Canad. Math. Bull. Vol. **22** (1), 1979

A NOTE ON REDUCTIVE OPERATORS

BY
P. A. FILLMORE

For a bounded linear operator T on a Hilbert space $\mathcal{H}$, denote by $\mathrm{Lat}_0\, T$ the lattice of all linear submanifolds $\mathcal{M}$ of $\mathcal{H}$ such that $T\mathcal{M} \subset \mathcal{M}$, and by $\mathrm{Lat}_{\frac{1}{2}}\, T$ (resp. $\mathrm{Lat}\, T$) the sublattice consisting of operator ranges (resp. closed subspaces). The operator T is said to be reductive if $\mathrm{Lat}\, T \subset \mathrm{Lat}\, T^*$. Dyer, Pedersen, and Porcelli [3] asked whether reductive operators must be normal, and showed that an affirmative answer would be equivalent to an affirmative solution of the invariant subspace problem (see also [1]).

Thus it may be of some interest to examine the implications of the (stronger) conditions $\mathrm{Lat}_0\, T \subset \mathrm{Lat}_0\, T^*$ and $\mathrm{Lat}_{\frac{1}{2}}\, T \subset \mathrm{Lat}_{\frac{1}{2}}\, T^*$. The former is known [4] to imply that T^* is a polynomial in T, and in particular that T is normal. However it seems not to have been noticed that the latter implies the same conclusion.

THEOREM. *For any operator T the following conditions are mutally equivalent*:

1. $\mathrm{Lat}_0\, T \subset \mathrm{Lat}_0\, T^*$;
2. *$\mathrm{Lat}_{\frac{1}{2}}\, T \subset \mathrm{Lat}_{\frac{1}{2}}\, T^*$*;
3. *$T^* = u(T)$ for some entire function u*;
4. *$T^* = p(T)$ for some polynomial p*;
5. *Either T is normal and algebraic, or else $T = aH + bI$ for some self-adjoint operator H and complex numbers a and b.*

Moreover each of these conditions is equivalent to the symmetric condition obtained by interchanging T and T^*.

Proof. Obviously 1 implies 2. Assume 2 holds. If T is not algebraic (i.e. $p(T) \neq 0$ for all nonzero polynomials p), then 3 holds by [2, Th. 2]. If T is algebraic, we will show that it satisfies 1, so that (as remarked above) 4 and hence 3 will follow by [4]. Fix $\mathcal{M} \in \mathrm{Lat}_0\, T$ and $x \in \mathcal{M}$. Since T is algebraic the cyclic subspace $[x]$ generated by x is finite dimensional, so that $[x] \in \mathrm{Lat}_{\frac{1}{2}}\, T$, $[x] \in \mathrm{Lat}_{\frac{1}{2}}\, T^*$ by 2, and $T^*x \in [x] \subset \mathcal{M}$. Therefore $T^*\mathcal{M} \subset \mathcal{M}$ and $\mathcal{M} \in \mathrm{Lat}_0\, T^*$.

Next assume $T^* = u(T)$ for an entire function u. Then $u^*(z) = \overline{u(\bar{z})}$ defines an entire function u^*, and it is easy to see (from the power series expansion of u) that

$$T = (T^*)^* = (u(T))^* = u^*(T^*) = u^*(u(T)).$$

Received by the editors May 15, 1978 and, in revised form, June 23, 1978.

Consider the entire function $v(z)=u^*(u(z))-z$; we have that $v(T)=0$. If $v\neq 0$, then T is algebraic [5, p. 860], so assume $v=0$. Then u is a homeomorphism, and in particular $u^{-1}(K)$ is compact for every compact set K, so that $\lim_{z\to\infty}|u(z)|=\infty$ and u has a pole at infinity. Thus u is a univalent polynomial, and so is linear. We now have $T^*=aT+bI$, and it follows easily that T is a linear function of a self-adjoint operator of the form $cT+\bar{c}T^*$. Therefore 3 implies 5.

Now assume 5. If T is normal and algebraic, then it is of the form $\sum_{i=1}^n \lambda_i E_i$ for suitable pairwise orthogonal projections E_i and complex numbers λ_i, and therefore $p(T)=T^*$ for any polynomial p with $p(0)=0$, $p(1)=1$, and $p(\lambda_i)=\bar{\lambda}_i$ for all i. If $T=aH+bI$ with H self-adjoint, then T^* is a linear function of H, and H is a linear function of T, so T^* is a linear function of T. Thus 5 implies 4. Since 4 obviously implies 1, and since 5 is equivalent to the symmetric condition obtained by interchanging T and T^*, the proof is complete.

References

1. E. A. Azoff and F. Gilfeather, *Measurable choice and the invariant subspace problem*, Bull. Amer. Math. Soc. **80** (1974) 893–895.

2. R. G. Douglas and C. Foias, *Infinite dimensional versions of a theorem of Brickman-Fillmore*, Indiana University Math. J. **25** (1976) 315–320.

3. J. A. Dyer, E. A. Pedersen, and P. Porcelli, *An equivalent formulation of the invariant subspace conjecture*, Bull. Amer. Math. Soc. **78** (1972) 1020–1023.

4. P. A. Fillmore, *On invariant linear manifolds*, Proc. Amer. Math. Soc. **41** (1973) 501–505.

5. P. R. Halmos, *Capacity in Banach algebras*, Indiana Univ. Math. J. **20** (1971) 855–863.

Dalhousie University
Halifax
Nova Scotia B3H 3J5