \alpha_p \frac{u^p}{p} + \Lambda[u] \subseteq \Lambda[u].$$

Since $v_p(a_p) > 0$, we see that $T_{\leq h}\left(a_p f^h\right) \in \mathfrak{m}_F[u]$ in every case.

By the prior paragraph, to show (c) it remains to show that $T_{\leq h}\left(p^h z f^h\right) \in \mathfrak{m}_F[u]$ as well. By definition, we can write $f^h = \sum_{i=0}^{\infty} \beta_i \frac{E^i}{p^{\lfloor \frac{i}{p} \rfloor}}$ with $\beta_i \in \Lambda$, and we recall that $z = \sum_{j=1}^{h-1} x_j E^j$. Thus

$$p^h z f^h = \sum_{n=1}^{\infty} \left(\sum_{i+j=n} p^h x_j \beta_i p^{-\lfloor \frac{i}{p} \rfloor}\right) E^n. \tag{4.7}$$

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Using the binomial expansion of $E^n = (u+p)^n$, we see that the u^m-term of equation (4.7) is exactly equal to

$$\sum_{n=m}^{\infty}\left(\sum_{i+j=n} p^h x_j \beta_i p^{-\left\lfloor \frac{i}{p}\right\rfloor}\right)\binom{n}{m} p^{n-m}.$$

We must show that this has positive p-adic valuation for $m \leq h$. Since $\beta_i \in \Lambda$ and binomial coefficients are integers, it is enough to show that for all $m \leq h$ and $j < h$, if $n \geq m, j$, then

$$v_p(x_j) + h + n - m - \left\lfloor \frac{n-j}{p} \right\rfloor > 0. \tag{4.8}$$

By formula (4.3) we have $v_p(x_j) > -j-1$, and so

$$v_p(x_j) + h + n - m - \left\lfloor \frac{n-j}{p} \right\rfloor > h - m - 1 + n - j - \left\lfloor \frac{n-j}{p} \right\rfloor. \tag{4.9}$$

But the right-hand side of this inequality is nonnegative. Indeed, when $h > m$, this is clear because $n \geq j$. When $h = m$, on the other hand, we have $n \geq m = h > j$. So the right-hand side in that case has the form $x - \lfloor x/p \rfloor - 1$ with $x \geq 1$, which is also nonnegative. □

4.3. Proof of Theorem 4.1

Finally, we give the proof of the main theorem:

Assume that $h \geq 3$ and $p \neq 2$. Then if $\mathcal{L}$ satisfies

$$v_p\left(\mathcal{L}^{-1}\right) > \frac{h-1}{2} - 1 + v_p((h-1)!),$$

we have $\overline{V}_{h+1,\mathcal{L}} \cong \mathrm{Ind}_{G_{\mathbb{Q}_p^2}}^{G_{\mathbb{Q}_p}}\left(\omega_2^h \chi\right)$.

Proof. First, if $h = 3$ and $p \geq 5$, then the assumption is that $v_p(\mathcal{L}) < 0$. The verification that $\overline{V}_{4,\mathcal{L}} \cong \mathrm{Ind}_{G_{\mathbb{Q}_p^2}}^{G_{\mathbb{Q}_p}}\left(\omega_2^3 \chi\right)$ is the first bullet point of [7, Theorem 4.2.4.7(iii)], where the reader should take $k = 4 < p$ and $\ell = v_p(\mathcal{L}) < 0$.

Now we assume that either $h \geq 4$ or $p = h = 3$. Then, applying Proposition 4.5, Lemma 4.7, and Proposition 4.6, we deduce that there exists a basis of $\mathcal{M}_2$ in which the matrix of $\varphi_{\mathcal{M}_2}$ is given by $A = \left(\begin{smallmatrix} P & -1 \\ E^h & 0 \end{smallmatrix}\right)$ and $P \in \mathfrak{m}_F[u]$. Define $\mathfrak{M} = \mathfrak{S}_\Lambda^{\oplus 2}$ with the matrix of φ being given by A. Clearly $\mathfrak{M}$ is a Kisin module over $\mathfrak{S}_\Lambda$ of height $\leq h$, and

$$\mathfrak{M} \otimes_{\mathfrak{S}_\Lambda} R_2 \cong \mathcal{M}_2 = \underline{\mathcal{M}}_{\mathcal{O}_F}(D_{h+1,\mathcal{L}}) \otimes_{\mathcal{O}_F} R_2$$

as φ-modules over R_2. Thus, by Proposition 2.1 we deduce $\mathfrak{M} = \mathfrak{M}(T)$ for some lattice $T \subseteq V_{h+1,\mathcal{L}}$. Furthermore, $\mathfrak{M} \otimes_{\mathfrak{S}_\Lambda} \mathbb{F}\left[u^{-1}\right]$ is a φ-module over $\mathbb{F}((u))$ with Frobenius given by $\left(\begin{smallmatrix} 0 & -1 \\ u^h & 0 \end{smallmatrix}\right)$. This shows, in particular, that $\overline{V}_{h+1,\mathcal{L}}$ is the same for any $\mathcal{L}$ satisfying formula (4.1) (see [3, Corollary 2.3.2]).

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Let $V_{h+1,\infty}$ be as in the introduction. By [3, Corollary 5.2.2], for $V_{h+1,\infty}$ there exists a Kisin module $\mathfrak{M}'$ such that $M' := \mathfrak{M}' \otimes_{\mathfrak{S}_\Lambda} \mathbb{F}\left[u^{-1}\right]$ has Frobenius also given by $\begin{pmatrix} 0 & -1 \\ u^h & 0 \end{pmatrix}$ and M' determines $\overline{V}_{h+1,\infty} \cong \mathrm{Ind}_{G_{\mathbb{Q}_p^2}}^{G_{\mathbb{Q}_p}}\left(\omega_2^h \chi\right)$. Therefore, $\overline{V}_{h+1,\mathcal{L}} \cong \overline{V}_{h+1,\infty} \cong \mathrm{Ind}_{G_{\mathbb{Q}_p^2}}^{G_{\mathbb{Q}_p}}\left(\omega_2^h \chi\right)$. □

Remark 4.8. We return to Remark 4.2. Suppose we replace formula (4.1) with

$$v_p\left(\mathcal{L}^{-1}\right) > \frac{h-1}{2} + v_p((h-1)!). \tag{4.10}$$

This has the impact of scaling z by a p-adic unit multiple of p, thus increasing $v_{R_2}(z)$ by 2 throughout our estimates in §4.1. The reader may check that Proposition 4.5 holds with these new estimates, and so the proof goes through for all $h \geq 2$ and $p \geq 3$ under the assumption (4.10). Of course, this bound is not the sharpest possible when $h = 2$ or $h = 3$. For instance, we have already noted that for $h = 3$ and $p \geq 5$, Breuil and Mézard confirmed Theorem 4.1 with the stronger bound (4.1).

The situation is more complicated when $h = 2$. In that case, for $p \geq 5$, Guerberoff and Park showed that $\overline{V}_{3,\mathcal{L}} \cong \mathrm{Ind}_{G_{\mathbb{Q}_p^2}}^{G_{\mathbb{Q}_p}}\left(\omega_2^2 \chi\right)$ exactly on $v_p(\mathcal{L} - 1) < \frac{1}{2}$ [17, Theorem 5.0.5]. Thus, the bound $v_p(\mathcal{L}) < \frac{1}{2}$ from Theorem 4.1 produces too large a region of $\mathcal{L}$-invariants, whereas formula (4.10) produces a region too small. For the interested reader, Guerberoff and Park also determined, for any $\mathcal{L}$, the restriction of $\overline{V}_{3,\mathcal{L}}$ to the inertia subgroup. The restriction to inertia was recently removed by Chitrao, Ghate, and Yasuda using a completely different method [10, Theorem 1.3]. Thus we have a complete picture of $\overline{V}_{3,\mathcal{L}}$. It would be amusing to understand if that picture can be recovered from the method here.

Acknowledgments. We owe the heuristic reframing in §1.2 to comments by Laurent Berger and Christophe Breuil during the conference "Géométrie arithmétique, théorie des représentations et applications" at the Centre International de Rencontres Mathématiques (CIRM) in Luminy, France. Part of this collaboration also took place during the workshop "Moduli spaces and modularity" at Casa Matemática Oaxaca (CMO). We thank both Berger and Breuil for their comments and both CIRM and CMO for their hospitality. The first author also thanks the Max Planck Institut für Mathematik in Bonn, Germany for hospitality during a period when this paper was revised.

Finally, acknowledgments are due for the discussion in §1.3. First, the data reported in Table 1 in the first preprint version of this article was inaccurate. Because of that, we drew faulty conclusions, which have now been removed, on the strength of the bound in Theorem 1.1. We thank Robert Pollack for calculating the original data and then alerting us to the error. We also especially thank Peter Gräf for replicating the newly reported data, using his alternative method.

The first author was partially supported by an NSF Grant (DMS-1402005) and a Simons Collaboration Grant (#713782). The second author was supported by a grant from the Simons Foundation/SFARI (#585753).

Competing Interests. None.

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